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Some Results on Divisior Cordial Graphs

Mohammed Abdel Azim Seoud[1], Shakir Mahmoud Salman[2]

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Abstract — In this paper, we introduce some results on divisor cordial graphs where we find some upper bound for the labeling of any simple graph and $r-regular$ graph and describe the divisor cordial labeling for some families of graphs such the jellyfish graph, shell graph and the bow and butterfly graphs.

Keywords — *labeling, divisor cordial, shell graph*

Introduction

In this paper by a simple graph, we mean a finite, undirected graph without loops and multiple edges, for terms not defined here, we refer to Harary [1]. Graph labeling, mean that the vertices and edges are assigned real values or subsets of a set, subject to certain conditions. For a dynamic survey on various graph labeling problems we refer to Gallian [2]. The concept of cordial labeling was introduced by Cahit [3], in [4], Varatharajan et al. introduce the concept of divisor cordial labeling of graph. The divisor cordial labeling of various types of graphs are presented in [4–12]. The brief summaries of definitions which are necessary for the present investigation are provided below. For standard terminology and notations related to number theory we refer to Burton [13].

Definition 1.1. [4] Let $G = (V(G), E(G))$ be a simple graph and $f : V(G) \longrightarrow \{1, 2, ..., |V(G)|\}$ be a bijection. For each edge uv, assign the label 1 if $f(u)|f(v)$ or $f(v)|f(u)$ and the label 0 otherwise. The function f is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$. A graph with a divisor cordial labeling is called a divisor cordial graph.

Definition 1.2. [1] The neighborhood of a vertex u is the set $N_u(G)$ consisting of all vertices v which are adjacent with u. The closed neighborhood is $N_u[G] = N_u(G) \bigcup \{u\}$.

Definition 1.3. [1] The number $\delta(G) = \min\{d(v) \mid v \in V\}$ is the minimum degree of the vertices in the graph G, the number $\Delta(G) = \max\{d(v) \mid v \in V\}$ is the maximum degree of the vertices in the graph G, the number $d(G) = \frac{1}{|V|} \sum\limits_{v \in V} d(v)$ is the average degree of the vertices in the graph G.

Definition 1.4. [14] The Jelly fish graph $J(m, n)$ is obtained from a $4-cycle$ v_1, v_2, v_3, v_4 by joining v_1 and v_3 with an edge and appending m pendent edges to v_2 and n pendent edges to v_4.

[1]m.a.seoud@hotmail.com; [2]s.m.salman@outlook.com (Corresponding Author)
[1]Ain Shams University, Faculty of Science, Department of mathematics, Abbasia, Cairo, Egypt
[2]Diyala University, Department of Mathematics, Basic Education College, Diyala, Iraq

Definition 1.5. [15] A shell graph is defined as a cycle C_n with $(n-3)$ chords sharing a common end point called the apex, shell graphs are denoted as $C(n, n-3)$.

Definition 1.6. [16] A bow graph is defined to be a double shell in which each shell has any order.

Definition 1.7. [15] Define a Butterfly graph as a bow graph with exactly two pendent edges at the apex.

The Results

Proposition 2.1. For any simple graph $G(p, q)$, the maximum value of $e_f(1)$ is
$min \left\{ \triangle(G) + \sum_{i=2}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1), q \right\}$, where $p \geq 4$.

PROOF. Let $G(p, q)$ be a simple connected graph and let the vertex v_k be of maximum degree $\triangle(G)$, if we labeled this vertex by 1 then we will achieve $\triangle(G)$ edges labeled 1, and from division algorithm the maximum numbers of the multiples of labels of vertices are:
for 2 is $\lfloor \frac{p}{2} \rfloor - 1$,
for 3 is $\lfloor \frac{p}{3} \rfloor - 1$,
for 4 is $\lfloor \frac{p}{4} \rfloor - 1$,
.
.
.
for $\lfloor \frac{p}{2} \rfloor$ is $\lfloor \frac{p}{\lfloor \frac{p}{2} \rfloor} \rfloor - 1$ which must equal 1

hence the maximum value for $e_f(1)$ equals $\triangle(G) + \sum_{i=2}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$ in any graph $G(p, q)$. □

Corollary 2.2. For each $r-regular$ graph the maximum value of $e_f(1)$ is $kr + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$; where $k = \lfloor \frac{p}{r+1} \rfloor$ and $p \geq 4$.

