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Cancellative Elements in Finite AG-groupoids

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Abstract − An Abel-Grassmann's groupoid (briefly AG-groupoid) is a groupoid S satisfying the left invertive law: $(xy)z = (zy)x \ \forall \ x, y, z \in S$. In the present paper, we discuss the left and right cancellative property of elements of the finite AG-groupoid S. For an AG-groupoid with left identity it is known that every left cancellative element is right cancellative. We prove a problem (for finite AG-groupoids) that every left cancellative element of an AG-groupoid (with out left identity) is right cancellative. Moreover, we generalize various results of finite AG-groupoids by removing the condition of existence of left identity.

Keywords − *AG-groupoid, AG-subgroupoid, Cancellative elements, non-cancellative elements*

1. Introduction

An AG-groupoid S is a groupoid which satisfies the left invertive law $(xy)z = (zy)x \ \forall \ x, y, z \in S$, this is non-associative in general. In literature, different authors used different names for this structure, e.g. left invertive groupoid in [1], Left Almost Semigroup (briefly LA-semigroup) in [2, 3], while right modular groupoid in [4, Line 35]. Cho et al. [4] proved that, an AG-groupoid S always satisfies the medial law: $(wx)(yz) = (wy)(xz) \ \forall \ w, x, y, z \in S$, while an AG-groupoid S with left identity always satisfies the paramedial law: $(wx)(yz) = (zx)(yw) \ \forall \ w, x, y, z \in S$. An AG-groupoid S with left identity is called a left almost group (briefly LA-group) or an AG-group, if each element of S has its inverse element [5]. For more study we refer [6, 7]. A non-empty subset H of an AG-groupoid is called an AG-subgroupoid if it is closed with respect to the binary operation. A left ideal I (respectively, right) of an AG-groupoid S is a subset of S which satisfies the property $SI \subset I$ (respectively, $IS \subset I$). A two sided ideal of S is an ideal which is both left and right ideal. An element $c \in S$ is called left cancellative if $cx = cy \implies x = y \ \forall \ x, y \in S$. Similarly, $c \in S$ is said to be right cancellative if $xc = yc \implies x = y \ \forall \ x, y \in S$. An element c of the AG-groupoid S is said to be cancellative if it is both left and right cancellative. From now onward, we will use LC for left cancellative, RC for right cancellative, TC for two sided cancellative and NC for non-cancellative elements. An AG-groupoid S is called LC (respectively, RC) if all element of S is LC (respectively, RC). LC, RC and TC play an important role in the theory of quasigroups and many results occur in this structure due to these properties. Every AG-groupoid is not necessarily TC but some or all of its elements may be TC and hence can enjoy some special properties that a general AG-groupoid cannot possess.

In this paper, we study the LC and RC property of a finite AG-groupoids. Moreover, we solve a problem proposed by Shah et al. [8], that every LC element of an AG-groupoid is also RC. We also

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generalize several results of [8] and remove the condition of existence of left identity. We prove that TC and NC elements of a finite AG-groupoid (not necessarily have left identity) S partition S and the two sub-classes of S are AG-subgroupoids. If a finite AG-groupoid S have at least one NC element then set of NC elements form a maximal ideal.

2. Characterization of AG-groupoid due to Cancellativity

In this section, we show that every LC element of a finite AG-groupoid is TC. The following lemma will be useful.

Lemma 2.1. If a finite AG-groupoid S has LC (respectively, RC) element then $SS = S$.

Proof. Let S be a finite AG-groupoid. Then clearly $SS \subseteq S$. On the other hand, let $S = \{s_1, s_2, \ldots, s_n\}$ be a finite AG-groupoid and $a \in S$ be a LC (respectively ,RC) element. Then $aS = \{as_1, as_2, \ldots, as_n\}$ (respectively, $Sa = \{s_1a, s_2a, \ldots, s_na\}$). We have to show that aS has n distinct elements. Let on the contrary there exist a_i and a_j of S such that $aa_i = aa_j$. Then since a is LC. This gives $a_i = a_j$, which implies that all elements of aS are distinct. Let $x \in S$ be any arbitrary element. Then there exist $a_i \in S$ such that $x = aa_i \in SS$. This gives $S \subseteq SS$. Hence $SS = S$. □

Remark 2.2. Lemma 2.1 does not hold for infinite AG-groupoid as $(N, +)$ is an infinite AG-groupoid but $N + N \neq N$.

The following theorems will be useful.

Theorem 2.3 (Shah et al. [8])**.** In AG-groupoid, if an element is RC then it is TC.

Theorem 2.4 (Shah et al. [8])**.** Let $x, y \in S$, where S is an AG-groupoid. We define a relation $\sim$ on S as

$$x \sim y, \quad x \text{ and } y \text{ are both TC or NC.}$$

Then the relation $\sim$ is an equivalence relation.

Theorem 2.5. Let S be a finite AG-groupoid and $c \in S$ such that $c = c_1c_2$. Then c is LC if and only if c_1 and c_2 are TC.

