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On the univalence of an integral operator

Dorina Răducanu *

Abstract

In this paper the method of Loewner chains is used to derive a fairly general and flexible univalence criterion for an integral operator. Two examples involving Bessel and hypergeometric functions are given. Our results include a number of known or new univalence criteria.

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1. Introduction

Let $\mathcal{U}_r = \{z \in \mathbb{C} : |z| < r,\ 0 < r \leq 1\}$ be the open disk of radius r centered at the origin and let $\mathcal{U} = \mathcal{U}_1$ be the open unit disk.

Denote by $\mathcal{A}$ the class of analytic functions in $\mathcal{U}$ which satisfy the usual normalization $f(0) = f'(0) - 1 = 0$.

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions.

There are known numerous criteria which ensure that a function $f \in \mathcal{A}$ is in the class $\mathcal{S}$. In Theorem 1.1 some of these criteria are listed.

1.1. Theorem. *Let $f \in \mathcal{A}$. Then, each of the following three conditions implies that $f \in \mathcal{S}$:*

$$(1.1) \qquad (1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \quad z \in \mathcal{U};$$

$$(1.2) \qquad \left|c|z|^2 + (1-|z|^2)\frac{zf''(z)}{f'(z)}\right| \leq 1, \quad z \in \mathcal{U}$$

for some $c \in \mathbb{C}, |c| \leq 1, c \neq -1$;

$$(1.3) \qquad \left||z|^2\left[(c+1)f'(z)e^{-\int_0^z a(\tau)d\tau} - 1\right] + z(1-|z|^2)a(z)\right| \leq 1, \quad z \in \mathcal{U}$$

*Faculty of Mathematics and Computer Science, Transilvania University of Braşov, 50091, Iuliu Maniu 50, Braşov, Romania
Email: draducanu@unitbv.ro

for some $c \in \mathbb{C}, |c| \le 1, c \ne -1$ *and for* $a(z)$ *analytic function in* $\mathcal{U}$.

The univalence criterion given in (1.2) (see [1]) is an extension of Becker's univalence criterion (see [3], [4]) given in (1.1). The univalence criterion (1.3) was obtained by D. Tan (see [19]).

An extension of Becker's criterion, due to N. N. Pascu ensures the univalence of an integral operator.

1.2. Theorem. *([12]) Let* $f \in \mathcal{A}$ *and let* $\alpha \in \mathbb{C}$ *with* $\Re\alpha > 0$. *If*

$$\text{(1.4)} \qquad \frac{1-|z|^{2\Re\alpha}}{\Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \quad z \in \mathcal{U}$$

then, the integral operator

$$\text{(1.5)} \qquad F_\alpha(z) = \left[\alpha\int_0^z \tau^{\alpha-1}f'(\tau)d\tau\right]^{1/\alpha}$$

is analytic and univalent in $\mathcal{U}$.

During the time many authors (see [5], [6], [7], [8], [9], [11], [18], etc.) have obtained numerous and various conditions which guarantee the univalence of a function in the class $\mathcal{A}$ or the univalence of an integral operator.

In this paper we are mainly interested on the integral operator

$$\text{(1.6)} \qquad F_{\alpha,\beta}(z) = \left[\alpha\int_0^z \tau^{\alpha-1}(f'(\tau))^\beta d\tau\right]^{1/\alpha}$$

where the function f belongs to the class $\mathcal{A}$ and the parameters α and β are complex numbers such that the integral exists. Here and in the sequel every many-valued function is taken with the principal branch.

For the integral operator $F_{\alpha,\beta}(z)$ we establish a fairly general and flexible univalence criterion which contains a number of known or new results.

2. Univalence criterion

Before proving our main result we need a brief summary of the theory of Loewner chains.

A function $L(z,t) : \mathcal{U} \times [0,\infty) \to \mathbb{C}$ is said to be a *Loewner chain* or a *subordination chain* if:

(i) $L(z,t)$ is analytic and univalent in $\mathcal{U}$ for all $t \ge 0$.
(ii) $L(z,t) \prec L(z,s)$ for all $0 \le t \le s < \infty$, where the symbol " $\prec$ " stands for subordination.

