Araştırma Makalesi **BAUN Fen Bil. Enst. Dergisi, 22(2), 403-414, (2020)**

DOI:10.25092/baunfbed.707344 **J. BAUN Inst. Sci. Technol., 22(2), 403-414, (2020)**

Ranks of some subsemigroups of full contraction mappings on a finite chain

Kemal TOKER[*]

Harran University Faculty of Science and Literature,
Department of Mathematics, Osmanbey Campus, Şanlıurfa.

Geliş Tarihi (Received Date): 28.09.2019
Kabul Tarihi (Accepted Date): 13.03.2020

Abstract

Let Z^+ denotes the set of all positive integers. Let $X_n = \{1,2,\dots,n\}$ be the finite chain for $n \in Z^+$ and let T_n be the full transformation semigroup on X_n. Also let OCT_n and $ORCT_n$ be the semigroup of order-preserving full contraction mappings, and the semigroup of order-preserving or order-reversing full contraction mappings on X_n, respectively. It is well-known that OCT_n and $ORCT_n$ are subsemigroups of T_n. In this paper we obtain ranks of the semigroups OCT_n and $ORCT_n$.

Keywords: *Order-preserving/order-reversing contraction mappings, generating set, rank.*

Sonlu zincir üzerindeki tam daralma dönüşümlerinin bazı alt yarıgruplarının rankları

Öz

Z^+, tüm pozitif tamsayıların kümesi olsun. $n \in Z^+$ için $X_n = \{1,2,\dots,n\}$ sonlu bir zincir ve T_n, X_n üzerindeki tam dönüşümler yarıgrubu olsun. Ayrıca OCT_n ve $ORCT_n$ sırasıyla X_n üzerindeki sıra-koruyan tam daralma dönüşümler yarıgrubu ve sıra-koruyan veya sıra-çeviren tam daralma dönüşüler yarıgrubu olsun. OCT_n ve $ORCT_n$ yarıgruplarının T_n yarıgrubunun altyarıgrupları olduğu bilinmektedir. Bu çalışmada OCT_n ve $ORCT_n$ yarıgruplarının rankları araştırılmıştır.

Anahtar kelimeler: *Sıra-koruyan/sıra-çeviren daralma dönüşümleri, doğuray kümeleri, rank.*

[*] Kemal TOKER, ktoker@harran.edu.tr, https://orcid.org/0000-0003-3696-1324

1. Introduction

Let Z^+denotes the set of all positive integers. Let $X_n = \{1,2, \dots, n\}$ be the finite chain for $n \in Z^+$ and let T_n and S_n be the full transformation semigroup and the symmetric group on X_n, respectively. Also let

$$O_n = \{\alpha \in T_n | (\forall x, y \in X_n)\ x \leq y \Rightarrow x\alpha \leq y\alpha \}, \tag{1}$$

the semigroup of all order-preserving full transformations on X_n. For any $\alpha \in T_n$, if $|x\alpha - y\alpha| \leq |x - y|$ for all $x, y \in X_n$ then α is called a full contraction mapping on X_n. Then let CT_n be the set of all full contraction mappings on X_n, say

$$CT_n = \{\alpha \in T_n | (\forall x, y \in X_n)\ |x\alpha - y\alpha| \leq |x - y|\}, \tag{2}$$

and let OCT_n be the set of all order-preserving full contraction mappings on X_n, say $OCT_n = O_n \cap CT_n$ which are clearly subsemigroups of T_n. Also, let OR_n be the semigroup of all order-preserving or order-reversing transformations on X_n, and let $ORCT_n = OR_n \cap CT_n$, which is clearly a subsemigroup of T_n consisting of all order-preserving or order-reversing full contraction mappings on X_n. Recall that, Garba et al. have presented characterisations of Green's relations on $CT_n \backslash S_n$ and $OCT_n \backslash S_n$ in [1], and that Adeshola and Umar have investigated the cardinalities of some equivalences on OCT_n and $ORCT_n$ in [2] which lead naturally to obtaining the orders of these subsemigroups.

Let S be any semigroup, and let A be any non-empty subset of S. Then the subsemigroup generated by A, that is the smallest subsemigroup of S containing A, is denoted by $\langle A \rangle$. If there exists a finite subset A of a semigroup S with $\langle A \rangle = S$, then S is called a finitely generated semigroup. The rank of a finitely generated semigroup S is defined by

$$\text{rank}(S) = \min\{|A| : \langle A \rangle = S\}. \tag{3}$$

There are many studies on various generating sets and ranks of any semigroup. Now we give some examples of recent studies. Let $Sing_n = T_n \backslash S_n$, the subsemigroup of all singular mappings. Gomes and Howie proved that $\text{rank}(Sing_n) = \frac{n(n-1)}{2}$ in [3] and Ayık et al. found the necessary and sufficient conditions for any set of transformations with $n-1$ image in $Sing_n$ to be a (minimal) generating set for $Sing_n$ in [4]. Let I_n be the symmetric inverse semigroup on X_n, and let

