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Bounds on the Path Energy

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Abstract − In this paper, the path energy is investigated for path matrix. Some bounds are explored for the path energy in terms of the eigenvalues and vertices. Also, some relations are obtained for tree connected graphs.

Keywords − *Path matrix, path energy*

1. Introduction

The path matrix is a popular matrix in graph theory, recently and it had started to develop in 2016. The path matrix of a graph G is defined as a real and symmetric matrix whose (i,j)-entry is the maximum number of internally disjoint paths between the vertices v_i and v_j when $i \neq j$ and is zero when $i = j$. Its eigenvalues are real and they are called path eigenvalues of G. The spectral radius of $P(G)$ is represented by $\rho = \rho(G)$. The concept of path matrix deals with vertices whose mathematical properties are reported in [1].

The path energy is described as the sum of the absolute values of path eigenvalues and it is denoted by $PE = PE(G)$. For several positive eigenvalues of order n, $PE(G) \geq 2(n-1)$. If G is a k-connected tree graph then $\rho(G) \geq k(n-1) \geq k^2$. Also, $PE(G) \geq 2\rho(G)$ for the spectral radius $\rho(G)$. The survey of properties of path energy is given in [2], [3].

The purpose of this paper is to examine different bounds for path energy in terms of defining relations. These bounds are important for they can be used in many areas of graph theory. Considering these cases, known and related results are given in second section. Then, main bounds are obtained using the vertices, the edges and the eigenvalues for path energy in the third section. These bounds are sharp.

2. Preliminaries

In order to prove the main results, some lemmas are needed:

Lemma 2.1. [4] If $a_1, a_2, ..., a_n \in R$ and $0 < m \leq a_i \leq M$ then,

$$(\frac{1}{n}\sum_{i=1}^{n} a_i)(\sum_{i=1}^{n}\frac{1}{a_i}) \leq \frac{M+n}{4Mn}$$

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Lemma 2.2. [5] Let $p = (p_i)$ and $a = (a_i)$, $i = 1, 2, ..., n$ be real sequences with $p_1 + p_2 + ... + p_n = 1$ and $r \leq a_i \leq R$. For such sequences,

$$0 \leq \sum_{i=1}^{n} p_i(a_i)^2 - (\sum_{i=1}^{n} p_i a_i)^2 \leq \frac{1}{2}(R - r) \sum_{i=1}^{n} p_i |a_i - \sum_{j=1}^{n} p_j a_j|$$

See [6], [7] for details.

3. MAIN RESULTS

In this section, some relations and bounds for energy of path matrix are established. These sharp results are surveyed with some fixed parameters. In addition, a relation is determined for tree connected graphs under the assumption of Lemma 2.2.

Theorem 3.1. Let G be a connected graph with eigenvalues of path matix; $\lambda_1^P, \lambda_2^P, ..., \lambda_n^P$. Then,

$$PE(G) \leq \sqrt{\frac{n^2}{4(n-1)}(\rho - \eta)}$$

where $\eta = \eta(G) = |\lambda_n^P|$.

PROOF. Let $a_i = |\lambda_i^P|$, $b_i = 1$. Assume that all the path eigenvalues of G are non-zero. A classical lemma (the Ozeki's inequality) refered in the article [8] implies that

$$n \sum_{i=1}^{n} |\lambda_i^P|^2 - (\sum_{i=1}^{n} |\lambda_i^P|)^2 \leq \frac{n^2}{4}(\rho - \eta)$$

That is;

$$(\sum_{i=1}^{n} |\lambda_i^P|)^2 \leq \frac{n}{4}(\rho - \eta) + \frac{(PE(G))^2}{n}$$

By the arrangements, the above ineguality transforms into

$$\frac{(n-1)}{n}(PE(G))^2 \leq \frac{n}{4}(\rho - \eta)$$

Consequently,

$$PE(G) \leq \sqrt{\frac{n^2(\rho - \eta)}{4(n-1)}}$$

□

Theorem 3.2. Let G be a connected graph consisting n vertices. Then,

$$PE(G) \leq \sqrt{\frac{n(\rho + \eta)}{4\rho\eta}}$$

where $\eta = \eta(G) = |\lambda_n^P|$.

PROOF. Let $a_i = |\lambda_i^P|$, $m = \eta$, $M = \rho$. By the Lemma 2.1, the following inequality gives that

$$\begin{aligned}(\frac{1}{n} \sum_{i=1}^{n} |\lambda_i^P|)(\sum_{i=1}^{n} \frac{1}{|\lambda_i^P|}) \quad &\leq \quad \frac{1}{n}(\sum_{i=1}^{n} |\lambda_i^P|)^2 \\ &= \frac{1}{n}(PE(G))^2\end{aligned}$$

On the other hand,

$$\frac{1}{n}(PE(G))^2 \leq \frac{\rho+\eta}{4\rho\eta}$$

Thus, the proof is completed with

$$PE(G) \leq \sqrt{\frac{n(\rho+\eta)}{4\rho\eta}}$$

□

Theorem 3.3. If G is a connected graph and G has n vertices, then

$$PE(G) \geq \sqrt{4(\rho(G))^2 + n(n-1)|[P(G)]|^{\frac{2}{n}}}$$

where $|[P(G)]|$ is the determinant of $[P(G)]$.