The following result due to Pommerenke is often used to obtain univalence criteria.

2.1. Theorem. *([15], [16]) Let* $L(z,t) = a_1(t)z + \dots$ *be an analytic function in* $\mathcal{U}_r$ $(0 < r \le 1)$ *for all* $t \ge 0$. *Suppose that:*

(i) $L(z,t)$ *is a locally absolutely continuous function of* $t \in [0,\infty)$, *locally uniform with respect to* $z \in \mathcal{U}_r$.
(ii) $a_1(t)$ *is a complex valued continuous function on* $[0,\infty)$ *such that* $a_1(t) \ne 0$, $\lim_{t\to\infty}|a_1(t)| = \infty$ *and*

$$\left\{\frac{L(z,t)}{a_1(t)}\right\}_{t\ge 0}$$

is a normal family of functions in $\mathcal{U}_r$.

(iii) *There exists an analytic function* $p : \mathcal{U} \times [0,\infty) \to \mathbb{C}$ *satisfying* $\Re p(z,t) > 0$ *for all* $(z,t) \in \mathcal{U} \times [0,\infty)$ *and*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \; z \in \mathcal{U}_r \text{ , a.e } \; t \geq 0. \tag{2.1}$$

Then, for each $t \geq 0$, *the function* $L(z,t)$ *has an analytic and univalent extension to the whole disk* $\mathcal{U}$, *i.e* $L(z,t)$ *is a Loewner chain.*

Our main result contains sufficient conditions for the univalence of the integral operator $F_{\alpha,\beta}(z)$ defined by (1.6).

2.2. Theorem. *Let* $a(z)$ *be an analytic function in* $\mathcal{U}$ *and let* $f \in \mathcal{A}$. *Consider three complex numbers* α, β *and* c *such that* $\Re\alpha > 0, \beta \neq 0$ *and* $|c| \leq 1, c \neq -1$. *Suppose that:*

$$\left|(c+1)(f'(z))^{\beta}e^{-\int_0^z a(\tau)d\tau} - 1\right| \leq 1, \; z \in \mathcal{U} \tag{2.2}$$

and

$$\left||z|^{2\alpha}\left[(c+1)(f'(z))^{\beta}e^{-\int_0^z a(\tau)d\tau} - 1\right] + z\frac{1-|z|^{2\alpha}}{\alpha}a(z)\right| \leq 1, \quad z \in \mathcal{U}\setminus\{0\}. \tag{2.3}$$

Then, the integral operator

$$F_{\alpha,\beta}(z) = \left[\alpha\int_0^z \tau^{\alpha-1}(f'(\tau))^{\beta}d\tau\right]^{1/\alpha}$$

is univalent in $\mathcal{U}$, *i.e. is in the class* $\mathcal{S}$.

Proof. Define the function

$$f_1(z,t) = \alpha\int_0^{e^{-t}z} \tau^{\alpha-1}(f'(\tau))^{\beta}d\tau \quad z \in \mathcal{U}, \; t \geq 0.$$

Since $f \in \mathcal{A}$, $e^{-t}z \in \mathcal{U}$ for all $t \geq 0$ and $z \in \mathcal{U}$, it follows that

$$f_1(z,t) = \left(e^{-t}z\right)^{\alpha} + \sum_{n=2}^{\infty} b_n\left(e^{-t}z\right)^{n+\alpha-1}$$

where $b_n \in \mathbb{C}, n \geq 2$. Consider the function $f_2(z,t)$ such that

$$f_1(z,t) = z^{\alpha}f_2(z,t) \quad z \in \mathcal{U}, \; t \geq 0.$$

It is easy to check that $f_2(z,t)$ is analytic in $\mathcal{U}$ for all $t \geq 0$ and

$$f_2(z,t) = e^{-\alpha t} + \sum_{n=2}^{\infty} b_n e^{-t(n+\alpha-1)}z^{n-1}.$$

Since the function $a(z)$ is analytic in $\mathcal{U}$ it follows that the function $f_3(z,t)$ defined by

$$f_3(z,t) = \left(e^{\alpha t} - e^{-\alpha t}\right)e^{\int_0^{e^{-t}z} a(\tau)d\tau}$$

is analytic in $\mathcal{U}$ for all $t \geq 0$.

