On the Pólya-Szegö operator inequality

Trung Hoa Dinh[1] , Hamid Reza Moradi[2] , Mohammad Sababheh*[3]

[1] *Department of Mathematics, Troy University, 600 University Ave, Troy, AL 36082, USA*
[2] *Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-4697, Tehran, Iran*
[3] *Department of Basic Sciences, Princess Sumaya University for Technology, Amman 11941, Jordan*

Abstract

In this paper, we present generalized Pólya-Szegö type inequalities for positive invertible operators on a Hilbert space for arbitrary operator means between the arithmetic and the harmonic means. As applications, we present operator Grüss, Diaz–Metcalf, and Klamkin–McLenaghan inequalities.

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1. Introduction

Let Φ be a positive linear map on $\mathcal{B}(\mathcal{H})$; the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. Ando [1] proved the inequality

$$\Phi(A\sharp B) \leq \Phi(A)\sharp\Phi(B), \tag{1.1}$$

for any positive linear map Φ and positive operators A, B, where "$\sharp$" is the geometric mean in the sense of Kubo-Ando theory [11]. That is,

$$A\sharp B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}.$$

Speaking of means, the arithmetic mean $A\nabla B$ and the harmonic mean $A!B$ of two invertible positive operators $A, B \in \mathcal{B}(\mathcal{H})$ are defined, respectively, by

$$A\nabla B = \frac{A+B}{2} \quad \text{and} \quad A!B = \left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}.$$

It is well-known that $A!B \leq A\sharp B \leq A\nabla B$. In fact, if σ is a symmetric operator mean (in the sense that $A\sigma B = B\sigma A$), then $A!B \leq A\sigma B \leq A\nabla B$, for the invertible positive operators $A, B \in \mathcal{B}(\mathcal{H})$.

The operator Pólya-Szegö inequality presents a converse of Ando's inequality (1.1), as follows.

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*Corresponding Author.
Email addresses: thdinh@troy.edu (T.H. Dinh), hrmoradi@mshdiau.ac.ir (H. Moradi),
sababheh@yahoo.com; sababheh@psut.edu.jo (M. Sababheh)

Theorem 1.1. *Let* Φ *be a positive linear map and* $A, B \in \mathcal{B}(\mathcal{H})$ *be such that* $mI \leq A, B \leq MI$ *for some scalars* $0 < m < M$ *(I stands for the identity operator). Then*

$$\Phi(A)\,\sharp\,\Phi(B) \leq \frac{M+m}{2\sqrt{Mm}}\Phi(A\sharp B). \tag{1.2}$$

The inequality (1.2) was first proved in [12, Theorem 4] under the sandwich condition $sA \leq B \leq tA$ $(0 < s \leq t)$ for matrices (see also [3]).

Recall that a continuous real-valued function f defined on an interval J is said to be operator monotone if $A \leq B$ implies $f(A) \leq f(B)$ for all self-adjoint operators A, B with spectra in J. Very recently, Dinh et al. [10, Theorem 2.12] proved the following converse of (1.1) that extends (1.2).

Theorem 1.2. *Let* Φ *be a positive linear map,* f *be an operator monotone function on* $[0,\infty)$, τ, σ *operator means such that* $! \leq \tau, \sigma \leq \nabla$, *and* $0 < m < M$. *Then for any positive operators* A, B *satisfying* $mI \leq A, B \leq MI$, *the following inequality holds*

$$f(\Phi(A))\,\tau f(\Phi(B)) \leq \frac{(M+m)^2}{4Mm}f(\Phi(A\sigma B)). \tag{1.3}$$

The first target of this article is to present a generalized form of Pólya-Szegö inequality. In particular, we present relations between

$$\Phi(f(A))\,\tau\,\Phi(f(B)) \quad \text{and} \quad \Phi(f(A\sigma B))$$

under the sandwich condition $sA \leq B \leq tA$, for the operator monotone function f and the symmetric operator means σ, τ. Similar discussion will be presented for operator monotone decreasing functions. See Theorem 2.4 below for the exact statements. We refer the reader to the recent references [6, 8] treating similar topics.

