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Some New Results on Absolute Summability Factors

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Abstract

In this paper, we establish the general summability factor theorems related to generalized absolute Cesàro summability $|C,\alpha,\beta|_k$ and absolute factorable matrix summability $\left|A_f,\varphi_n\right|_k$ methods for $k\geq 1$, $\alpha+\beta>-1$, where (φ_n) is arbitrary sequence of positive real constants and $A_f=(a_{nv})$ is a factorable matrix such that $a_{nv}=\hat{a}_n a_v$ for $0\leq v\leq n$, $a_{nv}=0$ for $v>n$, $(\hat{a}_n)$ and (a_n) are any sequences of real numbers. Also, absolute factorable summability method includes all absolute Riesz summability and absolute weighted summability methods in the special cases. Therefore, not only some well known results but also several new results for absolute Cesàro and weighted means are obtained as corollaries.

Keywords: Absolute Cesàro summability, Factorable matrix, Matrix methods, Sequence spaces, Summability factors.

1. Introduction

Let $\sum x_n$ be a given infinite series with sequence of partial sums (s_n) and $A=(a_{nv})$ be an infinite matrix of complex numbers. By $A(s)=\big(A_n(s)\big)$, we denote the A-transform of the sequence $s=(s_n)$, i.e.,

$$A_n(s)=\sum_{v=0}^{\infty}a_{nv}s_v$$

which converges for $n\geq 0$.
The nth $(\bar{N},p_n)$ weighted mean of the sequence (s_n) is given by

$$T_n=\frac{1}{P_n}\sum_{v=0}^{n}p_v s_v,$$

where (p_n) is a sequence of positive real constants such that $P_n=\sum_{v=0}^{n}p_v\to\infty$ as $n\to\infty$ $(P_{-1}=p_{-1}=0)$. Let (φ_n) be any sequence of positive real constants. Then the series $\sum x_n$ is said to be summable $|\bar{N},p_n,\varphi_n|_k$, $k\geq 1$, if (see [1])

$$\sum_{n=1}^{\infty}(\varphi_n)^{k-1}\,|T_n-T_{n-1}|^k<\infty. \tag{1.1}$$

Note that $|\bar{N},p_n,P_n/p_n|_k=|\bar{N},p_n|_k$ and $|\bar{N},p_n,n|_k=|R,p_n|_k$, which are defined by Bor and Sarıgöl in [2,3].

Taking account of

$$T_n-T_{n-1}=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}x_v$$

the relation (1.1) can be stated as

$$\sum_{n=1}^{\infty}(\varphi_n)^{k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}x_v\right|^k<\infty. \tag{1.2}$$

An appropriate extension of (1.2) to a factorable matrix would be as follows [4]. Let $A_f=(a_{nv})$ denote the factorable matrix defined by

$$a_{nv}=\begin{cases}\hat{a}_n a_v, & 0\leq v\leq n,\\ 0, & v>n,\end{cases}$$

where $(\hat{a}_n)$ and (a_n) are any sequences of real numbers. Then the series $\sum x_n$ is said to be summable $\left|A_f,\varphi_n\right|_k$, $k\geq 1$, if (see [4])

$$\sum_{n=1}^{\infty}(\varphi_n)^{k-1}\left|\hat{a}_n\sum_{v=1}^{n}a_v x_v\right|^k<\infty.$$

If we take $\hat{a}_n=\frac{p_n}{P_nP_{n-1}}$ and $a_v=P_{v-1}$, then $\left|A_f,\varphi_n\right|_k$ summability is equivalent to $|\bar{N},p_n,\varphi_n|_k$ summability.

Borwein [5] has introduced the nth generalized Cesàro mean (C,α,β) of order (α,β) with $\alpha+\beta>-1$, of the sequence (s_n) by

$$\sigma_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}s_v,$$

where $A_n^{\alpha+\beta}=O\left(n^{\alpha+\beta}\right), \alpha+\beta>-1,\ A_0^{\alpha+\beta}=1,$
$A_n^{\alpha}=\frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!}$ $and\ A_{-n}^{\alpha+\beta}=0, n\geq 1.$

Obviously, $(C,\alpha,0)$ is the same as (C,α) whereas $(C,0,\beta)$ is $(C,0)$.