Proof. Let $c \in S$ be any LC element of the finite AG-groupoid S. Then $\forall\, x, y \in S$, we have

$$cx = cy \implies x = y$$

Let $c = c_1c_2$, we have to show that both c_1 and c_2 are TC. For this it is enough to show that they are RC. Let $xc_2 = yc_2$ for any $x, y \in S$. Then by repeated use of left invertive law, we get

$$\begin{aligned} cx &= (c_1c_2)x = (xc_2)c_1 \\ &= (yc_2)c_1 = (c_1c_2)y \\ &= cy \end{aligned}$$

This gives $x = y$. This implies that c_2 is RC and hence TC. Next we have to show that c_1 is RC, for this let $xc_1 = yc_1$ for any $x, y \in S$. Since we have proved that c_2 is RC, thus there exist $x_1, y_1 \in S$ such that $x = x_1c_2$ and $y = y_1c_2$. Now as

$$\begin{aligned} xc_1 &= yc_1 \\ (x_1c_2)c_1 &= (y_1c_2)c_1 \\ (c_1c_2)x_1 &= (c_1c_2)y_1 \quad \text{by left invertive law} \end{aligned}$$

This implies that $cx_1 = cy_1$ which further implies that $x_1 = y_1$ and hence $x = y$. Hence c_1 is RC. Thus c is the product of two TC elements.
Conversely, let $c_1, c_2 \in S$ be two TC elements, we have to show that their product c_1c_2 is LC. For this consider

$$\begin{aligned} (c_1c_2)x &= (c_1c_2)y \\ (xc_2)c_1 &= (yc_2)c_1 \quad \text{by left invertive law} \end{aligned}$$

As c_1 and c_2 are RC, so we get $x = y$. □

Theorem 2.6. Every RC element in a finite AG-groupoid is the product of two TC elements.

PROOF. Let $c \in S$ be any arbitrary RC element of the finite AG-groupoid S. Then $\forall\ x, y \in S$, we have

$$xc = yc \implies x = y$$

Let $c = c_1c_2$. We have $xc_1 = yc_1$ and consider

$$\begin{aligned}(xc)c &= (xc)(c_1c_2) = (xc_1)(cc_2) \text{ by medial law}\\ &= (yc_1)(cc_2) = (yc)(c_1c_2) \text{ again by medial law}\\ &= (yc)c\end{aligned}$$

By repeated use of RC property of c, we get $x = y$.
Next we show that c_2 is RC. For this let $xc_2 = yc_2$. Then consider

$$\begin{aligned}(c_1x)c &= (c_1x)(c_1c_2) = (c_1c_1)(xc_2) \text{ by medial law}\\ &= (c_1x)(c_1c_2) = (c_1c_1)(yc_2) \text{ again by medial law}\\ &= (c_1y)c\end{aligned}$$

By use of TC property of c and c_1, we get $x = y$. This implies that c_2 is RC. □

Theorem 2.7. Every LC element in a finite AG-groupoid is RC element.

PROOF. The proof follows from Theorem 2.5 and Theorem 2.6. □

Corollary 2.8. For a finite AG-groupoid S, the following two conditions are equivalent for any $c \in S$.

(1) c is RC

(2) c is LC

Theorem 2.9. The set of all TC elements of a finite AG-groupoid S is either an AG-subgroupoid of S or an empty set.

PROOF. Let S be a finite AG-groupoid and H be the set of all TC elements of S. If H is empty then there is nothing to prove and if H is non-empty then let $c_1, c_2 \in H$. Let on the contrary $c = c_1c_2$ is NC then this implies that one of c_1 or c_2 or both are NC, which is a contradiction. Hence H is an AG-subgroupoid. □

Corollary 2.10. If S is a finite AG-groupoid then the product of one TC element and one NC element or product of two NC elements is always NC.

Lemma 2.11. If S is a finite AG-groupoid then the set of all NC elements of S is either an AG-subgroupoid of S or an empty set.

PROOF. Given that S is a finite AG-groupoid. Let K be the set of all NC elements. Clearly, K is empty if S is TC. So let us suppose S is not TC and let $c_1, c_2 \in K$ and on contrary that $c = c_1c_2$ is TC then by Theorem 2.5 both c_1 and c_2 are LC, which by Theorem 2.7 c_1 and c_2 are TC. Hence $c = c_1c_2$ is NC. Thus K is an AG-subgroupoid. □

Theorem 2.12. TC elements and NC elements of a finite AG-groupoid S partition S into two AG-subgroupoid of S.

Corollary 2.13. If S is a finite AG-groupoid then a proper right (respectively, left) ideal of S cannot be a subset of H.

PROOF. Proof follows from Theorem 2.9. □

Corollary 2.14. For a finite AG-groupoid S having at least one NC element, K is always a maximal ideal.

PROOF. Proof follows from Lemma 2.11. □

In the following theorem, we construct TC AG-groupoids from abelian group.

Theorem 2.15. *Let $(G, +)$ be an abelian group under addition and let $\alpha, \beta \in Auto(G)$ satisfying $\alpha^2 = \beta$. Then define new binary operation on G by $x \cdot y = \alpha(x) + \beta(y)\ \forall\ x, y \in G$. Then $G_{\alpha,\beta}$ is an AG-groupoid.*

Proof. Let x, y and z be any three arbitrary elements of the abelian group G. Then consider

$$\begin{aligned} (x \cdot y) \cdot z &= (\alpha(x) + \beta(y)) \cdot z \\ &= \alpha^2(x) + \alpha\beta(y) + \beta(z) \end{aligned} \tag{1}$$

On the other hand

$$\begin{aligned} (z \cdot y) \cdot x &= (\alpha(z) + \beta(y)) \cdot x \\ &= \alpha^2(z) + \alpha\beta(y) + \beta(x) \\ &= \alpha^2(x) + \alpha\beta(y) + \beta(z), \quad \alpha^2 = \beta \end{aligned} \tag{2}$$

From (1) and (2) $(x \cdot y) \cdot z = (z \cdot y) \cdot x$. This implies that $G_{\alpha,\beta} = (G, \cdot)$ is an AG-groupoid. It is easy to see that $G_{\alpha,\beta}$ is TC. □

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