Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (5) (2020), 1726 – 1734
DOI : 10.15672/hujms.605621

Research Article

On a subclass of the generalized Janowski type functions of complex order

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Abstract

In this paper, we introduce the class $\mathcal{JR}_b^{\lambda}(\alpha,\beta,\delta,A,B)$ of generalized Janowski type functions of complex order defined by using the Ruscheweyh derivative operator in the open unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. The bound for the n-th coefficient and subordination relation are obtained for the functions belonging to this class. Some consequences of our main theorems are same as the results obtained in the earlier studies.

Mathematics Subject Classification (2010). 30C45

Keywords. analytic function, subordination, λ-spirallike function, λ-Robertson function, λ-close-to-spirallike function, λ-close-to-Robertson function, Ruscheweyh derivative operator

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ which are univalent in $\mathbb{D}$.

The hadamard product or convolution of two functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{A}$ and $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{A}$ denoted by $f*g$, is defined by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n$$

for $z\in\mathbb{D}$.

In 1975, Ruscheweyh [10] introduced a linear operator

$$\mathcal{D}_{\mathcal{R}}^{\alpha}f(z)=\frac{z}{(1-z)^{\alpha+1}}*f(z)=z+\sum_{n=2}^{\infty}\varphi_n(\alpha)\,a_nz^n \tag{1.2}$$

with

$$\varphi_n(\alpha)=\frac{(\alpha+1)_{n-1}}{(n-1)!}$$

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Received: 16.08.2019; Accepted: 19.12.2019

for $\alpha > -1$ and $(a)_n$ is Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

for $a \in \mathbb{C}$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Notice that

$$\mathcal{D}^0_{\mathfrak{R}} f(z) = f(z),$$
$$\mathcal{D}^1_{\mathfrak{R}} f(z) = z f'(z)$$

and

$$\mathcal{D}^m_{\mathfrak{R}} f(z) = \frac{z\left(z^{m-1} f(z)\right)^m}{m!} = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+m)}{\Gamma(m+1)(n-1)!} a_n z^n$$

for all $\alpha = m \in N_0 = \{0, 1, 2, \ldots\}$.

In recent years, several authors obtained many interesting results for various subclasses of analytic functions defined by using the Ruscheweyh derivative operator.

Given two functions f and F, which are analytic in the unit disk $\mathbb{D}$, we say that the function f is subordinated to F , and write $f \prec F$ or $f(z) \prec F(z)$, if there exists a function ω analytic in $\mathbb{D}$ such that $|\omega(z)| < 1$ and $\omega(0) = 0$, with $f(z) = F(\omega(z))$ in $\mathbb{D}$.

In particular, if F is univalent in $\mathbb{D}$, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathbb{D}) \subseteq F(\mathbb{D})$.

Let $\mathcal{P}$ denote the class of all functions of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ that are analytic in $\mathbb{D}$ and for which $\Re\, p(z) > 0$ in $\mathbb{D}$.

For arbitrary fixed numbers A and B with $-1 \leq B < A \leq 1$, Janowski [5] introduced the class $\mathcal{P}(A, B)$, defined by the subordination principle as follows:

$$\mathcal{P}(A, B) = \left\{ p : \; p(z) \prec \frac{1 + Az}{1 + Bz}, \quad p(z) = 1 + p_1 z + p_2 z^2 + \ldots \right\}.$$

Also, if we take $A = 1$ and $B = -1$, we obtain the well-known class $\mathcal{P}$ of functions with positive real part.

In 2006, Polatoglu [8] introduced the class $\mathcal{P}(A, B, \delta)$ of the generalization of Janowski functions as follows:

$$\mathcal{P}(A, B, \delta) = \left\{ p : \; p(z) \prec (1 - \delta) \frac{1 + Az}{1 + Bz} + \delta, \quad p(z) = 1 + p_1 z + p_2 z^2 + \ldots \right\}. \tag{1.3}$$

for arbitrary fixed numbers A and B with $-1 \leq B < A \leq 1$, $0 \leq \delta < 1$, $z \in \mathbb{D}$.