We write $\tau_n^{\alpha,\beta}$ as the (C,α,β) transform of the sequence (nx_n), i.e.,

$$\tau_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}vx_v.$$

Then, the series $\sum x_n$ is said to be summable $|C,\alpha,\beta|_k$, $k\geq 1$, for $\alpha+\beta>-1$, if (see [6])

$$\sum_{n=1}^{\infty}\frac{1}{n}\left|\tau_n^{\alpha,\beta}\right|^k<\infty.$$

The summability $|C,\alpha,\beta|_k$ includes all Cesàro methods in the special cases. For example, if we take $\beta=0$, $\alpha=0$ and $\alpha=1$, then the summability $|C,\alpha,\beta|_k$ reduces to $|C,\alpha|_k$ defined by Flett in [7], to $|C,0|_k$ and the absolute Riesz summability $|R,p_n|_k$ with $p_n=A_n^{\beta}$ for $\beta\geq 0$ [3].

Throughout this paper, k^* denotes the conjugate of $k>1$, i.e., $1/k+1/k^*=1$, and $1/k^*=0$ for $k=1$. Let X and Y be two summability methods. If $\sum\varepsilon_n x_n$ is summable by the method Y whenever $\sum x_n$ is summable by the method X, then we say that the sequence $\varepsilon=(\varepsilon_n)$ is a summability factor of type (X,Y) and we write $\varepsilon\in(X,Y)$. Also, note that if $\varepsilon=1$, then $1\in(X,Y)$ means the comparisons of these methods, where $1=(1,1,\ldots)$, i.e., $X\subset Y$.

Absolute summability factors and comparison of the methods related to $|\bar{N},p_n|_k$ and $|C,\alpha|_k$ were widely studied by many authors [8-12]. We refer the reader to [11-13] for the most recent work in this topic. Also the Cesàro series spaces have been defined as the set of all series summable by absolute Cesàro summability methods in [14-16].

2. Results and Discussion

The aim of this paper is to characterize the sets $(|C,\alpha,\beta|,\left|A_f,\varphi_n\right|_k)$, $k\geq 1$ and $(\left|A_f,\varphi_n\right|_k,|C,\alpha,\beta|)$, $k>1$ for $\alpha+\beta>-1$. As a direct consequence of these results, we also obtain various new results as corollaries.

We use the following lemmas to prove our results.

Lemma 2.1. Let $1<k<\infty$. Then, $A(x)\in\ell$ whenever $x\in\ell_k$ if and only if

$$\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}|a_{nv}|\right)^{k^*}<\infty$$

where $\ell_k=\{x=(x_v):\sum_v|x_v|^k<\infty\}$, $\ell_1=\ell$, [17].

Lemma 2.2. Let $1\leq k<\infty$. Then, $A(x)\in\ell_k$ whenever $x\in\ell$ if and only if

$$\sup_v\sum_{n=0}^{\infty}|a_{nv}|^k<\infty,$$

[18].

Lemma 2.3. Let $\mu>-1$, $1\leq k<\infty$ and $\lambda<\mu$. Then, for $k=1$,

$$E_v=\begin{cases}O(v^{-\mu-1}), & \lambda\leq -1\\ O\left(v^{-\mu+\lambda}\right), & \lambda>-1\end{cases}$$

and

$$E_v=\begin{cases}O(v^{-k\mu-1}), & \lambda<-1/k\\ O(v^{-k\mu-1}logv), & \lambda=-1/k\\ O\left(v^{-k\mu+k\lambda}\right), & \lambda>-1/k\end{cases}$$

for $1<k<\infty$, where $E_v=\sum_{n=v}^{\infty}\frac{\left|A_{n-v}^{\lambda}\right|^k}{n\left(A_n^{\mu}\right)^k}$ for $v\geq 1$, [9].

Now, we are ready to prove the main theorems.