2. Main results

In this section we present relations between

$$\Phi(f(A))\,\tau\,\Phi(f(B)) \quad \text{and} \quad \Phi(f(A\sigma B))$$

as generalized forms of Pólya-Szegö inequality. Then we show some applications including Grüss, Diaz–Metcalf, and Klamkin–McLenaghan type inequalities.

The first main result in this direction will be presented in Theorem 2.4 below. However, we will need some lemmas first.

Lemma 2.1. *Let* $A, B \in \mathcal{B}(\mathcal{H})$ *such that* $sA \leq B \leq tA$ *for some scalars* $0 < s \leq t$.

(a) *If* $st \geq 1$, *then*

$$\frac{2}{\sqrt{s}+\sqrt{t}}A\nabla B \leq A\sharp B \leq \frac{\sqrt{s}+\sqrt{t}}{2}A!B. \tag{2.1}$$

(b) *If* $st \leq 1$, *then*

$$\frac{2\sqrt{st}}{\sqrt{s}+\sqrt{t}}A\nabla B \leq A\sharp B \leq \frac{\sqrt{s}+\sqrt{t}}{2\sqrt{st}}A!B. \tag{2.2}$$

Proof. By appealing to functional calculus, it suffices to show the corresponding scalar inequalities. We define $f(x) := \frac{x+1}{2\sqrt{x}}$ where $0 < s \leq x \leq t$. It is straightforward to see that

$$f(x) \leq \frac{1}{2}\max\left\{\sqrt{s}+\frac{1}{\sqrt{s}}, \sqrt{t}+\frac{1}{\sqrt{t}}\right\}.$$

Consequently,

$$\frac{x+1}{2} \leq \begin{cases} \frac{\sqrt{s}+\sqrt{t}}{2}\sqrt{x} & \text{if } st \geq 1, \\ \frac{\sqrt{s}+\sqrt{t}}{2\sqrt{st}}\sqrt{x} & \text{if } st \leq 1, \end{cases} \tag{2.3}$$

for $0 < s \leq x \leq t$, and

$$\frac{1/x + 1}{2} \leq \begin{cases} \frac{\sqrt{s}+\sqrt{t}}{2} \frac{1}{\sqrt{x}} & \text{if } st \geq 1, \\ \frac{\sqrt{s}+\sqrt{t}}{2\sqrt{st}} \frac{1}{\sqrt{x}} & \text{if } st \leq 1, \end{cases} \tag{2.4}$$

for $0 < \frac{1}{t} \leq \frac{1}{x} \leq \frac{1}{s}$. Now, if $0 < s \leq x \leq t$, the inequalities (2.3) and (2.4) imply

$$\frac{2}{\sqrt{s}+\sqrt{t}} \left(\frac{x+1}{2} \right) \leq \sqrt{x} \leq \frac{\sqrt{s}+\sqrt{t}}{2} \left(\frac{1/x + 1}{2} \right)^{-1}$$

whenever $st \geq 1$, and

$$\frac{2\sqrt{st}}{\sqrt{s}+\sqrt{t}} \left(\frac{x+1}{2} \right) \leq \sqrt{x} \leq \frac{\sqrt{s}+\sqrt{t}}{2\sqrt{st}} \left(\frac{1/x + 1}{2} \right)^{-1}$$

whenever $st \leq 1$. This completes the proof of the lemma. $\qquad\square$

Remark 2.2. The substitution of $s = \frac{m}{M}$ and $t = \frac{M}{m}$ in Lemma 2.1 implies the celebrated result [5, Theorem 13]

$$\frac{2\sqrt{Mm}}{M+m} A\nabla B \leq A\sharp B \leq \frac{M+m}{2\sqrt{Mm}} A!B.$$

The next elementary lemma is given for completeness.

Lemma 2.3. *Let $\alpha \geq 1$.*

(a) *If $f : [0, \infty) \to [0, \infty)$ is an operator monotone function, then*

$$f(\alpha t) \leq \alpha f(t).$$

(b) *If $g : [0, \infty) \to [0, \infty)$ is an operator monotone decreasing function, then*

$$g(\alpha t) \geq \frac{1}{\alpha} g(t).$$

Now we are ready to prove the first main result,.