Let $\mathcal{S}^*$ and $\mathcal{C}$ be the subclasses of $\mathcal{S}$ of all starlike functions and convex functions in $\mathbb{D}$, respectively. We also denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ the class of starlike functions of order α and the class of convex functions of order α, where $0 \leq \alpha < 1$, respectively.

In particular, we note that $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{C} := \mathcal{C}(0)$.

In [9], Reade introduced the class $\mathcal{CS}^*$ of close-to-star functions as follows:

$$\mathcal{CS}^* = \left\{ f \in \mathcal{A} : \Re \frac{f(z)}{g(z)} > 0 \text{ and } \; g \in \mathcal{S}^* \right\}$$

for all $z \in \mathbb{D}$. Also, we denote by $\mathcal{CS}^*(\beta)$ the class of close-to-star functions of order β where $0 \leq \beta < 1$. (See Goodman [3]).

In [6], Kaplan introduced the class $\mathcal{CC}$ of close-to-convex functions as follows:

$$\mathcal{CC} = \left\{ f \in \mathcal{A} : \; \Re \frac{f'(z)}{g'(z)} > 0 \text{ and } \; g \in \mathcal{C} \right\}$$

for all $z \in \mathbb{D}$. Also, we denote by $\mathcal{CC}(\beta)$ the class of close-to-convex functions of order β where $0 \leq \beta < 1$. (See Goodman [2]).

Clearly, we note that $\mathcal{CS}^* := \mathcal{CS}^*(0)$ and $\mathcal{CC} := \mathcal{CC}(0)$.

$f \in \mathcal{A}$ is an λ-spirallike function , $\mathcal{SP}^\lambda$, if and only if

$$\Re\left[e^{i\lambda}\,\frac{zf'(z)}{f(z)}\right] > 0$$

for some $|\lambda| < \frac{\pi}{2}$, $z \in \mathbb{D}$. The class of λ-spirallike functions was introduced by Špaček in [11].

Also, $f \in \mathcal{SP}^\lambda$ if and only if there exists a function $p \in \mathcal{P}$ such that

$$f(z) = z\exp\left\{cos\lambda e^{-i\lambda}\int_0^z \frac{p(t)-1}{t}dt\right\}.$$

We note that the extremal function for the class of $\mathcal{SP}^\lambda$

$$f(z) = \frac{z}{(1-z)^{2s}} \qquad \text{where} \qquad s = e^{-i\lambda}cos\lambda,$$

the λ-spiral koebe function.

$f \in \mathcal{A}$ is an λ-Robertson function , $\mathcal{R}^\lambda$, if and only if

$$\Re\left[e^{i\lambda}\,\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > 0$$

for some $|\lambda| < \frac{\pi}{2}$, $z \in \mathbb{D}$.

Lemma 1.1. *$f \in \mathcal{R}^\lambda$ if and only if there exists a function $p \in \mathcal{P}$ such that*

$$f'(z) = \exp\left\{e^{-i\lambda}\int_0^z \frac{p(t)\ cos\lambda - e^{i\lambda}}{t\ cos\lambda}dt\right\} \tag{1.4}$$

for some $|\lambda| < \frac{\pi}{2}$, $z \in \mathbb{D}$.

Proof. Suppose that $f \in \mathcal{R}^\lambda$. Since it is a λ-Robertson function, there exists a function $p \in \mathcal{P}$ such that

$$e^{i\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right) = p(z)\ cos\lambda \qquad \left(|\lambda| < \frac{\pi}{2}, z \in \mathbb{D}\right).$$

From this equality, we can easily obtain (1.4).