$$DP_n = \{\alpha \in I_n |\ \forall x, y \in \text{dom}(\alpha), |x\alpha - y\alpha| = |x - y|\} \tag{4}$$

be the subsemigroup of I_n consisting of all partial isometries on X_n. Also, let

$$ODP_n = \{\alpha \in DP_n | \forall x, y \in \text{dom}(\alpha),\ x \leq y \Rightarrow x\alpha \leq y\alpha\} \tag{5}$$

be the subsemigroup of DP_n consisting of all order-preserving partial isometries on X_n. Bugay et al. examined the subsemigroups

$$DP_{n,r} = \{\alpha \in DP_n | |\text{im}(\alpha)| \leq r\} \tag{6}$$

and

$$ODP_{n,r} = \{\alpha \in ODP_n \,||\mathrm{im}(\alpha)| \leq r\} \tag{7}$$

for $2 \leq r \leq n-1$, and showed that $\mathrm{rank}(DP_{n,r}) = \mathrm{rank}(ODP_{n,r}) = \binom{n}{r}$ in [5]. For any $\emptyset \neq Y \subseteq X_n$, let

$$T_{(X_n,Y)} = \{\, \alpha \in T_n \,|\, Y\alpha = Y\}. \tag{8}$$

Clearly $T_{(X_n,Y)}$ is a subsemigroup of T_n. Toker et al. examined the subsemigroups $T_{(n,m)} = \{\alpha \in T_n : X_m\alpha = X_m\}$ and showed that

$$\mathrm{rank}(T_{(n,m)}) = \begin{cases} 2, & \text{if } (n,m) = (2,1) \text{ or } (3,2) \\ 3, & \text{if } (n,m) = (3,1) \text{ or } 4 \leq n \text{ and } m = n-1 \\ 4, & \text{if } 4 \leq n \text{ and } 1 \leq m \leq n-2 \end{cases} \tag{9}$$

in [6]. Now, in this paper we examine OCT_n and $ORCT_n$, and show that

$$\mathrm{rank}\,(OCT_n) = \begin{cases} 3, & \text{if } n = 2 \\ n, & \text{if } n = 1 \text{ or if } n \geq 3. \end{cases} \tag{10}$$

and

$$\mathrm{rank}\,(ORCT_n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is an odd number} \\ \frac{n+2}{2}, & \text{if } n \text{ is an even number}. \end{cases} \tag{11}$$

2. Preliminaries

The kernel and the image of $\alpha \in T_n$ are defined by

$$\ker(\alpha) = \{(x,y) \in X_n \times X_n : x\alpha = y\alpha\} \tag{12}$$

$$\mathrm{im}(\alpha) = \{x\alpha : x \in X_n\}, \tag{13}$$

respectively. For any $\alpha, \beta \in T_n$ it is well known that $\ker(\alpha) \subseteq \ker(\alpha\beta)$ and $\mathrm{im}(\alpha\beta) \subseteq \mathrm{im}(\beta)$.

Definition 2.1 Let A be a non-empty subset of X_n. If $x, y \in A$ and $x \leq z \leq y \Rightarrow z \in A$ for all $x, y \in A$, then A is called a convex subset of X_n.

Recall from [Theorem 2.2 [1]] that if $\alpha \in T_n$ is a contraction mapping then $\mathrm{im}(\alpha)$ is a convex subset of X_n. Thus, if $\alpha \in OCT_n$ or $\alpha \in ORCT_n$ then $\mathrm{im}(\alpha)$ is a convex subset of X_n, that is there exists $1 \leq k \leq m \leq n$ such that $\mathrm{im}(\alpha) = \{k, k+1, \ldots, m\}$. If $\alpha \in OCT_n$ then since α is order-preserving, it is easy to see that each equivalence class of

$\ker(\alpha)$ is a convex subset of X_n, and if $\alpha \in ORCT_n$ then since α is order-preserving or order-reversing, it is easy to see that each equivalence class of $\ker(\alpha)$ is also a convex subset of X_n.

On a semigroup S, $(a, b) \in L^*(S)$ if and only if the elements $a, b \in S$ are related by Green's relation L in some oversemigroup of S. The relation R^* is defined dually. The join of relations L^* and R^* is denoted by D^* and their intersection by H^*. Those relations are called starred Green's relations. Garba et al. have found starred Green's relations semigroups of $CT_n \backslash S_n$ and $OCT_n \backslash S_n$ in [1]. In particular, they proved the following theorem.

Theorem 2.2 [1] Let $S \in \{CT_n \backslash S_n, OCT_n \backslash S_n\}$ and let $\alpha, \beta \in S$. Then
- (i) $(\alpha, \beta) \in L^*(S)$ if and only if $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$,
- (ii) $(\alpha, \beta) \in R^*(S)$ if and only if $\ker(\alpha) = \ker(\beta)$,
- (iii) $(\alpha, \beta) \in H^*(S)$ if and only if $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$ and $\ker(\alpha) = \ker(\beta)$,
- (iv) $(\alpha, \beta) \in D^*(S)$ if and only if $|\mathrm{im}(\alpha)| = |\mathrm{im}(\beta)|$.