Theorem 2.4. *Let Φ be a positive linear map, τ, σ operator means such that $! \leq \tau, \sigma \leq \nabla$, and let $A, B \in \mathcal{B}(\mathcal{H})$ such that $sA \leq B \leq tA$ for some scalars $0 < s \leq t$.*

(i) *If f is an operator monotone increasing function on $[0, \infty)$, then*

$$\Phi(f(A)) \tau \Phi(f(B)) \leq \left(\frac{\sqrt{s}+\sqrt{t}}{2} \right)^2 \Phi(f(A\sigma B)) \tag{2.5}$$

whenever $st \geq 1$, and

$$\Phi(f(A)) \tau \Phi(f(B)) \leq \left(\frac{\sqrt{s}+\sqrt{t}}{2\sqrt{st}} \right)^2 \Phi(f(A\sigma B))$$

whenever $st \leq 1$.

(ii) *If g is an operator monotone decreasing function on $[0, \infty)$, then*

$$\Phi(g(A\tau B)) \leq \left(\frac{\sqrt{s}+\sqrt{t}}{2} \right)^2 \Phi(g(A)) \sigma \Phi(g(B)) \tag{2.6}$$

whenever $st \geq 1$, and

$$\Phi(g(A\tau B)) \leq \left(\frac{\sqrt{s}+\sqrt{t}}{2\sqrt{st}} \right)^2 (\Phi(g(A)) \sigma \Phi(g(B)))$$

whenever $st \leq 1$.

Proof. First assume that $st \geq 1$. We observe that

$$\Phi\left(f\left(A\right)\right)\tau\Phi\left(f\left(B\right)\right) \leq \Phi\left(f\left(A\right)\right)\nabla\Phi\left(f\left(B\right)\right) \quad \text{(since } \tau \leq \nabla\text{)}$$
$$= \Phi\left(f\left(A\right)\nabla f\left(B\right)\right)$$
$$\leq \Phi\left(f\left(A\nabla B\right)\right) \quad \text{(by [13, Corollary 1.12])}$$
$$\leq \Phi\left(f\left(\left(\frac{\sqrt{s}+\sqrt{t}}{2}\right)A\sharp B\right)\right) \quad \text{(by (2.1))}$$
$$\leq \frac{\sqrt{s}+\sqrt{t}}{2}\Phi\left(f\left(A\sharp B\right)\right) \quad \text{(by Lemma 2.3 (a))}.$$

On the other hand,

$$\Phi\left(f\left(A\sharp B\right)\right) \leq \Phi\left(f\left(\left(\frac{\sqrt{s}+\sqrt{t}}{2}\right)A!B\right)\right) \quad \text{(by RHS of (2.1))}$$
$$\leq \frac{\sqrt{s}+\sqrt{t}}{2}\Phi\left(f\left(A!B\right)\right) \quad \text{(by Lemma 2.3 (a))} \tag{2.7}$$
$$\leq \frac{\sqrt{s}+\sqrt{t}}{2}\Phi\left(f\left(A\sigma B\right)\right) \quad \text{(since } ! \leq \sigma\text{)}.$$

These two inequalities together imply (2.5). This completes the proof of the case of operator monotone functions and $st \geq 1$.

Now assume that g is operator monotone decreasing. We have

$$g\left(A\right)\sigma g\left(B\right) \geq g\left(A\nabla B\right) \quad \text{(by [2, Theorem 2.1])}$$
$$\geq g\left(\left(\frac{\sqrt{s}+\sqrt{t}}{2}\right)A\sharp B\right) \quad \text{(by of (2.1))} \tag{2.8}$$
$$\geq \frac{2}{\sqrt{s}+\sqrt{t}}g\left(A\sharp B\right) \quad \text{(by Lemma 2.3 (b))}.$$