PROOF. Let $G(p, q)$ be an $r-regular$ graph then $\triangle(G) = r$, and for each vertex v in graph G the maximum number of edges that label 1 in $N_v(G)$ is r, hence for all i in which $\lfloor \frac{p}{i} \rfloor - 1 \geq r$ we reduced it to r.

But from Proposition 2.1 the maximum value of $e_f(1)$ is $\triangle(G) + \sum_{i=2}^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{i} \rfloor - 1)$, then the maximum value in an $r-regular$ graph is:

$$= r + \sum_{i=2}^{k} (r) + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} (\left\lfloor \frac{p}{i} \right\rfloor - 1)$$

$$= r + (k-1)r + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} (\left\lfloor \frac{p}{i} \right\rfloor - 1)$$

$$= kr + \sum_{i=k+1}^{\lfloor \frac{p}{2} \rfloor} (\left\lfloor \frac{p}{i} \right\rfloor - 1)$$

□

Proposition 2.3. For any divisor cordial graph $G(p, q)$, $q \leq 2(\triangle(G) + \sum_{i=3}^{\lfloor \frac{p}{2} \rfloor} \lfloor \frac{p}{i} \rfloor) + 3$, where $p \geq 6$.

PROOF. Let $G(p, q)$ be a divisor cordial graph, then $|e_f(0) - e_f(1)| \leq 1$, means $e_f(0) = e_f(1) - 1$ or $e_f(0) = e_f(1)$ or $e_f(0) = e_f(1) + 1$, by Proposition 2.1,

$$q \leq 2e_f(1) + 1$$

$$q \leq 2(\triangle G + \sum_{i=2}^{\lfloor \frac{p}{2} \rfloor}(\left\lfloor \frac{p}{i} \right\rfloor - 1)) + 1$$

$$q \leq 2(\triangle G + \sum_{i=3}^{\lfloor \frac{p}{2} \rfloor}(\left\lfloor \frac{p}{i} \right\rfloor)) + 3$$

□

Divisor Cordial Labeling for Some Families of Graphs

In this section we introduce the divisor cordial labeling for some types of graphs.

The Jelly Fish Graph

Proposition 3.1. For $m, n \geq 1$, Jelly fish graph $J(m, n)$ is a divisor cordial graph.

PROOF. Let $G(V, E) = J(m, n)$. Then G has $(m + n + 4)$ vertices and $(m + n + 5)$ edges. Without losing of generality, let $m \leq n$. Let $V(G) = V_1 \cup V_2$ where $V_1 = \{x, u, y, v\}$, $V_2 = \{u_i, v_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E = E_1 \cup E_2$, where $E_1 = \{xu, uy, yv, vx, xy\}$, $E_2 = \{uu_i, vv_j; 1 \leq i \leq m, 1 \leq j \leq n\}$.
Define $f : V \to \{1, 2, ..., (m + n + 4)\}$ as follows:
$f(u) = 1, f(v) = 2, f(x) = m + n + 4, f(y) = m + n + 3$ and
$f(u_i) = 2(i + 1); i = 1, 2, ..., m$,

$$f(v_i) = \begin{cases} 2i + 1 & , i = 1, 2, ..., m \\ i + m + 2 & , i = m + 1, m + 2, ..., n \end{cases}$$

From the function f there are $m + 2$ edges labeled 1 sice $f(u) = 1$, and since $f(v) = 2$, then there are exactly $\lfloor \frac{1}{2}(n - m) \rfloor$ of pendent edges from v labeled 1 and only one from vx or vy. means $e_f(1) = m + 3 + \lfloor \frac{1}{2}(n - m) \rfloor$ and

$$e_f(0) = m + n + 5 - (m + 3 + \lfloor \frac{1}{2}(n - m) \rfloor)$$
$$= n + 2 - \lfloor \frac{1}{2}(n - m) \rfloor,$$

Case 1: m, n are odd
The $|E|$ is odd and $\lfloor n - m \rfloor$ are even, hence, $|e_f(0) - e_f(1)| = 1$

Case 2: m, n are even
The $|E|$ is odd and $\lfloor n - m \rfloor$ is even, hence, $|e_f(0) - e_f(1)| = 1$

Case 3: m is odd and n is even
The $|E|$ is even and $\lfloor n - m \rfloor$ is odd, hence, $|e_f(0) - e_f(1)| = 0$