PROOF. By the Arithmetic-Geometric Mean inequality, the definition of path energy turns into

$$\begin{aligned}(PE(G))^2 &= \sum_{i=1}^{n}|\lambda_i^P|^2 + 2\sum_{1\leq i<j\leq n}|\lambda_i^P||\lambda_j^P| \\ &\geq \sum_{i=1}^{n}|\lambda_i^P|^2 + n(n-1)(\prod_{i=1}^{n}|\lambda_i^P|)^{\frac{2}{n}}\end{aligned}$$

Since $\sum_{i=1}^{n}|\lambda_i^P|^2 \geq (PE(G))^2$, then $\sum_{i=1}^{n}|\lambda_i^P|^2 \geq (PE(G))^2 \geq 4(\rho(G))^2$. Hence, the inequality gives that

$$(PE(G))^2 \geq 4(\rho(G))^2 + n(n-1)|[P(G)]|^{\frac{2}{n}}$$

Thus,

$$PE(G) \geq \sqrt{4(\rho(G))^2 + n(n-1)|[P(G)]|^{\frac{2}{n}}}$$

□

Corollary 3.4. Let G be a k-connected tree graph with n vertices. Then,

$$PE(G) \geq \sqrt{(n-1)[4k^2(n-1) + n|[P(G)]|^{\frac{2}{n}}]}$$

PROOF. As noted in [1], $\rho(G) \geq k(n-1)$ in this case. Therefore the corollary is clear. □

Corollary 3.5. Let G be a k-connected tree graph of order n. Then

i)

$$PE(G^C) \geq \sqrt{(n-1)[4k^2(n-1) + n|[P(G)]|^{\frac{2}{n}}]}$$

where G^C is the complement of G.

ii)

$$\rho(G^{'}) \geq \sqrt{(n-2)[4k^2(n-2) + (n-1)|[P(G)]|^{\frac{2}{n-1}}]}$$

where $G^{'}$ is formed from G by deleting edge ij.

Theorem 3.6. Let G be a connected graph of order n then,

$$PE(G) \leq m + \sqrt{2mn}$$

PROOF. Minkowski inequality gives that

$$(\sum_{i=1}^{n}(|\lambda_i^P|+1)^2)^{\frac{1}{2}} \leq (\sum_{i=1}^{n}|\lambda_i^P|^2)^{\frac{1}{2}} + (\sum_{i=1}^{n}1)^{\frac{1}{2}}$$

By the help of Bernoulli inequality, it is stated that

$$(n+2PE(G))^{\frac{1}{2}} \leq (\sum_{i=1}^{n}|\lambda_i^P|^2)^{\frac{1}{2}} + n^{\frac{1}{2}}$$

Since $\sum_{i=1}^{n}(\lambda_i^P)^2 = 2m$, then

$$n + 2PE(G) \leq (\sqrt{2m}+\sqrt{n})^2$$

Hence,

$$PE(G) \leq m + \sqrt{2mn}$$

□

Corollary 3.7. Let G be a connected graph with n vertices and m edges. Then, $PE(G^C) \leq \dfrac{n(n-1)-2m}{2} + \sqrt{(n(n-1)-2m)n}$ where G^C is the complement of G.

PROOF. By the Theorem 3.6, $PE(G^C) \leq m^C + \sqrt{2m^C n}$. Since $2(m+m^C = n(n-1))$, then $PE(G^C) \leq \dfrac{n(n-1)}{2} - m + \sqrt{2(\dfrac{n(n-1)}{2} - m)n}$. Hence, $PE(G^C) \leq \dfrac{n(n-1)-2m}{2} + \sqrt{(n(n-1)-2m)n}$.

□

Theorem 3.8. Let G be a connected tree graph of order n. Then,

$$PE(G) \geq \frac{4mn - 8(n-1)^2}{n(\rho-\eta)} + 2\rho$$

where $\eta = \eta(G) = |\lambda_n^P|$.

PROOF. Let $p_i = \frac{1}{n}$, $a_i = |\lambda_i^P|$, $r = \eta$, $R = \rho$. Lemma 2.2 implies that

$$\frac{1}{n}\sum_{i=1}^{n}|\lambda_i^P|^2 - (\sum_{i=1}^{n}\frac{1}{n}|\lambda_i^P|)^2 \leq \frac{1}{2}(\rho-\eta)(\sum_{i=1}^{n}\frac{1}{n}||\lambda_i^P| - \frac{1}{n}\sum_{j=1}^{n}|\lambda_j^P||)$$

This requires

$$\frac{2m}{n} - \frac{1}{n^2}(PE(G))^2 \leq \frac{1}{2}(\rho-\eta)(\frac{1}{n}PE(G) - 2n\frac{1}{n^2}\rho)$$

Since $PE(G) \geq 2(n-1)$, we have

$$\frac{2m}{n} - \frac{4(n-1)^2}{n^2} \leq \frac{(\rho-\eta)}{2n}(PE(G)) - \frac{\rho(\rho-\eta)}{n}$$

Hence,

$$PE(G) \geq \frac{4mn - 8(n-1)^2}{n(\rho-\eta)} + 2\rho$$

□

4. Conclusion

In this paper, the path energy is studied using the path matrix. Different bounds are obtained for the path energy with some fixed parameters.

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