In this paper we use the same notations with Howie's book [7].

3. The rank of OCT_n

In this section, we find a minimal generating set of OCT_n and so we obtain the rank of OCT_n. It is clear that $OCT_1 = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$ and so rank $(OCT_1) = 1$, it is also clear that

$$OCT_2 = \{\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}\}. \tag{14}$$

If $\{\alpha, \beta\} \in OCT_2$ then we observe that $< \alpha, \beta >= \{\alpha, \beta\}$, and so rank $(OCT_2) = 3$. Hence in this paper we consider the case $n \geq 3$. Let

$$D_k^* = \{\alpha \in OCT_n : |\ \mathrm{im}\ (\alpha)| = k\} \tag{15}$$

for $1 \leq k \leq n$. Notice that $\mathrm{D}_\mathrm{n}^* = \{\epsilon = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}\}$.

Lemma 3.1 If $\alpha \in D_r^*$ then $\alpha \in < D_{r+1}^* >$ for each $1 \leq r \leq n - 2$.

Proof. Let $\alpha \in D_r^*$ for $1 \leq r \leq n - 2$, then there exists a partition $\{A_1, A_2, \dots, A_r\}$ of X_n and there exists $1 \leq k \leq n - r + 1$ such that

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_i & \dots & A_r \\ k & k+1 & \dots & k-1+i & \dots & k-1+r \end{pmatrix}. \tag{16}$$

It is clear that $|A_i| \geq 2$ at least for one $1 \leq i \leq r$ since $r \leq n - 2$. Without loss of generality let $A_i = \{a_1, a_2, \dots, a_m\}$ for $m \geq 2$ and let x_i be the maximum element in A_i. If $k > 1$ and $k + r - 1 < n$, let

$$\beta = \begin{pmatrix} A_1 \backslash \{x_1\} & \{x_1\} & A_2 & \dots & A_r \\ k & k+1 & k+2 & \dots & k+r \end{pmatrix}, \tag{17}$$

for $i = 1$, let

$$\beta = \begin{pmatrix} A_1 & A_2 & \dots & A_{r-1} & A_r \backslash \{x_r\} & \{x_r\} \\ k & k+1 & \dots & k+r-2 & k+r-1 & k+r \end{pmatrix}, \tag{18}$$

for $i = r$, and let

$$\beta = \begin{pmatrix} A_1 & \dots & A_{i-1} & A_i \backslash \{x_i\} & \{x_i\} & A_{i+1} & \dots & A_r \\ k & \dots & k+i-2 & k+i-1 & k+i & k+i+1 & \dots & k+r \end{pmatrix}, \tag{19}$$

for $2 \leq i \leq r-1$. Also, let γ be the mapping defined as

$$j\gamma = \begin{cases} k-1; & if \quad 1 \leq j \leq k-1 \\ j; & if \quad k \leq j \leq k+i-1 \\ j-1; & if \quad k+i \leq j \leq k+r \\ k+r-1; & if \quad j > k+r, \end{cases} \tag{20}$$

for $1 \leq i \leq r$. Then $\beta, \gamma \in D^*_{r+1}$ and $\alpha = \beta\gamma$.

If $k = 1$, let

$$\beta = \begin{pmatrix} A_1 \backslash \{x_1\} & \{x_1\} & A_2 & \dots & A_r \\ 1 & 2 & 3 & \dots & r+1 \end{pmatrix}, \tag{21}$$

for $i = 1$, let

$$\beta = \begin{pmatrix} A_1 & A_2 & \dots & A_{r-1} & A_r \backslash \{x_r\} & \{x_r\} \\ 1 & 2 & \dots & r-1 & r & r+1 \end{pmatrix}, \tag{22}$$

for $i = r$, and let

$$\beta = \begin{pmatrix} A_1 & \dots & A_{i-1} & A_i \backslash \{x_i\} & \{x_i\} & A_{i+1} & \dots & A_r \\ 1 & \dots & i-1 & i & i+1 & i+2 & \dots & r+1 \end{pmatrix}, \tag{23}$$

for $2 \leq i \leq r-1$. Also, let γ be the mapping defined as

$$j\gamma = \begin{cases} j; & if \quad j \leq i \\ j-1; & if \quad i+1 \leq j \leq r+1 \\ r+1; & if \quad r+2 \leq j \leq n, \end{cases} \tag{24}$$

for $1 \leq i \leq r$. Then, similarly $\beta, \gamma \in D^*_{r+1}$ and $\alpha = \beta\gamma$.