Then, the function $f_4(z,t)$ given by

$$f_4(z,t) = f_2(z,t) + \frac{1}{c+1}f_3(z,t) \quad z \in \mathcal{U}, \; t \geq 0$$

is also analytic in $\mathcal{U}$.

We have

$$f_4(0,t) = f_2(0,t) + \frac{1}{c+1} f_3(0,t) = \frac{e^{\alpha t}}{c+1}(1 + ce^{-2\alpha t}).$$

The conditions $\Re\alpha > 0$ and $|c| \le 1, c \ne -1$ yield $f_4(0,t) \ne 0$ for all $t \ge 0$. Thus, there exists an open disk $\mathcal{U}_{r_1}$ $(0 < r_1 \le 1)$ in which $f_4(z,t) \ne 0$ for all $t \ge 0$. Therefore, we can choose an analytic branch of $[f_4(z,t)]^{1/\alpha}$, which will be denoted by $f_5(z,t)$.

Making use of the previous results, we obtain that the function

$$L(z,t) = zf_5(z,t)$$

or

$$L(z,t) = \left[\alpha \int_0^{e^{-t}z} \tau^{\alpha-1}(f'(\tau))^\beta d\tau + \frac{1}{c+1}\left(e^{\alpha t} - e^{-\alpha t}\right) z^\alpha e^{\int_0^{e^{-t}z} a(\tau)d\tau} \right]^{1/\alpha}$$

is analytic in $\mathcal{U}_{r_1}$ for all $t \ge 0$.

We have $L(z,t) = a_1(t)z + \dots$ for $z \in \mathcal{U}_{r_1}$ and $t \ge 0$, where

$$a_1(t) = e^t \left(\frac{1 + ce^{-2\alpha t}}{c+1} \right)^{1/\alpha}, \quad t \ge 0.$$

From $\Re\alpha > 0$ and $|c| \le 1, c \ne -1$ we obtain $a_1(t) \ne 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$.

Let $r_2 \in (0, r_1]$ and let $K = \overline{\{z \in \mathbb{C} : |z| \le r_2\}}$. Since the function $L(z,t)$ is analytic in $\mathcal{U}_{r_1}$, there exists $M > 0$ such that $|L(z,t)| \le Me^t$ for $z \in K$ and $t \ge 0$. Also, for $t \ge 0$, it is easy to see that there exists $N > 0$ such that $|a_1(t)| > Ne^t$. It follows that

$$\left| \frac{L(z,t)}{a_1(t)} \right| \le \frac{M}{N} \text{ ,for } z \in K \text{ and } t \ge 0.$$

Thus, $\{L(z,t)/a_1(t)\}_{t\ge 0}$ is a normal family of functions in $\mathcal{U}_{r_1}$.

Elementary calculations show that $\frac{\partial L}{\partial z}(z,t)$ is analytic in $\mathcal{U}_{r_1}$. It follows that $\left|\frac{\partial L}{\partial z}(z,t)\right|$ is bounded on $[0,T]$ for any fixed $T > 0$ and $z \in \mathcal{U}_{r_3}$ $(0 < r_3 \le r_1)$. Therefore, the function $L(z,t)$ is locally absolutely continuous on $[0,\infty)$ locally uniform with respect to $z \in \mathcal{U}_{r_1}$.

Consider the function $p(z,t)$ defined by

$$p(z,t) = z\frac{\partial L}{\partial z}(z,t) / \frac{\partial L}{\partial t}(z,t).$$

In order to prove that the function $p(z,t)$ has an analytic extension in $\mathcal{U}$ and $\Re p(z,t) > 0$ for all $t \ge 0$, we will show that the function $w(z,t)$ given by

$$w(z,t) = \frac{p(z,t) - 1}{p(z,t) + 1} \quad z \in \mathcal{U}_{r_1},\ t \ge 0$$

has an analytic extension in $\mathcal{U}$ and $|w(z,t)| < 1$, for all $z \in \mathcal{U}$ and $t \ge 0$.