Theorem 2.4. Let $k\geq 1$ and $\alpha+\beta>-1$. Then the necessary and sufficient condition for $\varepsilon\in(|C,\alpha,\beta|,\left|A_f,\varphi_n\right|_k)$ is that

$$\sup_r\left\{\sum_{n=r}^{\infty}\left|\varphi_n^{1/k^*}\hat{a}_n rA_r^{\alpha+\beta}\sum_{v=r}^{n}\frac{a_v\varepsilon_vA_{v-r}^{-\alpha-1}}{vA_v^{\beta}}\right|^k\right\}<\infty. \quad (2.1)$$

Proof. Let $\tau_n^{\alpha,\beta}$ be the nth (C,α,β) mean of the sequence (nx_n) and define the sequence (y_n) by

$$y_n=\frac{\tau_n^{\alpha,\beta}}{n}=\frac{1}{nA_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}vx_v,$$
$$n\geq 1\quad and\quad y_0=x_0. \quad (2.2)$$

So, $\sum x_n$ is summable $|C,\alpha,\beta|$ iff $y=(y_n)\in\ell$. Also, by inversion of (2.2), we have for $n\geq 1$

$$x_n=\frac{1}{nA_n^{\beta}}\sum_{v=1}^{n}A_{n-v}^{-\alpha-1}vA_v^{\alpha+\beta}y_v. \quad (2.3)$$

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Using definition of factorable matrix A_f, we define the sequence $(\tilde{y}_n)$ by

$$\tilde{y}_n = \varphi_n^{1/k^*}\hat{a}_n \sum_{v=1}^{n} a_v x_v \varepsilon_v \ , \qquad \tilde{y}_0 = \varepsilon_0 x_0.$$

This gives us that $\sum \varepsilon_n x_n$ is summable $\left|A_f, \varphi_n\right|_k$ iff $\tilde{y} = (\tilde{y}_n) \in \ell_k$.

Hence, in view of (2.3), we get for $n \geq 1$,

$$\begin{aligned}\tilde{y}_n &= \varphi_n^{1/k^*}\hat{a}_n \sum_{v=1}^{n} a_v \varepsilon_v x_v \\ &= \varphi_n^{1/k^*}\hat{a}_n \sum_{v=1}^{n} a_v \varepsilon_v \frac{1}{vA_v^{\beta}} \sum_{r=1}^{v} A_{v-r}^{-\alpha-1} r A_r^{\alpha+\beta} y_r \\ &= \varphi_n^{1/k^*}\hat{a}_n \sum_{r=1}^{n} \left(rA_r^{\alpha+\beta} \sum_{v=r}^{n} \frac{a_v \varepsilon_v A_{v-r}^{-\alpha-1}}{vA_v^{\beta}} \right) y_r \\ &= \sum_{r=1}^{n} d_{nr}\, y_r\end{aligned}$$

where

$$d_{nr} = \begin{cases} \varphi_n^{1/k^*}\hat{a}_n r A_r^{\alpha+\beta} \sum_{v=r}^{n} \frac{a_v \varepsilon_v A_{v-r}^{-\alpha-1}}{vA_v^{\beta}} , & 1 \leq r \leq n \\ 0, & r > n. \end{cases}$$

Then, $\sum \varepsilon_n x_n$ is summable $\left|A_f, \varphi_n\right|_k$ whenever $\sum x_n$ is summable $|C, \alpha, \beta|$ if and only if $\tilde{y} \in \ell_k$ whenever $y \in \ell$. Hence using Lemma 2.2, we obtain that $\varepsilon \in \left(|C, \alpha, \beta|, \left|A_f, \varphi_n\right|_k\right)$ if and only if

$$\sup_r \left\{ \sum_{n=r}^{\infty} \left| \varphi_n^{1/k^*}\hat{a}_n r A_r^{\alpha+\beta} \sum_{v=r}^{n} \frac{a_v \varepsilon_v A_{v-r}^{-\alpha-1}}{vA_v^{\beta}} \right|^k \right\} < \infty$$

which completes the proof.