If $k + r - 1 = n$, let

$$\beta = \begin{pmatrix} A_1 \backslash \{x_1\} & \{x_1\} & A_2 & \dots & A_r \\ k-1 & k & k+1 & \dots & n \end{pmatrix}, \tag{25}$$

for $i = 1$, let

$$\beta = \begin{pmatrix} A_1 & A_2 & \dots & A_{r-1} & A_r \backslash \{x_r\} & \{x_r\} \\ k-1 & k & \dots & n-2 & n-1 & n \end{pmatrix}, \tag{26}$$

for $i = r$, and let

$$\beta = \begin{pmatrix} A_1 & \dots & A_{i-1} & A_i \backslash \{x_i\} & \{x_i\} & A_{i+1} & \dots & A_r \\ k-1 & \dots & k+i-3 & k+i-2 & k+i-1 & k+i & \dots & n \end{pmatrix}, \tag{27}$$

for $2 \leq i \leq r-1$. Also, let γ be the mapping defined as

$$j\gamma = \begin{cases} k-1; & if \quad 1 \leq j \leq k-2 \\ j+1; & if \quad k-1 \leq j \leq k+i-2 \\ j; & if \quad k+i-1 \leq j \leq n, \end{cases} \tag{28}$$

for $1 \leq i \leq r$. Then, similarly $\beta, \gamma \in D^*_{r+1}$ and $\alpha = \beta\gamma$.

Corollary 3.2 $D^*_i \subseteq < D^*_{n-1} >$ for each $1 \leq \mathrm{i} \leq \mathrm{n} - 1$.

Let $OCT_{(n,r)} = \{\alpha \in OCT_n : |\operatorname{im}(\alpha)| \leq r\}$ for $1 \leq r < n$. It is clear that $OCT_{(n,r)}$ is an ideal of OCT_n. Thus we have

$$< D^*_{n-1} > = OC\mathrm{T}_{(n,n-1)} = OCT_n \backslash S_n = OCT_n \backslash \{\epsilon\}. \tag{29}$$

If $\alpha \in D^*_{n-1}$ then $\operatorname{im}(\alpha) = \{1,2,\dots,n-1\}$ or $\operatorname{im}(\alpha) = \{2,3,\dots,n\}$ since $\operatorname{im}(\alpha)$ is a convex subset of X_n. Moreover since kernel classes of α are convex subsets of X_n, there exists $1 \leq i \leq n-1$ such that

$$\ker(\alpha) = \bigcup_{j=1}^{n} \{(j,j)\} \cup \{(i+1,i),(i,i+1)\}. \tag{30}$$

In this case, we denote $\ker(\alpha)$ by $[i, i+1]$ instead of $\bigcup_{j=1}^{n} \{(j,j)\} \cup \{(i+1,i),(i,i+1)\}$ for convenience.

It is clear that $|D^*_{n-1}| = 2(n-1)$ for $n \geq 3$ and so rank $(OCT_{(n,n-1)}) \leq 2n-2$ from Corollary 3.2. Notice that, since $OCT_{(n,n-2)}$ is an ideal of $OCT_{(n,n-1)}$ for $n \geq 3$, $\alpha \in D^*_{n-1}$ can be written as a product of only the elements of D^*_{n-1}. Moreover, since there are $n-1$ R^*-classes (kernel classes) in D^*_{n-1}, we have rank $(OCT_{(n,n-1)}) \geq n-1$ for $n \geq 3$.

Let $\alpha_{i,i+1} \in D^*_{n-1}$ be the element with $\operatorname{im}(\alpha_{i,i+1}) = \{1,2,\dots,n-1\}$ and $\ker(\alpha_{i,i+1}) = [i, i+1]$, that is

$$\alpha_{i,i+1} = \begin{pmatrix} 1 & \dots & i & i+1 & i+2 & \dots & n \\ 1 & \dots & i & i & i+1 & \dots & n-1 \end{pmatrix}, \tag{31}$$

for $1 \leq i \leq n-2$, and

$$\alpha_{n-1,n} = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 1 & 2 & \dots & n-1 & n-1 \end{pmatrix}. \tag{32}$$

Let $\beta_{i,i+1} \in D^*_{n-1}$ be the element with $\mathrm{im}\,(\beta_{i,i+1}) = \{2,3,\dots,n\}$ with $\ker(\beta_{i,i+1}) = [i, i+1]$, that is

$$\beta_{1,2} = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 2 & 3 & \dots & n \end{pmatrix}, \tag{33}$$

$$\beta_{i,i+1} = \begin{pmatrix} 1 & 2 & \dots & i & i+1 & \dots & n \\ 2 & 3 & \dots & i+1 & i+1 & \dots & n \end{pmatrix} \tag{34}$$

for $2 \le i \le n-1$.

Theorem 3.3 rank $(OCT_{(n,n-1)}) = \mathrm{n} - 1$ for $\mathrm{n} \ge 3$.