On the other hand,

$$g\left(A\sharp B\right) \geq g\left(\left(\frac{\sqrt{s}+\sqrt{t}}{2}\right)A!B\right) \quad \text{(by (2.1))}$$
$$\geq \frac{2}{\sqrt{s}+\sqrt{t}}g\left(A!B\right) \quad \text{(by Lemma 2.3 (b))} \tag{2.9}$$
$$\geq \frac{2}{\sqrt{s}+\sqrt{t}}g\left(A\tau B\right) \quad \text{(since } ! \leq \tau\text{)}.$$

Combining (2.8) and (2.9) yields

$$g\left(A\tau B\right) \leq \frac{\left(\sqrt{s}+\sqrt{t}\right)^2}{4}\left(g\left(A\right)\sigma g\left(B\right)\right).$$

Applying Φ, we obtain that

$$\Phi\left(g\left(A\tau B\right)\right) \leq \frac{\left(\sqrt{s}+\sqrt{t}\right)^2}{4}\Phi\left(g\left(A\right)\sigma g\left(B\right)\right)$$
$$\leq \frac{\left(\sqrt{s}+\sqrt{t}\right)^2}{4}\Phi\left(g\left(A\right)\right)\sigma\Phi\left(g\left(B\right)\right) \quad \text{(by [1, Theorem 3])}.$$

This completes the proof for operator monotone decreasing functions in case $st \geq 1$.

The proof of the case $st \leq 1$ is similar to that $st \geq 1$; except instead of inequality (2.1) we use the inequality (2.2). $\square$

As an application of Theorem 2.4, we have the following Grüss type inequalities.

Corollary 2.5. *Let Φ be a positive linear map, τ, σ operator means such that $! \leq \tau, \sigma \leq \nabla$, and let $A, B \in \mathcal{B}(\mathcal{H})$ be such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M$.*

(i) *If f is an operator monotone increasing function on $[0, \infty)$, then*

$$\Phi\left(f\left(A\right)\right)\tau\Phi\left(f\left(B\right)\right) - \Phi\left(f\left(A\sigma B\right)\right) \leq \frac{\left(M - m\right)^2}{4Mm}f\left(M\right).$$

(ii) *If g is an operator monotone decreasing function, then*

$$\Phi\left(g\left(A\tau B\right)\right) - \Phi\left(g\left(A\right)\right)\sigma\Phi\left(g\left(B\right)\right) \leq \frac{\left(M - m\right)^2}{4Mm}g\left(m\right).$$

Proof. It follows from Theorem 2.4 (i) that

$$\Phi\left(f\left(A\right)\right)\tau\Phi\left(f\left(B\right)\right) \leq \frac{\left(M + m\right)^2}{4Mm}\Phi\left(f\left(A\sigma B\right)\right). \tag{2.10}$$

Hence

$$\Phi\left(f\left(A\right)\right)\tau\Phi\left(f\left(B\right)\right) - \Phi\left(f\left(A\sigma B\right)\right) \leq \left(\frac{\left(M + m\right)^2}{4Mm} - 1\right)\Phi\left(f\left(A\sigma B\right)\right)$$

$$\leq \left(\frac{\left(M + m\right)^2}{4Mm} - 1\right)f\left(M\right),$$

where in the first inequality we used (2.10) and the second inequality follows from the fact that $f\left(m\right)I \leq f\left(A\sigma B\right) \leq f\left(M\right)I$.

The other case can be obtained similarly by utilizing Theorem 2.4 (ii). □

In [7, Theorem 3], the inequality

$$\frac{\|g(A)\sharp g(B)\|}{\|A\sharp B\|} \leq 2S(h)^2 \left\|\frac{g(A\sharp B)}{A\sharp B}\right\| \tag{2.11}$$

was proved for the positive matrices A, B satisfying $mI \leq A, B \leq MI$, the operator convex function $g : [0, \infty) \to [0, \infty)$ satisfying $g(0) = 0$ and the Specht's ratio $S(h)$. Following the same ideas as in [7] one can prove the following more general form, which then implies a refinement of (2.11).