Theorem 2.5. Let $k > 1$, $\alpha + \beta > -1$ and $\beta > -1$. Then the necessary and sufficient condition for $\varepsilon \in \left(\left|A_f, \varphi_n\right|_k, |C, \alpha, \beta|\right)$ is that

$$\sum_{v=1}^{\infty} \left(\sum_{n=v}^{\infty} \left| \frac{1}{nA_n^{\alpha+\beta} \varphi_v^{1/k^*} \hat{a}_v} \Omega_{nv} \right| \right)^{k^*} < \infty, \qquad (2.4)$$

where $\Omega = (\Omega_{nv})$ is defined by

$$\Omega_{nv} = \begin{cases} \frac{A_{n-v}^{\alpha-1} A_v^{\beta} v \varepsilon_v}{a_v} - \frac{A_{n-v-1}^{\alpha-1} A_{v+1}^{\beta} (v+1) \varepsilon_{v+1}}{a_{v+1}}, & 1 \leq v \leq n, \\ 0, & v > n. \end{cases}$$

Proof. Let $(\tilde{y}_n)$ denote the sequence defined by

$$\tilde{y}_n = \varphi_n^{1/k^*}\hat{a}_n \sum_{v=1}^{n} a_v x_v \, , n \geq 1, and\ \tilde{y}_0 = x_0. \qquad (2.5)$$

So, we can write that $\sum x_n$ is summable $\left|A_f, \varphi_n\right|_k$ iff $\tilde{y} = (\tilde{y}_n) \in \ell_k$. By inversion of (2.5), we obtain for $n \geq 1$,

$$x_n = \frac{1}{a_n} \left(\frac{\tilde{y}_n}{\varphi_n^{1/k^*}\hat{a}_n} - \frac{\tilde{y}_{n-1}}{\varphi_{n-1}^{1/k^*}\hat{a}_{n-1}} \right). \qquad (2.6)$$

Also let $\left(u_n^{\alpha,\beta}\right)$ be the nth (C, α, β) mean of the sequence $(nx_n\varepsilon_n)$, i.e.,

$$u_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v \varepsilon_v x_v.$$

If we define $y = (y_n)$ by

$$y_n = \frac{u_n^{\alpha,\beta}}{n} = \frac{1}{nA_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v \varepsilon_v x_v,$$

then, we say that $\sum \varepsilon_n x_n$ is summable $|C, \alpha, \beta|$ iff $y = (y_n) \in \ell$. Hence, by virtue of the (2.6), we get for $n \geq 1$,

$$\begin{aligned} y_n &= \frac{1}{nA_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v \varepsilon_v x_v \\ &= \frac{1}{nA_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v \varepsilon_v \frac{1}{a_v} \left(\frac{\tilde{y}_v}{\varphi_v^{1/k^*}\hat{a}_v} - \frac{\tilde{y}_{v-1}}{\varphi_{v-1}^{1/k^*}\hat{a}_{v-1}} \right) \\ &= \frac{1}{nA_n^{\alpha+\beta}} \left(\sum_{v=1}^{n} \frac{A_{n-v}^{\alpha-1} A_v^{\beta} v \varepsilon_v \tilde{y}_v}{a_v \varphi_v^{1/k^*}\hat{a}_v} - \sum_{v=0}^{n-1} \frac{A_{n-v-1}^{\alpha-1} A_{v+1}^{\beta} (v+1) \varepsilon_{v+1} \tilde{y}_v}{a_{v+1} \varphi_v^{1/k^*}\hat{a}_v} \right) \\ &= -\frac{A_{n-1}^{\alpha-1} A_1^{\beta} \varepsilon_1 \tilde{y}_0}{nA_n^{\alpha+\beta} a_1 \varphi_0^{1/k^*}\hat{a}_0} + \frac{1}{nA_n^{\alpha+\beta}} \sum_{v=1}^{n} \left(\frac{A_{n-v}^{\alpha-1} A_v^{\beta} v \varepsilon_v}{a_v} - \frac{A_{n-v-1}^{\alpha-1} A_{v+1}^{\beta} (v+1) \varepsilon_{v+1}}{a_{v+1}} \right) \frac{\tilde{y}_v}{\varphi_v^{1/k^*}\hat{a}_v} = \sum_{v=0}^{n} d_{nv} \tilde{y}_v \end{aligned}$$

where $D = (d_{nv})$ is defined by

$$d_{nv} = \begin{cases} -\frac{A_{n-1}^{\alpha-1} A_1^{\beta} \varepsilon_1}{nA_n^{\alpha+\beta} a_1 \varphi_0^{1/k^*}\hat{a}_0}, & v = 0, n \geq 1, \\ \frac{1}{nA_n^{\alpha+\beta} \varphi_v^{1/k^*}\hat{a}_v} \Omega_{nv} , & 1 \leq v \leq n \\ 0, & v > n, \end{cases}$$

and $\Omega = (\Omega_{nv})$ is as in Theorem 2.5.