Lengthy but elementary calculations give

$$w(z,t) = e^{-2t\alpha} \left[(c+1)(f'(e^{-t}z))^\beta e^{-\int_0^{e^{-t}z} a(\tau)d\tau} - 1 \right] + \frac{1}{\alpha}(1 - e^{-2t\alpha})e^{-t}za(e^{-t}z).$$

It is easy to check that $w(z,t)$ is an analytic function in $\mathcal{U}$. We have $w(0,t) = ce^{-2t\alpha}$ and thus

$$|w(0,t)| = |c|e^{-2t\Re\alpha} < 1, \text{ for all } t > 0. \tag{2.4}$$

For $t = 0$ we obtain

$$w(z,0) = (c+1)(f'(z))^{\beta} e^{-\int_0^z a(\tau)d\tau} - 1, \quad z \in \mathcal{U}.$$

Inequality (2.2) from the hyphothesis, yields

$$(2.5) \qquad |w(z,0)| < 1 \quad z \in \mathcal{U}.$$

Let $t > 0$ and let $z \neq 0$. Since $|e^{-t}z| \leq e^{-t} < 1$ for all $z \in \bar{\mathcal{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$, it follows that $w(z,t)$ is analytic in $\bar{\mathcal{U}}$. Making use of the maximum modulus principle we obtain that, for each fixed $t > 0$, there exists $\theta \in \mathbb{R}$ such that :

$$|w(z,t)| < \max_{|z|=1} |w(z,t)| = |w(e^{i\theta},t)|.$$

Denote $u = e^{-t}e^{i\theta}$. Then, $|u| = e^{-t}$ and thus

$$|w(e^{i\theta},t)| = \left| |u|^{2\alpha} \left[(c+1)(f'(u))^{\beta} e^{-\int_0^u a(\tau)d\tau} - 1 \right] + \frac{1-|u|^{2\alpha}}{\alpha} u a(u) \right|.$$

Inequality (2.3), from the hyphothesis, shows that

$$(2.6) \qquad |w(e^{i\theta},t)| \leq 1.$$

Combining (2.4), (2.5) and (2.6) we immediately get $|w(z,t)| < 1$ for all $z \in \mathcal{U}$ and $t \geq 0$. Therefore, the function $p(z,t)$ has an analytic extension in $\mathcal{U}$ and $\Re p(z,t) > 0$ for $(z,t) \in \mathcal{U} \times [0,\infty)$.

Since all the conditions of Theorem 2.1 are satisfied we can conclude that the function $L(z,t)$ has an analytic and univalent extension in $\mathcal{U}$ for all $t \geq 0$. For $t = 0$, we have $L(z,0) = F_{\alpha,\beta}(z)$ and thus, the function $F_{\alpha,\beta}(z)$ given by (1.6) is analytic and univalent in $\mathcal{U}$. With this the proof is complete. □

Remark. The univalence condition (1.3) can be derived from Theorem 2.2 for $\alpha = \beta = 1$.

3. Specific univalence criteria

Many new or known univalence criteria can be generated with Theorem 2.2 and specific choiches of the functions $a(z)$ and $f(z)$. In this section some of these univalence criteria are listed.

1. Consider first

$$a(z) = \beta \frac{f''(z)}{f'(z)}, \quad z \in \mathcal{U}, \quad f \in \mathcal{A}.$$

Then, making use of Theorem 2.2 we immediately obtain the following result.

3.1. Theorem. *Let $f \in \mathcal{A}$ and let α, β, c be complex numbers such that $\Re\alpha > 0, \beta \neq 0$ and $|c| \leq 1, c \neq -1$. If*

$$(3.1) \qquad \left| c|z|^{2\alpha} + \frac{\beta}{\alpha}(1-|z|^{2\alpha}) \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad z \in \mathcal{U}$$

then the integral operator $F_{\alpha,\beta}(z)$ defined by (1.6) is in the class $\mathcal{S}$.

Remark.

(i) For $\beta = 1$, Theorem 3.1 reduces to a result obtained by V. Pescar [13].

(ii) Setting $\alpha = \beta = 1$ in Theorem 3.1, we obtain the univalence criterion given in (1.2).