Conversely, suppose that (1.4) holds. If we take the logarithmic derivative of (1.4), $f(z)$ belongs to $\mathcal{R}^\lambda$. So that, the proof is completed. □

We note that $f \in \mathcal{R}^\lambda$ if and only if $zf' \in \mathcal{SP}^\lambda$.

$f \in \mathcal{A}$ is an λ-close-to-spirallike function , $\mathcal{CSP}^\lambda$, if there exists a function $g \in \mathcal{SP}^\lambda$ such that

$$\Re\left[\frac{f(z)}{g(z)}\right] > 0$$

for some $|\lambda| < \frac{\pi}{2}$, $z \in \mathbb{D}$.

We note that the extremal function for the class of $\mathcal{CSP}^\lambda$

$$f(z) = \frac{z+z^2}{(1-z)^{2s+1}}, \qquad \text{where} \qquad s = e^{-i\lambda}cos\lambda,$$

the λ-close-to-spiral koebe function.

$f \in \mathcal{A}$ is an λ-close-to-Robertson function , $\mathcal{CR}^{\lambda}$, if there exists a function $g \in \mathcal{R}^{\lambda}$ such that

$$\Re\left[\frac{f'(z)}{g'(z)}\right] > 0$$

for some $|\lambda| < \frac{\pi}{2}$, $z \in \mathbb{D}$.

Haidan [4] introduced the class $\mathcal{SP}^{\lambda}(b)$ of λ-spirallike functions of complex order b as follows:

$$\mathcal{SP}^{\lambda}(b) = \left\{f \in \mathcal{A}:\ \Re\left\{1 + \frac{e^{i\lambda}}{bcos\lambda}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0\right\}$$

for some $|\lambda| < \frac{\pi}{2}$, $b \in \mathbb{C} - \{0\}, z \in \mathbb{D}$.

Haidan [4] introduced the class $\mathcal{R}^{\lambda}(b)$ of λ-Robertson functions of complex order b as follows:

$$\mathcal{R}^{\lambda}(b) = \left\{f \in \mathcal{A}:\ \Re\left\{1 + \frac{e^{i\lambda}}{bcos\lambda}\left(\frac{zf''(z)}{f'(z)}\right)\right\} > 0\right\}$$

for some $|\lambda| < \frac{\pi}{2}$, $b \in \mathbb{C} - \{0\}, z \in \mathbb{D}$.

Now, respectively, we introduce the classes of λ-close-to-spirallike functions of complex order b and λ-close-to-Robertson functions of complex order b, denoted by $\mathcal{CSP}^{\lambda}(b)$ and $\mathcal{CR}^{\lambda}(b)$, as follows:

$$\mathcal{CSP}^{\lambda}(b) = \left\{f \in \mathcal{A}:\ \Re\left\{1 + \frac{1}{b}\left(\frac{f(z)}{g(z)} - 1\right)\right\} > 0 \ \ ,\ g \in \mathcal{SP}^{\lambda}\right\}$$

and

$$\mathcal{CR}^{\lambda}(b) = \left\{f \in \mathcal{A}:\ \Re\left\{1 + \frac{1}{b}\left(\frac{f'(z)}{g'(z)} - 1\right)\right\} > 0 \ \ ,\ g \in \mathcal{R}^{\lambda}\right\}$$

for some $|\lambda| < \frac{\pi}{2}$, $b \in \mathbb{C} - \{0\}, z \in \mathbb{D}$.

Definition 1.2. The class of generalized Janowski functions which are defined by Ruscheweyh derivative operator in $z \in \mathbb{D}$, denoted by $\mathcal{JR}_b^{\lambda}(\alpha, \beta, \delta, A, B)$, is defined as

$$\mathcal{JR}_b^{\lambda}(\alpha, \beta, \delta, A, B) = \left\{f \in \mathcal{A}:\ 1 + \frac{e^{i\lambda}}{bcos\lambda}\left(\frac{\mathcal{D}_{\mathcal{R}}^{\alpha}f(z)}{\mathcal{D}_{\mathcal{R}}^{\beta}g(z)} - 1\right) \prec (1-\delta)\frac{1+Az}{1+Bz} + \delta \ \ , g \in \mathcal{SP}^{\lambda}\right\}$$

for some $|\lambda| < \frac{\pi}{2}$, $b \in \mathbb{C} - \{0\}$, $\alpha > -1$, $\beta > -1$, $0 \le \delta < 1$, $-1 \le B < A \le 1$, $z \in \mathbb{D}$.