Then, $\sum \varepsilon_n x_n$ is summable $|C,\alpha,\beta|$ whenever $\sum x_n$ is summable $\left|A_f,\varphi_n\right|_k$ if and only if $y \in \ell$ whenever $\tilde{y} \in \ell_k$. Hence in view of Lemma 2.1, we obtain that $\varepsilon \in \left(\left|A_f,\varphi_n\right|_k, |C,\alpha,\beta|\right)$ if and only if

$$\sum_{v=0}^{\infty}\left(\sum_{n=v}^{\infty}|d_{nv}|\right)^{k^*} < \infty$$

which gives that

$$\left(\sum_{n=1}^{\infty}|d_{n0}|\right)^{k^*} + \sum_{v=1}^{\infty}\left(\sum_{n=v}^{\infty}|d_{nv}|\right)^{k^*} = \left(\sum_{n=1}^{\infty}\left|\frac{A_{n-1}^{\alpha-1}A_1^{\beta}\varepsilon_1}{nA_n^{\alpha+\beta}a_1\varphi_0^{1/k^*}\hat{a}_0}\right|\right)^{k^*}$$

$$+\sum_{v=1}^{\infty}\left(\sum_{n=v}^{\infty}\left|\frac{1}{nA_n^{\alpha+\beta}\varphi_v^{1/k^*}\hat{a}_v}\left(\frac{A_{n-v}^{\alpha-1}A_v^{\beta}v\varepsilon_v}{a_v} - \frac{A_{n-v-1}^{\alpha-1}A_{v+1}^{\beta}(v+1)\varepsilon_{v+1}}{a_{v+1}}\right)\right|\right)^{k^*} < \infty.$$

Since $\sum_{n=1}^{\infty}\left|\frac{A_{n-1}^{\alpha-1}}{nA_n^{\alpha+\beta}}\right| < \infty$ from Lemma 2.3 , we get that (2.4) holds, which completes the proof.

3. Conclusion

Our results have several consequences depending on $\alpha, \beta, (\hat{a}_n)$ and (a_n) .

If we consider the special case $\varepsilon = 1$ in the Theorem 2.4 and Theorem 2.5, we have following results dealing with comparison of summability fields of methods $|C,\alpha,\beta|$ and $\left|A_f,\varphi_n\right|_k$.

Corollary 3.1. Let $k \geq 1$ and $\alpha+\beta > -1$. Then, $|C,\alpha,\beta| \subset \left|A_f,\varphi_n\right|_k$ if and only if

$$\sup_r\left\{\sum_{n=r}^{\infty}\left|\varphi_n^{1/k^*}\hat{a}_n r A_r^{\alpha+\beta}\sum_{v=r}^{n}\frac{a_v A_{v-r}^{-\alpha-1}}{vA_v^{\beta}}\right|^k\right\} < \infty.$$

Corollary 3.2. Let $k > 1$, $\alpha+\beta > -1$ and $\beta > -1$. Then $\left|A_f,\varphi_n\right|_k \subset |C,\alpha,\beta|$ if and only if

$$\sum_{v=1}^{\infty}\left(\sum_{n=v}^{\infty}\left|\frac{1}{nA_n^{\alpha+\beta}\varphi_v^{1/k^*}\hat{a}_v}\left(\frac{A_{n-v}^{\alpha-1}A_v^{\beta}v}{a_v} - \frac{A_{n-v-1}^{\alpha-1}A_{v+1}^{\beta}(v+1)}{a_{v+1}}\right)\right|\right)^{k^*} < \infty.$$

Taking $\hat{a}_n = \frac{p_n}{P_nP_{n-1}}$, $a_v = P_{v-1}$ in the Theorem 2.4 and Theorem 2.5, we get the following results, respectively.