(iii) With $c = 0$ and $\beta = 1$, inequality (3.1) specializes to

$$\left|\frac{1-|z|^{2\alpha}}{\alpha}\frac{zf''(z)}{f'(z)}\right| \le 1, \quad z \in \mathcal{U}. \tag{3.2}$$

Using the next inequality

$$\left|\frac{1-|z|^{2\alpha}}{\alpha}\right| \le \frac{1-|z|^{2\Re\alpha}}{\Re\alpha} \tag{3.3}$$

in (3.2) we get the univalence condition (1.4) which guarantees the univalence of the integral operator $F_\alpha(z)$ given by (1.5).

Let $g_\nu : \mathcal{U} \to \mathbb{C}$ be the normalized Bessel function of the first kind (see [2]) with Taylor expansion

$$g_\nu(z) = z + \sum_{n=1}^{\infty} \frac{(-1)^n z^{n+1}}{4^n n!(\nu+1)\ldots(\nu+n)}.$$

For $\nu = \frac{1}{2}$ we have $g_{\frac{1}{2}}(z) = \sqrt{z}\sin\sqrt{z}$.

The next result follows from Theorem 3.1 with $f(z) = g_\nu(z)$.

3.2. Corollary. *Let $\nu > 0$ and let α, β, c be complex numbers such that $0 < |\beta| \le \frac{2(4\nu^2+9\nu+3)}{4\nu+9}\Re\alpha$ and $|c| \le 1, c \ne -1$. Then the function*

$$F_{\alpha,\beta,\nu}(z) = \left[\alpha \int_0^z \tau^{\alpha-1}(g_\nu'(\tau))^\beta d\tau\right]^{1/\alpha}, \quad z \in \mathcal{U} \tag{3.4}$$

is in the class $\mathcal{S}$. In particular, if $0 < |\beta| \le \frac{17}{11}\Re\alpha$ and $|c| \le 1, c \ne -1$, then the function

$$F_{\alpha,\beta,\frac{1}{2}}(z) = \left[\alpha \int_0^z \tau^{\alpha-1}\left(\frac{\sin\sqrt{\tau} + \sqrt{\tau}\cos\sqrt{\tau}}{2\sqrt{\tau}}\right)^\beta d\tau\right]^{1/\alpha}$$

is in $\mathcal{S}$.

Proof. Replace $f(z) = g_\nu(z)$ in left-hand side of (3.1). Making use of the triangle inequality and (3.3) we have

$$\begin{aligned}
&\left|c|z|^{2\alpha} + \frac{\beta}{\alpha}(1-|z|^{2\alpha})\frac{zf''(z)}{f'(z)}\right| \\
&= \left|c|z|^{2\alpha} + \frac{\beta}{\alpha}(1-|z|^{2\alpha})\frac{zg_\nu''(z)}{g_\nu'(z)}\right| \\
&\le |c||z|^{2\Re\alpha} + \frac{|\beta|}{\Re\alpha}(1-|z|^{2\Re\alpha})\left|\frac{zg_\nu''(z)}{g_\nu'(z)}\right|.
\end{aligned}$$

Since $0 < |\beta| \le \frac{2(4\nu^2+9\nu+3)}{4\nu+9}\Re\alpha, |c| \le 1, c \ne -1$ and making use of

$$\left|\frac{zg_\nu''(z)}{g_\nu'(z)}\right| \le \frac{4\nu+9}{2(4\nu^2+9\nu+3)}, \quad z \in \mathcal{U}, \quad \nu > 0$$

(see [6]), we obtain that

$$\begin{aligned}
&|c||z|^{2\Re\alpha} + \frac{|\beta|}{\Re\alpha}(1-|z|^{2\Re\alpha})\left|\frac{zg_\nu''(z)}{g_\nu'(z)}\right| \\
&\le |z|^{2\Re\alpha} + \frac{|\beta|}{\Re\alpha}(1-|z|^{2\Re\alpha})\frac{4\nu+9}{2(4\nu^2+9\nu+3)} \le |z|^{2\Re\alpha} + 1 - |z|^{2\Re\alpha} = 1.
\end{aligned}$$

It follows that inequality (3.1) holds true and therefore, the function $F_{\alpha,\beta,\nu}(z)$ defined by (3.4) is in $\mathcal{S}$. The particular case follows from the first part by setting $\nu = \dfrac{1}{2}$. □

2. Let $g \in \mathcal{A}$. Choosing

$$f(z) = \int_0^z \frac{g(\tau)}{\tau} d\tau, \quad z \in \mathcal{U}$$

in Theorem 2.2 we obtain easily a univalence criterion for another well known integral operator.