Nothing that the class $\mathcal{JR}_b^{\lambda}(\alpha, \beta, \delta, A, B)$ include several subclasses which have important role in the analytic and geometric function theory.

By specializing the parameters α, β, δ, λ, b and A, B, we obtain the following subclasses studied earlier:

(1) $\mathcal{CS}_b^{*}(\delta, A, B) := \mathcal{JR}_b^{0}(0, 0, \delta, A, B)$ is the class of the generalized Janowski type close-to-star functions of complex order b,
(2) $\mathcal{CS}_b^{*}(A, B) := \mathcal{JR}_b^{0}(0, 0, 0, A, B)$ is the class of the Janowski type close-to-star functions of complex order b,
(3) $\mathcal{CS}^{*}(A, B) := \mathcal{JR}_1^{0}(0, 0, 0, A, B)$ is the class of the Janowski type close-to-star functions,
(4) $\mathcal{CS}^{*}(\eta) := \mathcal{JR}_1^{0}(0, 0, 0, 1-2\eta, -1)$ is the class of the close-to-star functions of order η,
(5) $\mathcal{CS}^{*} := \mathcal{JR}_1^{0}(0, 0, 0, 1, -1)$ is the class of the close-to-star functions,
(6) $\mathcal{CC}_b(\delta, A, B) := \mathcal{JR}_b^{0}(1, 0, \delta, A, B)$ is the class of the generalized Janowski type close-to-convex functions of complex order b,
(7) $\mathcal{CC}_b(A, B) := \mathcal{JR}_b^{0}(1, 0, 0, A, B)$ is the class of the Janowski type close-to-convex functions of complex order b,

(8) $\mathcal{CC}(A,B) := \mathcal{JR}_1^0(1,0,0,A,B)$ is the class of the Janowski type close-to-convex functions,
(9) $\mathcal{CC}(\eta) := \mathcal{JR}_1^0(1,0,0,1-2\eta,-1)$ is the class of the close-to-convex functions of order η,
(10) $\mathcal{CC} := \mathcal{JR}_1^0(1,0,0,1,-1)$ is the class of the close-to-convex functions.

Lemma 1.3. [1] *If the function $p(z)$ of the form*

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

is analytic in $\mathbb{D}$ and

$$p(z) \prec \frac{1+Az}{1+Bz},$$

then $|p_n| \leq A - B$, for $n \in \mathbb{N}, -1 \leq B < A \leq 1$.

Theorem 1.4. [3] *If $f \in \mathcal{SP}^{\lambda}$, then*

$$|a_n| \leq \prod_{k=1}^{n-1} \frac{|k+2s-1|}{k},$$

where $s = e^{-i\lambda}cos\lambda$, $|\lambda| < \frac{\pi}{2}$, $z \in \mathbb{D}$.