Proof. Let $n \ge 3$ and $W = \{\alpha_{1,2}\} \cup \{\beta_{i,i+1} | 2 \le i \le n-1\}$ where $\alpha_{1,2}, \beta_{i,i+1}$ $(2 \le i \le n-1)$ are the elements defined above. It is clear that $|W| = n-1$ and so for the proof it is enough to show that W is a generating set of $OCT_{(n,n-1)}$ since rank $(OCT_{(n,n-1)}) \ge n-1$. By using the multiplication it is a routine matter to show $\alpha_{1,2}\beta_{n-1,n} = \beta_{1,2}$ and $\beta_{i,i+1}\alpha_{1,2} = \alpha_{i,i+1}$ for $2 \le i \le n-1$. Thus, $D^*_{n-1} \subseteq < W >$ and so $< W > = OC\mathrm{T}_{(n,n-1)}$ from Corollary 3.2. Therefore, rank $(OCT_{(n,n-1)}) = n-1$ for $n \ge 3$, as required.

Theorem 3.4 rank $(OCT_n) = \begin{cases} 3, & \text{if } n = 2 \\ \mathrm{n}, & \text{if } \mathrm{n} = 1 \text{ or if } \mathrm{n} \ge 3. \end{cases}$

Proof. Recall that rank $(OCT_1) = 1$ and rank $(OCT_2) = 3$. For $n \ge 3$, it is clear that $OCT_n = OCT_{(n,n-1)} \cup \{\epsilon\}$ where ϵ is the identity mapping on OCT_n. Since OCT_n is a monoid and $OCT_{(n,n-1)}$ is a finitely generated semigroup, and since $\alpha\beta \ne \epsilon$ for all $\alpha, \beta \in OCT_{(n,n-1)}$, we have rank (OCT_n) = rank $(OCT_{(n,n-1)}) + 1 = n$ for $n \ge 3$.

4. The rank of $ORCT_n$

In this section, we find a generating set and the rank of $ORCT_n$. It is clear that $ORCT_1 = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$ and so rank $(ORCT_1) = 1$. It is also clear that

$$ORCT_2 = \left\{\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}\right\}. \tag{35}$$

Since

$$ORCT_2 = < \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} > \tag{36}$$

and since $ORCT_2$ is not a commutative semigroup, we have rank $(ORCT_2) = 2$. Now we consider the generating sets of $ORCT_n$ for $n \ge 3$. Let

$$F_k = \{\alpha \in ORCT_n : |\,\mathrm{im}\,(\alpha)| = k\} \tag{37}$$

for $1 \le k \le n$.

Lemma 4.1 If $\alpha \in F_r$ then $\alpha \in < F_{r+1} >$ for $1 \leq r \leq n-2$.

Proof. Let $1 \leq r \leq n-2$. If $\alpha \in OCT_n \cap F_r$, then the result follows from Lemma 3.1. Let $\alpha \in ORCT_n \backslash OCT_n$ and $\alpha \in F_r$. Then α is an order-reversing full contraction mappings and so there exists a partition $\{A_1, A_2, \ldots, A_r\}$ of X_n and there exists $r \leq k \leq n$ such that

$$\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_i & \ldots & A_r \\ k & k-1 & \ldots & k-i+1 & \ldots & k-r+1 \end{pmatrix}. \quad (38)$$

It is clear that $|A_i| \geq 2$ at least for one $1 \leq i \leq r$ since $r \leq n-2$. Without loss of generality let $A_i = \{a_1, a_2, \ldots, a_m\}$ for $m \geq 2$, and let x_i be the maximum element in A_i. If $k < n$ and $k-r \geq 1$, let

$$\beta = \begin{pmatrix} A_1 \backslash \{x_1\} & \{x_1\} & A_2 & \ldots & A_r \\ k & k-1 & k-2 & \ldots & k-r \end{pmatrix}, \quad (39)$$

for $i = 1$, let

$$\beta = \begin{pmatrix} A_1 & A_2 & \ldots & A_{r-1} & A_r \backslash \{x_r\} & \{x_r\} \\ k & k-1 & \ldots & k-r+2 & k-r+1 & k-r \end{pmatrix}, \quad (40)$$

for $i = r$, let

$$\beta = \begin{pmatrix} A_1 & A_2 & \ldots & A_{i-1} & A_i \backslash \{x_i\} & \{x_i\} & A_{i+1} & \ldots & A_r \\ k & k-1 & \ldots & k-i+2 & k-i+1 & k-i & k-i-1 & \ldots & k-r \end{pmatrix}, \quad (41)$$

for $2 \leq i \leq r-1$. Also, let γ be the mapping defined as

$$j\gamma = \begin{cases} k-r+1; & if \quad 1 \leq j \leq k-r \\ j+1; & if \quad k-r+1 \leq j \leq k-i \\ j; & if \quad k-i+1 \leq j \leq k \\ k+1; & if \quad k+1 \leq j \leq n. \end{cases} \quad (42)$$

Then $\beta, \gamma \in F_{r+1}$ and $\alpha = \beta\gamma$.