Case 4: m is even and n is odd
The $|E|$ is even and $\lfloor n - m \rfloor$ is odd, hence, $|e_f(0) - e_f(1)| = 0$

Then from Case 1, Case 2, Case 3 and Case 4 the jelly fish graph is divisor cordial. □

Example 3.2. The jelly fish graph $j(6, 11)$ and its divisor cordial labeling are shown in Fig.1

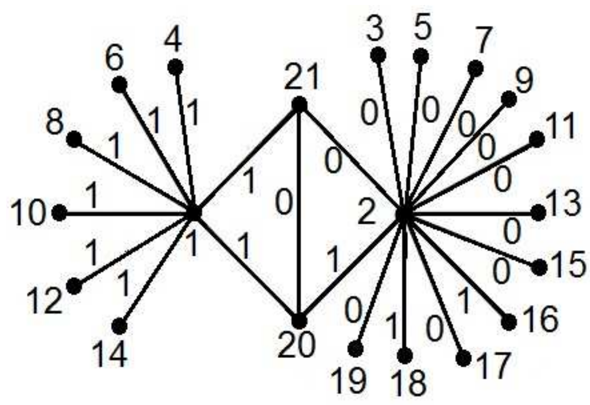

Fig. 1. A Jelly fish graph $j(6,11)$ and its divisor cordial labeling

The shell and The Bow Graph

Proposition 3.3. Every shell graph is divisor cordial.

PROOF. Let $G=(V,E)$ be a $C(n,n-3)$ graph with $|V|=n$, then $|E|=2n-3$ means $|E|$ is an odd number, and let v_0 be the apex and $v_1, v_2, ..., v_{n-1}$ other its vertices.
Define the labeling $f : V \longrightarrow \{1,2,...,n\}$ as:
$f(v_0)=2$, $f(v_1)=1$ and other vertices by the following:

$$\begin{array}{llllll} & 2\cdot 2, & 2\cdot 2^2, & \cdots, & 2\cdot 2^{k_1}, \\ 3, & 3\cdot 2, & 3\cdot 2^2, & \cdots, & 3\cdot 2^{k_2}, \\ 5, & 5\cdot 2, & 5\cdot 2^2, & \cdots, & 5\cdot 2^{k_3}, \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots, \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots, \end{array}$$

where $(2m-1)\cdot 2^{k_m} \leq n$ and $m \geq 1, k_m \geq 0$. We observe that $(2m-1)\cdot 2^a$ divides $(2m-1)\cdot 2^b$; $(a<b)$ and $(2m-1)\cdot 2^{k_i}$ does not divide $2m+1$.
In this labeling, there are $\lceil \frac{n-1}{2} \rceil$ edges label 1 passing through v_0, but other edges not passing through the apex make a path, hence there are also $\lfloor \frac{n-2}{2} \rfloor$ edges are labeled 1. Hence, $e_f(1) = \lceil \frac{n-1}{2} \rceil + \lfloor \frac{n-2}{2} \rfloor$

Case 1: n is odd, then $e_f(1) = \frac{n-1}{2} + \lfloor \frac{n-2}{2} \rfloor$ and $e_f(0) = \frac{n-1}{2} + \lceil \frac{n-2}{2} \rceil$

Case 2: n is even, then $e_f(1) = \lceil \frac{n-1}{2} \rceil + \frac{n-2}{2}$ and $e_f(0) = \lfloor \frac{n-1}{2} \rfloor + \frac{n-2}{2}$

In the two cases Case 1 and Case 2, the difference between $e_f(1)$ and $e_f(0)$ is 1 which means the shell graph is divisor cordial. □

Notice another divisor labeling for shell graphs can found with fan graphs [4]

Example 3.4. The shell graph $C(13,10)$ and its divisor cordial labeling are shown in Fig. 2

Proposition 3.5. All bow graphs are divisor cordial.