Corollary 3.3. Let $k \geq 1$ and $\alpha+\beta > -1$. Then the necessary and sufficient condition for $\varepsilon \in (|C,\alpha,\beta|, |\bar{N},p_n,\varphi_n|_k)$ is that

$$\sup_r\left\{\sum_{n=r}^{\infty}\left|\varphi_n^{1/k^*}\frac{p_n}{P_nP_{n-1}}rA_r^{\alpha+\beta}\sum_{v=r}^{n}\frac{P_{v-1}\varepsilon_v A_{v-r}^{-\alpha-1}}{vA_v^{\beta}}\right|^k\right\} < \infty.$$

Corollary 3.4. Let $k > 1$, $\alpha+\beta > -1$ and $\beta > -1$. Then the necessary and sufficient condition for $\varepsilon \in (|\bar{N},p_n,\varphi_n|_k, |C,\alpha,\beta|)$ is that

$$\sum_{v=1}^{\infty}\left(\sum_{n=v}^{\infty}\left|\frac{P_vP_{v-1}}{nA_n^{\alpha+\beta}\varphi_v^{1/k^*}p_v}\left(\frac{A_{n-v}^{\alpha-1}A_v^{\beta}v\varepsilon_v}{P_{v-1}} - \frac{A_{n-v-1}^{\alpha-1}A_{v+1}^{\beta}(v+1)\varepsilon_{v+1}}{P_v}\right)\right|\right)^{k^*} < \infty.$$

If we take $\beta = 0$, Theorem 2.4 and Theorem 2.5 reduce to the next results, respectively.

Corollary 3.5. Let $k \geq 1$ and $\alpha > -1$. Then the necessary and sufficient condition for $\varepsilon \in (|C,\alpha|, \left|A_f,\varphi_n\right|_k)$ is that

$$\sup_r\left\{\sum_{n=r}^{\infty}\left|\varphi_n^{1/k^*}\hat{a}_n r A_r^{\alpha}\sum_{v=r}^{n}\frac{a_v\varepsilon_v A_{v-r}^{-\alpha-1}}{v}\right|^k\right\} < \infty.$$

Corollary 3.6. Let $k > 1$ and $\alpha > -1$. Then the necessary and sufficient condition for $\varepsilon \in (\left|A_f,\varphi_n\right|_k, |C,\alpha|)$ is that

$$\sum_{v=1}^{\infty}\left(\sum_{n=v}^{\infty}\left|\frac{1}{nA_n^{\alpha}\varphi_v^{1/k^*}\hat{a}_v}\left(\frac{A_{n-v}^{\alpha-1}v\varepsilon_v}{a_v} - \frac{A_{n-v-1}^{\alpha-1}(v+1)\varepsilon_{v+1}}{a_{v+1}}\right)\right|\right)^{k^*} < \infty.$$

Also, taking $\hat{a}_n = \frac{p_n}{P_nP_{n-1}}$, $a_v = P_{v-1}$ in the Corollary 3.5. and Corollary 3.6, we have:

Corollary 3.7. Let $k \geq 1$ and $\alpha > -1$. Then the necessary and sufficient condition for $\varepsilon \in (|C,\alpha|, |\bar{N}, p_n, \varphi_n|_k)$ is that

$$\sup_r \left\{ \sum_{n=r}^{\infty} \left| \varphi_n^{1/k^*} \frac{p_n}{P_n P_{n-1}} r A_r^{\alpha} \sum_{v=r}^{n} \frac{P_{v-1}\varepsilon_v A_{v-r}^{-\alpha-1}}{v} \right|^k \right\} < \infty.$$

Corollary 3.8. Let $k > 1$ and $\alpha > -1$. Then the necessary and sufficient condition for $\varepsilon \in (|\bar{N}, p_n, \varphi_n|_k, |C,\alpha|)$ is that

$$\sum_{v=1}^{\infty} \left(\sum_{n=v}^{\infty} \left| \frac{P_v P_{v-1}}{n A_n^{\alpha} \varphi_v^{1/k^*} p_v} \left(\frac{A_{n-v}^{\alpha-1} v \varepsilon_v}{P_{v-1}} - \frac{A_{n-v-1}^{\alpha-1}(v+1)\varepsilon_{v+1}}{P_v} \right) \right| \right)^{k^*} < \infty.$$

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Ethics

There are no ethical issues after the publication of this manuscript.

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