If $k = n$, let

$$\beta = \begin{pmatrix} A_1 \backslash \{x_1\} & \{x_1\} & A_2 & \ldots & A_r \\ n & n-1 & n-2 & \ldots & n-r \end{pmatrix}, \quad (43)$$

for $i = 1$, let

$$\beta = \begin{pmatrix} A_1 & A_2 & \ldots & A_{r-1} & A_r \backslash \{x_r\} & \{x_r\} \\ n & n-1 & \ldots & n-r+2 & n-r+1 & n-r \end{pmatrix}, \quad (44)$$

for $i = r$, let

$$\beta = \begin{pmatrix} A_1 & A_2 & \ldots & A_{i-1} & A_i \backslash \{x_i\} & \{x_i\} & A_{i+1} & \ldots & A_r \\ n & n-1 & \ldots & n-i+2 & n-i+1 & n-i & n-i-1 & \ldots & n-r \end{pmatrix}, \quad (45)$$

for $2 \leq i \leq r-1$. Also, let γ be the mapping defined as

$$j\gamma = \begin{cases} n-r; & if \quad 1 \leq j \leq n-r-1 \\ j+1; & if \quad n-r \leq j \leq n-i \\ j; & if \quad n-i+1 \leq j \leq n. \end{cases} \tag{46}$$

Then $\beta, \gamma \in F_{r+1}$ and $\alpha = \beta\gamma$.

If $k = r$, let

$$\beta = \begin{pmatrix} A_1 \backslash \{x_1\} & \{x_1\} & A_2 & \dots & A_r \\ r+1 & r & r-1 & \dots & 1 \end{pmatrix}, \tag{47}$$

for $i = 1$, let

$$\beta = \begin{pmatrix} A_1 & A_2 & \dots & A_{r-1} & A_r \backslash \{x_r\} & \{x_r\} \\ r+1 & r & \dots & 3 & 2 & 1 \end{pmatrix}, \tag{48}$$

for $i = r$, let

$$\beta = \begin{pmatrix} A_1 & A_2 & \dots & A_{i-1} & A_i \backslash \{x_i\} & \{x_i\} & A_{i+1} & \dots & A_r \\ r+1 & r & \dots & r-i+3 & r-i+2 & r-i+1 & r-i & \dots & 1 \end{pmatrix}, \tag{49}$$

for $2 \leq i \leq r-1$. Also, let γ be the mapping defined as

$$j\gamma = \begin{cases} j; & if \quad 1 \leq j \leq r-i+1 \\ j-1; & if \quad r-i+2 \leq j \leq r+1 \\ r+1; & if \quad r+2 \leq j \leq n. \end{cases} \tag{50}$$

Then $\beta, \gamma \in F_{r+1}$ and $\alpha = \beta\gamma$.

Corollary 4.2 If $\alpha \in F_i$ for $1 \leq i \leq n-1$ then $\alpha \in < F_{n-1} >$ for $n \geq 3$.

Let $ORCT_{(n,r)} = \{\alpha \in ORCT_n : |\operatorname{im}(\alpha)| \leq r\}$ for $1 \leq r < n$. It is clear that $ORCT_{(n,r)}$ is an ideal of $ORCT_n$. Moreover we have

$$F_n = \{\epsilon = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}, \theta = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}\}, \tag{51}$$

and notice that

$$< F_{n-1} > = ORCT_{(n,n-1)} = ORCT_n \backslash S_n = ORCT_n \backslash \{\epsilon, \theta\} \tag{52}$$

where ϵ is the identity element of $ORCT_n$ and that $\theta^2 = \epsilon$.

Corollary 4.3 $ORCT_n = < F_{n-1} \cup \{\theta\} >$ for $n \geq 3$.

If $\alpha \in F_{n-1}$, since $\operatorname{im}(\alpha)$ is a convex subset of X_n, we have $\operatorname{im}(\alpha) = \{1,2,\dots,n-1\}$ or $\operatorname{im}(\alpha) = \{2,3,\dots,n\}$. Moreover there are $n-1$ different kernel classes in F_{n-1} and there exist 4 elements in F_{n-1} which has the same kernel classes. Thus $|F_{n-1}| = 4(n-1)$ for $n \geq 3$.

Let $\alpha_{i,i+1}$ and $\beta_{i,i+1}$ be the order-preserving full contraction mappings defined before Theorem 3.3 for each $1 \leq i \leq n-1$. Moreover let $\lambda_{i,i+1} \in F_{n-1}$ be the order-reversing full contraction mappings such that im $(\lambda_{i,i+1}) = \{1,2,\dots,n-1\}$ and $\ker(\lambda) = [i, i+1]$, that is

$$\lambda_{i,i+1} = \begin{pmatrix} 1 & \dots & i & i+1 & i+2 & \dots & n \\ n-1 & \dots & n-i & n-i & n-i-1 & \dots & 1 \end{pmatrix}, \tag{53}$$

for $1 \leq i \leq n-2$, and

$$\lambda_{n-1,n} = \begin{pmatrix} 1 & 2 & \dots & n-2 & n-1 & n \\ n-1 & n-2 & \dots & 2 & 1 & 1 \end{pmatrix}. \tag{54}$$