PROOF. Let G be a bow graph with two shells of order m and n excluding the apex. Then the number of vertices in G is $p=m+n+1$ and the edges $q=2(m+n-1)$. The apex of the bow graph is denoted by v_0, denote the vertices in the right wing of the bow graph from bottom to top by $v_1, v_2, ..., v_m$, and the vertices in the left wing of the bow graph are denoted from top to bottom by $v_{m+1}, v_{m+2}, ..., v_{m+n}$. Without losing of generality, suppose $m \leq n$.
Define the labeling $f : V \longrightarrow \{1,2,...,m+n+1\}$ by:
$f(v_0)=2$, $f(v_1)=1$ and label the vertices of the wings by the following:

$$\begin{array}{llllll} & 2\cdot 2, & 2\cdot 2^2, & \cdots, & 2\cdot 2^{k_1}, \\ 3, & 3\cdot 2, & 3\cdot 2^2, & \cdots, & 3\cdot 2^{k_2}, \\ 5, & 5\cdot 2, & 5\cdot 2^2, & \cdots, & 5\cdot 2^{k_3}, \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

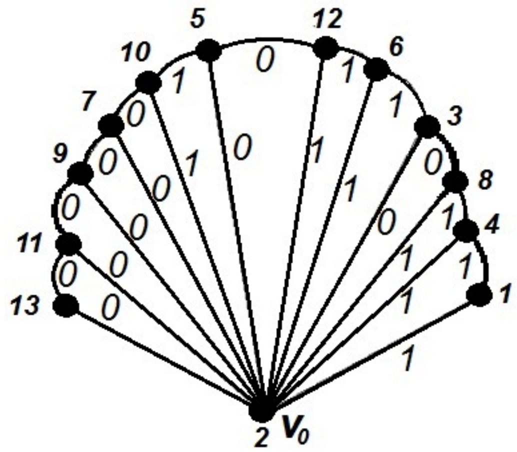

Fig. 2. A shell graph $C(13, 10)$ and its divisor cordial labeling

where $(2m-1)\cdot 2^{k_m} \leq p$ and $m \geq 1, k_m \geq 0$. We observe that $(2m-1)\cdot 2^a$ divides $(2m-1)\cdot 2^b (a < b)$ and $(2m-1)\cdot 2^{k_i}$ does not divide $2m+1$.

Let G' be a graph obtained from the bow graph G by adding the edge $v_m v_{m+1}$.
The graph G' has an odd number of edges and it is a shell graph, then by Proposition 3.3 the graph G' is divisor cordial. The graph $G = G' - v_m v_{m+1}$ with even edges, then G is divisor cordial since:

Case 1: If $m+n$ is even, then $e_f(0) = e_f(1) + 1$ hence the deleted edge $v_m v_{m+1}$ must be labeled 0.

Subcase i: If $f(v_m) = (2t-1)\cdot 2^{k_i}$ for some i, then the deleted edge $v_m v_{m+1}$ is labeled 0.

Subcase ii: If $f(v_m) \neq (2t-1)\cdot 2^{k_i}$ for some i, then we will shift the labels of vertices $v_2, v_3, ..., v_{m+n-l}$ in the wings, by l where l is the smallest integer satisfying $f(v_{m+1}) = (2t-1)\cdot 2^{k_i}$ for some i, and shift the labels of the vertices $v_{m+n-l+1}, v_{m+n-l+2}, ..., v_{m+n}$, by $l+1$ and take it modulo $(m+n+1)$.

Case 2: If $m+n$ is odd, then $e_f(1) = e_f(0) + 1$ hence the deleted edge $v_m v_{m+1}$ must be labeled 1.

Subcase i: If $f(v_m) = (2t-1)\cdot 2^{k_i}$ for some i, then we will shift the labels of vertices $v_2, v_3, ..., v_{m+n-1}$ in the wings, by one step and shift the label of vertex v_{m+n} by two and take it modulo $(m+n+1)$.

Subcase ii: If $f(v_m) \neq (2t-1)\cdot 2^{k_i}$ for some i, then the edge $v_m v_{m+1}$ is labeled 1.

Then the bow graph G with two wings of m and n vertices is a divisor cordial graph for each m and n. □

Example 3.6. The bow graph with two wings of 13 *and* 16 vertices respectively and its divisor cordial labeling are shown in Fig. 3

Butterfly Graphs

Proposition 3.7. The butterfly graphs are divisor cordial.