Corollary 2.6. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that $sA \leq B \leq tA$ for some scalars $0 < s \leq t$ with $st \geq 1$, and let g be an operator convex function with $g\left(0\right) = 0$. Then for any $\tau \geq \sharp$, $\sigma \leq \sharp$ and for any unitarily invariant norm $\|\cdot\|_u$,*

$$\frac{\|g(A)\tau g(B))\|_u}{\|A\tau B\|_u} \leq \left(\frac{\sqrt{s} + \sqrt{t}}{2}\right)^2 \left\|\frac{g(A\sharp B)}{A\sharp B}\right\|_u, \tag{2.12}$$

and

$$\frac{\|g(A)\sharp g(B))\|_u}{\|A\sharp B\|_u} \leq \left(\frac{\sqrt{s} + \sqrt{t}}{2}\right)^2 \left\|\frac{g(A\sigma B)}{A\sigma B}\right\|_u. \tag{2.13}$$

In particular, if $! \leq \tau, \sigma \leq \nabla$,

$$\frac{\|g(A)\tau g(B))\|_u}{\|A\tau B\|_u} \leq \left(\frac{\sqrt{s} + \sqrt{t}}{2}\right)^4 \left\|\frac{g(A\sigma B)}{A\sigma B}\right\|_u.$$

Proof. By Theorem 2.4,

$$\frac{\|g(A)\tau g(B))\|_u}{\|A\tau B\|_u} \leq \left(\frac{\sqrt{s} + \sqrt{t}}{2}\right)^2 \|g(A\sharp B)\|_u.$$

Consequently, the following double inequality is valid:

$$\frac{\|g(A)\tau g(B))\|_u}{\|A\tau B\|_u} \leq \left(\frac{\sqrt{s}+\sqrt{t}}{2}\right)^2 \frac{\|g(A\sharp B))\|_u}{\|A\sharp B\|_u} \leq \left(\frac{\sqrt{s}+\sqrt{t}}{2}\right)^2 \left\|\frac{g(A\sharp B)}{A\sharp B}\right\|_u.$$

The second inequality is obtained by similar arguments. □

The case $st \leq 1$ in Corollary 2.6 is also valid if we employ inequality (2.2) instead of (2.1).

Remark 2.7. In the special cases when $s = \frac{m}{M}$, $t = \frac{M}{m}$, and $\tau = \sharp$ (respectively $\sigma = \sharp$), (2.12) (respectively (2.13)) reduces to

$$\frac{\|g(A)\sharp g(B)\|_u}{\|A\sharp B\|_u} \leq 2\left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \left\|\frac{g(A\sharp B)}{A\sharp B}\right\|_u. \tag{2.14}$$

This shows that the inequality (2.14) is a refinement of [7, Theorem 3]. To see that (2.14) is a refinement of [7, Theorem 3], one has to recall that $\frac{M+m}{2} \leq S\left(\frac{M}{m}\right)\sqrt{Mm}$ (see [14]).

Remark 2.8. By choosing Φ as an identity map, $s = \frac{m}{M}$, $t = \frac{M}{m}$, and $\tau = \sigma = \sharp$ in (2.7) and (2.8), we have the following two cases:

(i) If f is an operator monotone increasing function, then

$$f(A)\sharp f(B) \leq \frac{M+m}{2\sqrt{Mm}}f(A\sharp B). \tag{2.15}$$

(ii) If g is an operator monotone decreasing function, then

$$g(A\sharp B) \leq \frac{M+m}{2\sqrt{Mm}}(g(A)\sharp g(B)). \tag{2.16}$$

As mentioned in [4, Theorem 6], if $A, B \in \mathcal{B}(\mathcal{H})$ are two positive operators such that $A \leq B$ and $mI \leq A \leq MI$ for some scalars $0 < m < M$ then,

$$A^2 \leq \frac{(M+m)^2}{4Mm}B^2.$$

Now, by the substitutions $A \to f(A)\sharp f(B)$ and $B \to \frac{M+m}{2\sqrt{Mm}}f(A\sharp B)$ in the above discussion, we get

$$(f(A)\sharp f(B))^2 \leq \left(\frac{(M+m)^2}{4Mm}\right)^2 f(A\sharp B)^2.$$

A similar approach gives

$$g(A\sharp B)^2 \leq \left(\frac{(M+m)^2}{4Mm}\right)^2 (g(A)\sharp g(B))^2.$$

We conclude this article by showing operator Diaz–Metcalf and Klamkin–McLenaghan inequalities.