2. Subordination result and their consequences

Theorem 2.1. *$f(z) \in \mathcal{JR}_b^{\lambda}(\alpha,\beta,\delta,A,B)$ if and only if*

$$\frac{\mathcal{D}_{\mathcal{R}}^{\alpha} f(z)}{\mathcal{D}_{\mathcal{R}}^{\beta} g(z)} - 1 \prec \frac{(1-\delta)(A-B)\, be^{-i\lambda}cos\lambda\ z}{1+Bz}. \tag{2.1}$$

Proof. Suppose that $f \in \mathcal{JR}_b^{\lambda}(\alpha,\beta,\delta,A,B)$. Using the subordination principle, we write

$$1 + \frac{e^{i\lambda}}{bcos\lambda}\left(\frac{\mathcal{D}_{\mathcal{R}}^{\alpha} f(z)}{\mathcal{D}_{\mathcal{R}}^{\beta} g(z)} - 1\right) = (1-\delta)\frac{1+A\omega(z)}{1+B\omega(z)} + \delta. \tag{2.2}$$

After simple calculations, we get

$$\frac{e^{i\lambda}}{bcos\lambda}\left(\frac{\mathcal{D}_{\mathcal{R}}^{\alpha} f(z)}{\mathcal{D}_{\mathcal{R}}^{\beta} g(z)} - 1\right) = \frac{(1-\delta)(A-B)\,\omega(z)}{1+B\omega(z)}.$$

Thus, this equality is equivalent to (2.1). Similarly, the other side is proved. □

In Theorem 2.1, if we choice special values for α, β, δ, λ, b and A, B we get the following corollaries.

Corollary 2.2. *$f(z) \in \mathcal{CSP}^{\lambda}(b)$ if and only if*

$$\frac{f(z)}{g(z)} - 1 \prec \frac{2be^{-i\lambda}cos\lambda\ z}{1-z}$$

and this result is as sharp as the function

$$\frac{2be^{-i\lambda}cos\lambda\ z}{(1-z)^{2s+1}}, \quad \textit{where} \quad s = e^{-i\lambda}cos\lambda.$$

Proof. We let $\alpha = \beta = \delta = 0$ and $A = 1, B = -1$ in Theorem 2.1. □

Corollary 2.3. *$f(z) \in \mathcal{CS}^*(A,B)$ if and only if*

$$\frac{f(z)}{g(z)} - 1 \prec \frac{(A-B)\,z}{1+Bz}$$

and this result is as sharp as the function

$$\frac{1+Az}{1+Bz}.\frac{z}{(1-z)^2}.$$

Proof. We let $\lambda = \alpha = \beta = \delta = 0$ and $b = 1$ in Theorem 2.1. ☐

Corollary 2.4. *$f(z) \in \mathcal{CS}^*$ if and only if*

$$\frac{f(z)}{g(z)} - 1 \prec \frac{2z}{1-z}$$

and this result is as sharp as the function

$$\frac{1+z}{1-z}.$$

Proof. We let $\lambda = \alpha = \beta = \delta = 0$ and $b = 1, A = 1, B = -1$ in Theorem 2.1. ☐

Corollary 2.5. *$f(z) \in \mathcal{R}^{\lambda}(b)$ if and only if*

$$\frac{zf'(z)}{g(z)} - 1 \prec \frac{2be^{-i\lambda}cos\lambda\ z}{1-z}.$$

Proof. We let $\alpha = 1, \beta = \delta = 0$ and $A = 1, B = -1$ in Theorem 2.1. ☐

Corollary 2.6. *$f(z) \in \mathcal{CC}(A,B)$ if and only if*

$$\frac{zf'(z)}{g(z)} - 1 \prec \frac{(A-B)\,z}{1+Bz}.$$

Proof. We let $\lambda = \beta = \delta = 0$ and $\alpha = 1, b = 1$ in Theorem 2.1. ☐

Corollary 2.7. *$f(z) \in \mathcal{CC}$ if and only if*

$$\frac{zf'(z)}{g(z)} - 1 \prec \frac{2z}{1-z}$$

and this result is as sharp as the function

$$\frac{1+z}{1-z}.$$

Proof. We let $\lambda = \beta = \delta = 0$ and $\alpha = 1, b = 1, A = 1, B = -1$ in Theorem 2.1. ☐

3. Coefficient estimates and their consequences

Lemma 3.1. *If the function $\phi(z)$ of the form*

$$\phi(z) = 1 + \sum_{n=1}^{\infty} \phi_n z^n$$

is analytic in $\mathbb{D}$ and

$$\phi(z) \prec (1-\delta)\,\frac{1+Az}{1+Bz} + \delta,$$

then

$$|\phi_n| \leq (A-B)\,(1-\delta) \tag{3.1}$$

for $0 \leq \delta < 1, -1 \leq B < A \leq 1, n \in \mathbb{N}, z \in \mathbb{D}$.