Also, let $\mu_{i,i+1} \in F_{n-1}$ be the order-reversing full contraction mappings with im $(\mu_{i,i+1}) = \{2,3,\dots,n\}$ with $\ker(\mu_{i,i+1}) = [i, i+1]$, that is

$$\mu_{1,2} = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ n & n & n-1 & \dots & 3 & 2 \end{pmatrix}, \tag{55}$$

and

$$\mu_{i,i+1} = \begin{pmatrix} 1 & 2 & \dots & i & i+1 & \dots & n \\ n & n-1 & \dots & n-i+1 & n-i+1 & \dots & 2 \end{pmatrix}, \tag{56}$$

for $2 \leq i \leq n-1$. Also notice that $F_{n-1} = \{\alpha_{i,i+1}, \beta_{i,i+1}, \lambda_{i,i+1}, \mu_{i,i+1} | 1 \leq i \leq n-1\}$. We give some equations in the following lemmas.

Lemma 4.4 For $n \geq 3$ and $1 \leq i \leq n-1$,
(i). $\alpha_{i,i+1}\theta = \mu_{i,i+1}$
(ii). $\beta_{i,i+1}\theta = \lambda_{i,i+1}$
(iii). $\lambda_{i,i+1}\theta = \beta_{i,i+1}$
(iv). $\mu_{i,i+1}\theta = \alpha_{i,i+1}$.

Proof. By using the multiplication it is a routine matter to show (i) and (ii). Also, the results (iii) and (iv) follows from the fact $\theta^2 = \epsilon$.

Lemma 4.5 For $n \geq 3$ and $1 \leq i \leq n-1$,
(i). $\theta\alpha_{i,i+1} = \lambda_{n-i,n-i+1}$
(ii). $\theta\beta_{i,i+1} = \mu_{n-i,n-i+1}$
(iii). $\theta\lambda_{i,i+1} = \alpha_{n-i,n-i+1}$
(iv). $\theta\mu_{i,i+1} = \beta_{n-i,n-i+1}$.

Proof. (i) First notice that $1(\theta\alpha_{i,i+1}) = n\alpha_{i,i+1} = n-1$ and $n(\theta\alpha_{i,i+1}) = 1\alpha_{i,i+1} = 1$. Thus im $(\theta\alpha_{i,i+1}) = \{1,2,\dots,n-1\}$ and clearly $\theta\alpha_{i,i+1}$ is an order-reversing full contraction mappings. Moreover

$$(n-i)(\theta\alpha_{i,i+1}) = (i+1)\alpha_{i,i+1} = i\alpha_{i,i+1} = (n-i+1)(\theta\alpha_{i,i+1}) \tag{57}$$

and so we have $\ker(\theta\alpha_{i,i+1}) = [n-i, n-i+1]$. Thus, $\theta\alpha_{i,i+1} = \lambda_{n-i,n-i+1}$, as required.
(ii), (iii) and (iv) can be shown similarly.

Lemma 4.6 For $n \geq 3$ and $1 \leq i \leq n-1$ we have $\alpha_{i,i+1}\beta_{n-1,n} = \beta_{i,i+1}$.

Proof. By using the multiplication it is a routine matter to prove the claim.

Proposition 4.7 Let $n \geq 3$ and let A be a generating set for $ORCT_n$. If n is an odd number then A must include at least $\frac{n-1}{2}$ elements from F_{n-1}, and if n is an even number then A must include at least $\frac{n}{2}$ elements from F_{n-1}.
Proof. Let $n \geq 3$ and let A be a generating set for $ORCT_n$. Recall that $F_n = \{\epsilon, \theta\}$ and $\theta^2 = \epsilon$ where ϵ is the identity element of $ORCT_n$. Also $ORCT_{(n,n-2)}$ is an ideal of $ORCT_n$ and there are $n-1$ different kernel classes in F_{n-1}. Let $\alpha \in F_{n-1}$ then there exists $1 \leq k \leq n-1$ such that $\ker(\alpha) = [k, k+1]$. Let $m \in Z^+$ and suppose that $\alpha = \alpha_1\alpha_2 \dots \alpha_m$ where $\alpha_i \in ORCT_n$ for each $1 \leq i \leq m$. Then every $\alpha_i \in F_{n-1} \cup F_n$ since $ORCT_{(n,n-2)}$ is an ideal of $ORCT_n$. If $\alpha_1 \in F_{n-1}$ then it is clear that $\ker(\alpha_1) = \ker(\alpha)$. If $\alpha_1 \in F_n$ then we can assume that $\alpha_1 = \theta$ since ϵ is the identity element. Then we can assume that $\alpha_2 \in F_{n-1}$ since $\theta^2 = \epsilon$ and so $\ker(\alpha_2) = [n-k, n-k+1]$ from Lemma 4.5. Thus if n is an odd number then A must include at least $\frac{n-1}{2}$ elements from F_{n-1} and if n is an even number then A must include at least $\frac{n}{2}$ elements from F_{n-1}.