3.3. Theorem. *Let $g \in \mathcal{A}$ and let α, β, c be complex numbers such that $\Re\alpha > 0, \beta \neq 0$ and $|c| \leq 1, c \neq -1$. Suppose that*

$$\left|(c+1)\left(\frac{g(z)}{z}\right)^{\beta} e^{-\int_0^z a(\tau)d\tau} - 1\right| \leq 1, \quad z \in \mathcal{U}$$

and

$$\left| |z|^{2\alpha}\left[(c+1)\left(\frac{g(z)}{z}\right)^{\beta} e^{-\int_0^z a(\tau)d\tau} - 1\right] + \frac{1-|z|^{2\alpha}}{\alpha} za(z)\right| \leq 1, \quad z \in \mathcal{U} \setminus \{0\}.$$

Then the integral operator

$$G_{\alpha,\beta}(z) = \left[\alpha \int_0^z \tau^{\alpha-1}\left(\frac{g(\tau)}{\tau}\right)^{\beta} d\tau\right]^{1/\alpha}, \quad z \in \mathcal{U} \tag{3.5}$$

is in the class $\mathcal{S}$.

3. Consider $a(z)$ defined by

$$a(z) = \beta\left(\frac{g'(z)}{g(z)} - \frac{1}{z}\right), \quad z \in \mathcal{U}, \quad g \in \mathcal{A}.$$

Then, making use of Theorem 3.2 we get the following result.

3.4. Corollary. *Let $g \in \mathcal{A}$ and let $\alpha, \beta, c \in \mathbb{C}$ with $\Re\alpha > 0, \beta \neq 0$ and $|c| \leq 1, c \neq -1$. If*

$$\left|c|z|^{2\alpha} + \frac{\beta}{\alpha}(1-|z|^{2\alpha})\left(\frac{zg'(z)}{g(z)} - 1\right)\right| \leq 1, \quad z \in \mathcal{U} \tag{3.6}$$

then the function $G_{\alpha,\beta}(z)$ defined by (3.5) is in the class $\mathcal{S}$.

Suppose that the function g in Corollary 3.2 is in $\mathcal{S}$. Then we have the following result which shows that the integral operator $G_{\alpha,\beta}(z)$ preserves univalency.

3.5. Corollary. *Let $g \in \mathcal{S}$ and let $\alpha, \beta, c \in \mathbb{C}$ with $c \neq -1, 0 < |\beta| \leq \min\left\{\frac{\Re\alpha}{2}, \frac{1}{4}\right\}$ and $\Re\alpha > 0$. If*

$$|c| \leq \begin{cases} 1 - \dfrac{2|\beta|}{\Re\alpha}, & \Re\alpha \in (0, \dfrac{1}{2}) \\ 1 - 4|\beta|, & \Re\alpha \in [\dfrac{1}{2}, \infty) \end{cases} \tag{3.7}$$

then the function $G_{\alpha,\beta}(z)$ is in $\mathcal{S}$.

Proof. Making use of the triangle inequality in left-hand side of (3.6) we obtain

$$\left|c|z|^{2\alpha} + \frac{\beta}{\alpha}(1-|z|^{2\alpha})\left(\frac{zg'(z)}{g(z)} - 1\right)\right|$$
$$\leq |c||z|^{2\Re\alpha} + \frac{|\beta|}{\Re\alpha}(1-|z|^{2\Re\alpha})\left[\left|\frac{zg'(z)}{g(z)}\right| + 1\right].$$

Let $g \in \mathcal{S}$. Then

$$\left|\frac{zg'(z)}{g(z)}\right| \leq \frac{1+|z|}{1-|z|}, \quad z \in \mathcal{U}.$$