Proof. Suppose that $\phi(z) \prec (1-\delta)\frac{1+Az}{1+Bz} + \delta$ for $\phi(z) = 1 + \sum_{n=1}^{\infty}\phi_n z^n$. Using the subordination principle, we write

$$\phi(z) = (1-\delta)\frac{1+A\omega(z)}{1+B\omega(z)} + \delta. \tag{3.2}$$

From (3.2), we get

$$\kappa(z) = \frac{\phi(z)-\delta}{(1-\delta)} = \frac{1+A\omega(z)}{1+B\omega(z)}.$$

By using Lemma 1.3 for the above function $\kappa(z)$, we get

$$\left|\frac{\phi_n}{1-\delta}\right| \leq A - B.$$

This inequality is equivalent to (3.1). □

Theorem 3.2. *If the function $f(z) \in \mathcal{A}$ be in the class $\mathcal{JR}_b^{\lambda}(\alpha, \beta, \delta, A, B)$, then*

$$|a_n| \leq \frac{1}{|b|\,\varphi_n(\alpha)} \tag{3.3}$$
$$\times \left(|b|\,\varphi_n(\beta)\prod_{k=1}^{n-1}\frac{|k+2s-1|}{k} + (A-B)(1-\delta)\left[\sum_{m=1}^{n-1}\varphi_{n-m}(\beta)\prod_{k=1}^{n-(m+1)}\frac{|k+2s-1|}{k}\right]\right),$$

where $s = e^{-i\lambda}cos\lambda$, $|\lambda| < \frac{\pi}{2}$, $b \in \mathbb{C}-\{0\}$, $\alpha > -1$, $\beta > -1$, $0 \leq \delta < 1$, $-1 \leq B < A \leq 1$, $z \in \mathbb{D}$.

Proof. Since $f \in \mathcal{JR}_b^{\lambda}(\alpha, \beta, \delta, A, B)$, there are analytic functions $g, \phi : \mathbb{D} \longmapsto \mathbb{D}$ such that $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{SP}^{\lambda}$, $\phi(z) = 1 + \sum_{n=1}^{\infty}\phi_n z^n$ and $\omega(z)$ is a Schwarz function as in Lemma 3.1 such that

$$1 + \frac{e^{i\lambda}}{bcos\lambda}\left(\frac{\mathcal{D}_{\mathcal{R}}^{\alpha}f(z)}{\mathcal{D}_{\mathcal{R}}^{\beta}g(z)} - 1\right) = (1-\delta)\frac{1+A\omega(z)}{1+B\omega(z)} + \delta = \phi(z) \tag{3.4}$$

for $z \in \mathbb{D}$. Then (3.4) can be written as

$$\mathcal{D}_{\mathcal{R}}^{\alpha}f(z) = \{1 + sb[\phi(z) - 1]\}\,\mathcal{D}_{\mathcal{R}}^{\beta}g(z)$$

or

$$z + \sum_{n=2}^{\infty}\varphi_n(\alpha)\,a_n z^n = z + \sum_{n=2}^{\infty}\left\{\varphi_n(\beta)\,b_n + sb\sum_{m=1}^{n-1}\varphi_{n-m}(\beta)\,b_{n-m}\,\phi_m\right\}z^n.$$