PROOF. Let G be a butterfly graph with shells of orders m and n excluding the apex then the number of vertices in G is $p = m+n+3$ and the edges $q = 2(m+n)$. The apex of the butterfly graph is denoted as v_0,denote the vertices in the right wing of the butterfly graph from bottom to top as $v_1, v_2, ..., v_m$, the vertices in the left wing of the butterfly graph are denoted from top to bottom as $v_{m+1}, v_{m+2}, ..., v_{m+n}$, and the vertices in the pendant edges are v_{m+n+1}, v_{m+n+2}.
Since the butterfly defined as a bow graph with exactly two pendent edges at the apex,then we define the labeling $f : V \longrightarrow \{1, 2, ..., m+n+3\}$ by:
$f(v_0) = 2$, $f(v_1) = 1$, $f(v_{m+n+1}) = m+n+2$, $f(v_{m+n+2}) = m+n+3$ and labeled the vertices of the

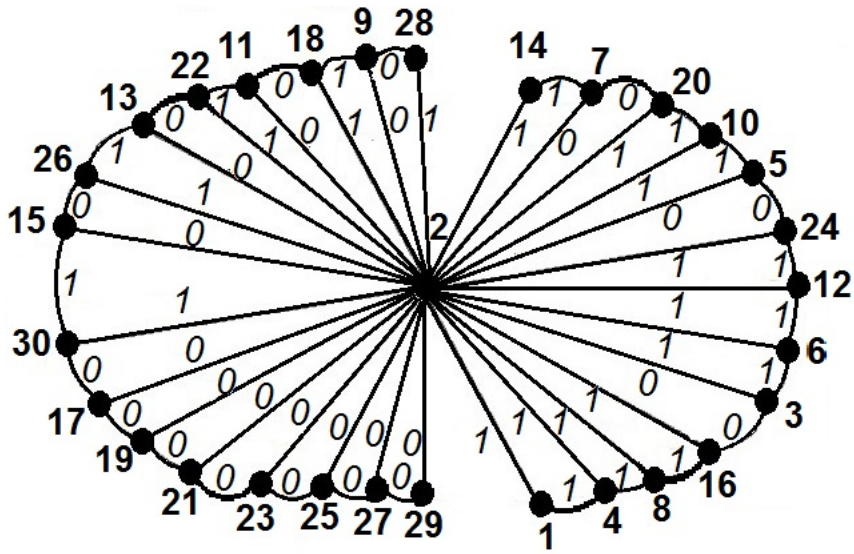

Fig. 3. A bow graph with $m = 13, n = 16$ and its divisor cordial labeling.

wings by the following:

$$\begin{array}{cccccc} & 2\cdot 2, & 2\cdot 2^2, & \cdots, & 2\cdot 2^{k_1}, \\ 3, & 3\cdot 2, & 3\cdot 2^2, & \cdots, & 3\cdot 2^{k_2}, \\ 5, & 5\cdot 2, & 5\cdot 2^2, & \cdots, & 5\cdot 2^{k_3}, \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

where $(2m-1)\cdot 2^{k_m} \leq p$ and $m \geq 1, k_m \geq 0$. We observe that $(2m-1)\cdot 2^a$ divides $(2m-1)\cdot 2^b (a < b)$ and $(2m-1)\cdot 2^{k_i}$ does not divide $2m+1$.
And we make the shift as in Proposition 3.5, for labeling of the vertices in the wings.
Since the only one of the numbers $m+n+2$ or $m+n+3$ must be even then the pendent edges will be labeled 1 and 0, hence the graph G is divisor cordial. □

Example 3.8. The butterfly graph G with two wings having $m = 9, n = 15$ vertices respectively, and its divisor cordial labeling is shown in Fig. 4.

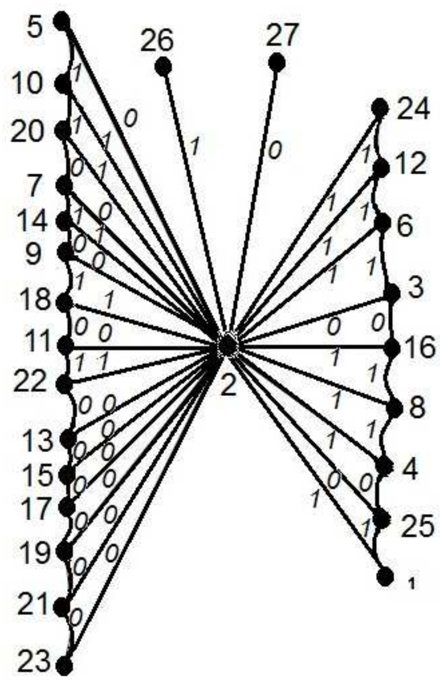

Fig. 4. A divisor cordial labeling for the butterfly with 27 vertices

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