Theorem 2.9. *Let Φ be a positive linear map, τ, σ operator means such that $! \leq \tau, \sigma \leq \nabla$, and let $A, B \in \mathcal{B}(\mathcal{H})$ such that $sA \leq B \leq tA$ for some scalars $0 < s \leq t$. If f is a non-negative operator monotone function then,*

(i) *(Operator Diaz–Metcalf type inequality) If $st \geq 1$, then*

$$\Phi\left(f\left(\sqrt{st}A\right)\right)\tau\Phi(f(B)) \leq \left(\frac{\sqrt{s}+\sqrt{t}}{2}\right)^2 \Phi(f(A\sigma B)). \tag{2.17}$$

On the other hand, if $st \leq 1$, then

$$\Phi\left(f\left(\sqrt{st}A\right)\right)\tau\Phi(f(B)) \leq \frac{\left(\sqrt{s}+\sqrt{t}\right)^2}{4\sqrt{st}}\Phi(f(A\sigma B)).$$

(ii) *(Operator Klamkin–McLenaghan type inequality) If $st \geq 1$, then*

$$\Phi(f(A\sigma B))^{-\frac{1}{2}}\Phi(f(B))\Phi(f(A\sigma B))^{-\frac{1}{2}} - \Phi(f(A\sigma B))^{\frac{1}{2}}\Phi\left(f\left(\sqrt{st}A\right)\right)^{-1}\Phi(f(A\sigma B))^{\frac{1}{2}}$$

$$\leq \frac{\left(\sqrt{s}+\sqrt{t}\right)^2}{2} - 2I - \left(\left(\Phi(f(A\sigma B))^{-\frac{1}{2}}\Phi\left(f\left(\sqrt{st}A\right)\right)\Phi(f(A\sigma B))^{-\frac{1}{2}}\right)^{\frac{1}{2}}\right.$$

$$\left. -\left(\Phi(f(A\sigma B))^{-\frac{1}{2}}\Phi\left(f\left(\sqrt{st}A\right)\right)\Phi(f(A\sigma B))^{-\frac{1}{2}}\right)^{-\frac{1}{2}}\right)^2. \tag{2.18}$$

On the other hand, if $st \leq 1$, then

$$\Phi(f(A\sigma B))^{-\frac{1}{2}}\Phi(f(B))\Phi(f(A\sigma B))^{-\frac{1}{2}} - \Phi(f(A\sigma B))^{\frac{1}{2}}\Phi\left(f\left(\sqrt{st}A\right)\right)^{-1}\Phi(f(A\sigma B))^{\frac{1}{2}}$$

$$\leq \frac{\left(\sqrt{s}+\sqrt{t}\right)^2}{2\sqrt{st}} - 2I - \left(\left(\Phi(f(A\sigma B))^{-\frac{1}{2}}\Phi\left(f\left(\sqrt{st}A\right)\right)\Phi(f(A\sigma B))^{-\frac{1}{2}}\right)^{\frac{1}{2}}\right.$$

$$\left. -\left(\Phi(f(A\sigma B))^{-\frac{1}{2}}\Phi\left(f\left(\sqrt{st}A\right)\right)\Phi(f(A\sigma B))^{-\frac{1}{2}}\right)^{-\frac{1}{2}}\right)^2. \tag{2.19}$$

Proof. We assume $st \geq 1$. From the assumption $sA \leq B \leq tA$, it follows that $\sqrt{s} \leq \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}} \leq \sqrt{t}$. Therefore,

$$\frac{\sqrt{st}A + B}{2} \leq \left(\frac{\sqrt{s}+\sqrt{t}}{2}\right)A\sharp B. \tag{2.20}$$