For $n \geq 3$ it is clear that $F_n = \{\epsilon, \theta\}$ is a subsemigroup generated by $\{\theta\}$ or $\{\theta, \epsilon\}$, and $ORCT_n \backslash F_n = ORCT_{(n,n-1)}$ is an ideal of $ORCT_n$. Hence every generating set of $ORCT_n$ must include the element θ. Thus, if n is an odd number then rank $(ORCT_n) \geq \frac{n+1}{2}$, and if n is an even number then rank $(ORCT_n) \geq \frac{n+2}{2}$ from Proposition 4.7.

Theorem 4.8 For $n \geq 1$,

$$rank\ (ORCT_n) = \begin{cases} \dfrac{n+1}{2}; & \text{if } n \text{ is an odd number} \\ \dfrac{n+2}{2}; & \text{if } n \text{ is an even number}. \end{cases}$$

Proof. If $n = 1$ or $n = 2$ then the result is clear. Let $n \geq 3$ and n be an odd number. Then we have rank $(ORCT_n) \geq \frac{n+1}{2}$. Let

$$W = \{\theta\} \cup \{\alpha_{i,i+1} | 1 \leq i \leq \tfrac{n-1}{2}\}, \tag{58}$$

and it is clear that $|W| = \frac{n+1}{2}$. Hence it is enough to show that W is a generating set of $ORCT_n$. For $1 \leq k \leq \frac{n-1}{2}$ then $\alpha_{k,k+1} \in W$ and so $\alpha_{1,2}\theta = \mu_{1,2}$ and $\theta\mu_{1,2} = \beta_{n-1,n}$. It follows that $\alpha_{k,k+1}\beta_{n-1,n} = \beta_{k,k+1}$, $\beta_{k,k+1}\theta = \lambda_{k,k+1}$ and $\alpha_{k,k+1}\theta = \mu_{k,k+1}$. Thus if $1 \leq k \leq \frac{n-1}{2}$ then

$\{\alpha_{k,k+1}, \beta_{k,k+1}, \lambda_{k,k+1}, \mu_{k,k+1}\} \in < W >$.

Now let $\frac{n-1}{2} < k \leq n-1$ and let $i = n-k$. Then it is clear that $\alpha_{i,i+1} \in W$. Moreover $\theta\alpha_{i,i+1} = \lambda_{n-i,n-i+1} = \lambda_{k,k+1}$ and $\lambda_{k,k+1}\theta = \beta_{k,k+1}$. Since $i \leq \frac{n-1}{2}$ we have $\lambda_{i,i+1} \in < W >$, and so $\theta\lambda_{i,i+1} = \alpha_{n-i,n-i+1} = \alpha_{k,k+1}$ and $\alpha_{k,k+1}\theta = \mu_{k,k+1}$. It follows that

$\{\alpha_{k,k+1}, \beta_{k,k+1}, \lambda_{k,k+1}, \mu_{k,k+1}\} \in < W >$.

So W is a generating set of $ORCT_n$ from Corollary 4.3. Thus if n is an odd number then we have rank $(ORCT_n) = \frac{n+1}{2}$. If n is an even number similarly it can be shown that $W = \{\theta\} \cup \{\alpha_{i,i+1} | 1 \leq i \leq \frac{n}{2}\}$ is a generating set of $ORCT_n$ and so rank $(ORCT_n) = \frac{n+2}{2}$, as required.

Acknowledgement

We would like to thank the referees for their valuable comments which helped to improve the manuscript.

References

[1] Garba, G.U., Ibrahim, M.J. and Imam, A.T., On certain semigroups of full contraction maps of a finite chain, **Turkish Journal of Mathematics**, 41, 3, 500-507, (2017).

[2] Adeshola, A.D. and Umar, A., Results for certain semigroups of order-preserving full contraction mappings of a finite chain, **Journal of Combinatorial Mathematics and Combinatorial Computing**, 106, 37-49, (2018).

[3] Gomes, G.M.S. and Howie, J.M., On the ranks of certain finite semigroups of transformations, **Mathematical Proceedings of the Cambridge Philosophical Society**, 101, 3, 395-403, (1987).

[4] Ayık, G., Ayık, H., Bugay, L. and Kelekci, O., Generating sets of finite singular transformation semigroups, **Semigroup Forum**, 86, 1, 59-66, (2013).

[5] Bugay, L., Yağcı, M. And Ayık, H., The ranks of certain semigroups of partial isometries, **Semigroup Forum**, 97, 2, 214-222, (2018).

[6] Toker, K. and Ayık, H., On the rank of transformation semigroup $T_{(n,m)}$, **Turkish Journal of Mathematics**, 42, 4, 1970-1977, (2018).

[7] Howie, J.M., **Fundamentals of Semigroup Theory**, Oxford University Press, USA, (1995).