Equating the coefficients of like powers of z, we get

$$\varphi_2(\alpha)\,a_2 = \varphi_2(\beta)\,b_2 + sb\,\phi_1,$$
$$\varphi_3(\alpha)\,a_3 = \varphi_3(\beta)\,b_3 + sb\left[\varphi_2(\beta)\,b_2\phi_1 + \phi_2\right]$$

and

$$\varphi_n(\alpha)\,a_n = \varphi_n(\beta)\,b_n + sb\left[\varphi_{n-1}(\beta)\,b_{n-1}\,\phi_1 + \varphi_{n-2}(\beta)\,b_{n-2}\,\phi_2 + \ldots + \phi_{n-1}\right].$$

By using Lemma 3.1 and Theorem 1.4, we get (3.3). □

Corollary 3.3. *Let $f(z) \in \mathcal{A}$ be in the class $\mathcal{CSP}^{\lambda}(b)$, then*

$$|a_n| \leq \frac{1}{|b|}\left(|b|\prod_{k=1}^{n-1}\frac{|k+2s-1|}{k} + 2\left[\sum_{m=1}^{n-1}\prod_{k=1}^{n-(m+1)}\frac{|k+2s-1|}{k}\right]\right),$$

where $s = e^{-i\lambda}cos\lambda$, $|\lambda| < \frac{\pi}{2}$, $b \in \mathbb{C}-\{0\}$, $z \in \mathbb{D}$.

Proof. In Theorem 3.2, we take $\alpha = \beta = \delta = 0$ and $A = 1, B = -1$. □

Corollary 3.4. [7] *Let* $f(z) \in \mathcal{A}$ *be in the class* $\mathcal{CS}^*(A,B)$*, then*

$$|a_n| \leq n + \frac{(A-B)(n-1)n}{2},$$

where $-1 \leq B < A \leq 1$*,* $z \in \mathbb{D}$*.*

Proof. In Theorem 3.2, we take $\alpha = \beta = \delta = \lambda = 0$ and $b = 1$. □

Corollary 3.5. [7] *Let* $f(z) \in \mathcal{A}$ *be in the class* $\mathcal{CS}^*$*, then*

$$|a_n| \leq n^2,$$

where $z \in \mathbb{D}$*.*

Proof. In Theorem 3.2, we take $\alpha = \beta = \delta = \lambda = 0$ and $b = 1$. □

Corollary 3.6. *Let* $f(z) \in \mathcal{A}$ *be in the class* $\mathcal{R}^{\lambda}(b)$*, then*

$$|a_n| \leq \frac{1}{|b|\,n}\left(|b|\prod_{k=1}^{n-1}\frac{|k+2s-1|}{k} + 2\sum_{m=1}^{n-1}\prod_{k=1}^{n-(m+1)}\frac{|k+2s-1|}{k}\right),$$

where $s = e^{-i\lambda}cos\lambda$*,* $|\lambda| < \frac{\pi}{2}$*,* $b \in \mathbb{C} - \{0\}$*,* $z \in \mathbb{D}$*.*

Proof. In Theorem 3.2, we take $\alpha = 1$, $\beta = \delta = 0$ and $A = 1, B = -1$. □

Corollary 3.7. [7] *Let* $f(z) \in \mathcal{A}$ *be in the class* $\mathcal{CC}(A,B)$*, then*

$$|a_n| \leq 1 + \frac{(A-B)(n-1)}{2},$$

where $-1 \leq B < A \leq 1$*,* $z \in \mathbb{D}$*.*

Proof. In Theorem 3.2, we take $\alpha = 1$, $\beta = \delta = \lambda = 0$ and $b = 1$. □

Corollary 3.8. [7] *Let* $f(z) \in \mathcal{A}$ *be in the class* $\mathcal{CC}$*, then*

$$|a_n| \leq n,$$

where $z \in \mathbb{D}$*.*

Proof. In Theorem 3.2, we take $\alpha = 1$, $\beta = \delta = \lambda = 0$ and $A = 1, B = -1, b = 1$. □

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