It follows that

$$\left|c|z|^{2\alpha} + \frac{\beta}{\alpha}(1-|z|^{2\alpha})\left(\frac{zg'(z)}{g(z)} - 1\right)\right| \leq |c| + \frac{2|\beta|}{\Re\alpha}\frac{1-|z|^{2\Re\alpha}}{1-|z|}. \tag{3.8}$$

Denote $x = |z|$ and $a = \Re\alpha$. Consider the function $\phi : [0,1) \to \mathbb{R}$ defined by

$$\phi(x) = \frac{1-x^{2a}}{1-x}.$$

It is easy to check that

$$\phi(x) \leq \begin{cases} 1, & a \in (0, \frac{1}{2}) \\ 2a, & a \in [\frac{1}{2}, \infty). \end{cases} \tag{3.9}$$

Combining (3.8) and (3.9) we have

$$\left|c|z|^{2\alpha} + \frac{\beta}{\alpha}(1-|z|^{2\alpha})\left(\frac{zg'(z)}{g(z)} - 1\right)\right| \leq \begin{cases} |c| + \frac{2|\beta|}{\Re\alpha}, & \Re\alpha \in (0, \frac{1}{2}) \\ |c| + 4|\beta|, & \Re\alpha \in [\frac{1}{2}, \infty) \end{cases}$$

Inequality (3.7) from hypothesis shows that the condition (3.6) is satisfied and thus, making use of Corollary 3.2 we obtain that the function $G_{\alpha,\beta}(z)$ is in $\mathcal{S}$. With this the proof is complete. □

3.6. Corollary. *Let $\alpha, \beta, c \in \mathbb{C}$ with $c \neq -1, 0 < |\beta| \leq \min\left\{\frac{\Re\alpha}{2}, \frac{1}{4}\right\}, \Re\alpha > 0$. If inequality (3.7) holds true, then the function $K_{\alpha,\beta}(z) = z\left[{}_2F_1(\alpha, 2\beta; 1+\alpha; z)\right]^{1/\alpha}$ is in the class $\mathcal{S}$. The symbol ${}_2F_1(a,b;c;z)$ denotes the well known hypergeometric function.*

Proof. The Koebe function $k(z) = \dfrac{z}{(1-z)^2}$ is in $\mathcal{S}$. Applying Corollary 3.3 we obtain that the function

$$K_{\alpha,\beta}(z) := \left[\alpha \int_0^z \tau^{\alpha-1}\left(\frac{k(\tau)}{\tau}\right)^{\beta} d\tau\right]^{1/\alpha} = \left[\alpha \int_0^z \tau^{\alpha-1}(1-\tau)^{-2\beta}\, d\tau\right]^{1/\alpha}$$

is also in $\mathcal{S}$. With the substitution $\tau = uz$ the function $K_{\alpha,\beta}(z)$ becomes

$$K_{\alpha,\beta}(z) = z\left[\alpha \int_0^1 u^{\alpha-1}(1-uz)^{-2\beta} du\right]^{1/\alpha} = z\left[{}_2F_1(\alpha, 2\beta; 1+\alpha; z)\right]^{1/\alpha}.$$

Thus, the proof is completed. □

Remark. Similar results with the one given in Corollary 3.3 can be found in [9], [14].

4. Let $g_1, \ldots, g_m \in \mathcal{A}$ and $\delta_1, \ldots, \delta_m \in \mathbb{C} \setminus \{0\}$. Setting

$$f(z) = \int_0^z \prod_{k=1}^{m}\left(\frac{g_k(\tau)}{\tau}\right)^{\frac{\delta_k}{\beta}} d\tau$$

in Theorem 2.2 or Theorem 3.1 we can easily obtain various univalence criteria for the integral operator

$$G_{\delta_1,\ldots,\delta_m}(z) = \left[\alpha \int_0^z \tau^{\alpha-1} \prod_{k=1}^{m} \left(\frac{g_k(\tau)}{\tau}\right)^{\delta_k} d\tau\right]^{1/\alpha}$$

which has been studied by many authors (see [2], [5], [6], [8], [18], etc.)

From the previous examples, it is clear that one can generate many univalence criteria with Theorem 2.2 and suitable choices of the functions $a(z)$ and $f(z)$.

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