Now, since f is an operator monotone increasing we have

$$\frac{f\left(\sqrt{st}A\right) + f(B)}{2} \leq f\left(\frac{\sqrt{st}A + B}{2}\right) \quad \text{(by [13, Corollary 1.12])}$$

$$\leq f\left(\left(\frac{\sqrt{s}+\sqrt{t}}{2}\right)A\sharp B\right) \quad \text{(by (2.20))}$$

$$\leq f\left(\left(\frac{\sqrt{s}+\sqrt{t}}{2}\right)^2 A!B\right) \quad \text{(by (2.1))}$$

$$\leq \left(\frac{\sqrt{s}+\sqrt{t}}{2}\right)^2 f(A!B) \quad \text{(by Lemma 2.3(a))}$$

$$\leq \left(\frac{\sqrt{s}+\sqrt{t}}{2}\right)^2 f(A\sigma B) \quad \text{(since $! \leq \sigma$)}.$$

It follows from the linearity of Φ and the fact $\tau \leq \nabla$ that

$$\Phi\left(f\left(\sqrt{st}A\right)\right)\tau\Phi(f(B)) \leq \frac{\Phi\left(f\left(\sqrt{st}A\right)\right) + \Phi(f(B))}{2}$$

$$\leq \left(\frac{\sqrt{s}+\sqrt{t}}{2}\right)^2 \Phi(f(A\sigma B)).$$

So we have (2.17). The case $st \leq 1$ can be obtained similarly.

From (2.17) we easily obtain that

$$\Phi\left(f\left(\sqrt{st}A\right)\right) + \Phi\left(f\left(B\right)\right) \leq \frac{\left(\sqrt{s}+\sqrt{t}\right)^2}{2}\Phi\left(f\left(A\sigma B\right)\right). \qquad (2.21)$$

The estimate (2.21) guarantees

$$\Phi(f(A\sigma B))^{-\frac{1}{2}}\Phi\left(f\left(B\right)\right)\Phi(f(A\sigma B))^{-\frac{1}{2}}$$

$$\leq \frac{\left(\sqrt{s}+\sqrt{t}\right)^2}{2} - \Phi(f(A\sigma B))^{-\frac{1}{2}}\Phi\left(f\left(\sqrt{st}A\right)\right)\Phi(f(A\sigma B))^{-\frac{1}{2}}.$$

We set

$$X := \Phi(f(A\sigma B))^{-\frac{1}{2}}\Phi\left(f\left(B\right)\right)\Phi(f(A\sigma B))^{-\frac{1}{2}}$$

$$- \Phi(f(A\sigma B))^{\frac{1}{2}}\Phi\left(f\left(\sqrt{st}A\right)\right)^{-1}\Phi(f(A\sigma B))^{\frac{1}{2}}$$

and observe

$$X \leq \frac{\left(\sqrt{s}+\sqrt{t}\right)^2}{2} - T - T^{-1}, \qquad (2.22)$$

where $T = \Phi(f(A\sigma B))^{-\frac{1}{2}}\Phi\left(f\left(\sqrt{st}A\right)\right)\Phi(f(A\sigma B))^{-\frac{1}{2}}$. Notice that

$$T + T^{-1} = \left(T^{\frac{1}{2}} - T^{-\frac{1}{2}}\right)^2 + 2I. \qquad (2.23)$$

Combining (2.22) and (2.23) we get

$$X \leq \frac{\left(\sqrt{s}+\sqrt{t}\right)^2}{2} - \left(T^{\frac{1}{2}} - T^{-\frac{1}{2}}\right)^2 - 2I,$$

which is equivalent to the inequality (2.18). The inequality (2.19) is obtained by similar arguments. $\qquad\square$

Remark 2.10. Assume $\sqrt{st} \geq 1$. Due to the monotonicity property of operator means, we have

$$\Phi\left(f\left(A\right)\right)\tau\Phi\left(f\left(B\right)\right) \leq \Phi\left(f\left(\sqrt{st}A\right)\right)\tau\Phi\left(f\left(B\right)\right) \leq \left(\frac{\sqrt{s}+\sqrt{t}}{2}\right)^2\Phi\left(f\left(A\sigma B\right)\right),$$

which is stronger than (2.5).

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