On M-injective and M-projective Modules

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Abstract
A left $R$-module $M$ is called max-injective (or m-injective for short) if for any maximal left ideal $I$, any homomorphism $f : I \to M$ can be extended to $g : R \to M$, if and only if $\text{Ext}_R^1(R/I, M) = 0$ for any maximal left ideal $I$. A left $R$-module $M$ is called max-projective (or m-projective for short) if $\text{Ext}_R^1(M, N) = 0$ for any max-injective left $R$-module $N$. We prove that every left $R$-module has a special m-projective precover and a special m-injective preenvelope. We characterize $C$-rings, SF rings and max-hereditary rings using m-projective and m-injective modules.

Keywords: M-injective modules; m-projective modules; max-hereditary rings.

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Throughout, $R$ will denote an associative ring with identity, and modules will be unital $R$-modules, unless otherwise stated. As usual, we denote by $\text{Mod} - R$ the category of right $R$-modules. For a module $M$, $E(M)$, $M^+$ denote the the injective hull and the character module $\text{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$ of $M$, respectively.

Given a class $\mathcal{C}$ of right $R$-modules, we will denote by $\mathcal{C}^\perp = \{ X : \text{Ext}_R^1(C, X) = 0 \text{ for all } C \in \mathcal{C} \}$ the right orthogonal class of $\mathcal{C}$, and by $^\perp \mathcal{C} = \{ X : \text{Ext}_R^1(X, C) = 0 \text{ for all } C \in \mathcal{C} \}$ the left orthogonal class of $\mathcal{C}$. Let $M$ be a right $R$-module. A homomorphism $\phi : M \to F$ with $F \in \mathcal{C}$ is called a $\mathcal{C}$-preenvelope of $M$ ([2]) if for any homomorphism $f : M \to G$ with $G \in \mathcal{C}$, there is a homomorphism $g : F \to G$ such that $g\phi = f$. Moreover, if the only such $g$ are automorphisms of $F$ when $F = G$ and $f = \phi$, the $\mathcal{C}$-preenvelope is called a $\mathcal{C}$-envelope of $M$. Following [2], a monomorphism $\alpha : M \to C$ with $C \in \mathcal{C}$ is said to be a special $\mathcal{C}$-preenvelope of $M$ if $\text{coker}(\alpha) \in ^\perp \mathcal{C}$. Dually, we have the definitions of a (special) $\mathcal{C}$-precover and a $\mathcal{C}$-cover. $\mathcal{C}$-envelopes ($\mathcal{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism. A pair $(\mathfrak{g}, C)$ of classes of left $R$-modules is called a cotorsion theory ([2]) if $\mathfrak{g}^\perp = \mathcal{C}$ and $^\perp \mathcal{C} = \mathfrak{g}$. A cotorsion theory $(\mathfrak{g}, C)$ is called perfect (complete) if every left $R$-module has a $\mathfrak{g}$-envelope and an $\mathfrak{g}$-cover (a special $\mathcal{C}$-preenvelope and a special $\mathfrak{g}$-precover). A cotorsion theory $(\mathfrak{g}, C)$ is said to be hereditary ([3]) if whenever $0 \to L' \to L \to L'' \to 0$ is exact with $L, L'' \in \mathfrak{g}$, then $L'$ is also in $\mathfrak{g}$. By [3], $(\mathfrak{g}, C)$ is hereditary if and only if $0 \to C' \to C \to C'' \to 0$ is exact with $C, C' \in C$, then $C''$ is also in $\mathcal{C}$.

A left $R$-module $M$ is called max-injective (or m-injective for short) if for any maximal left ideal $I$, any homomorphism $f : I \to M$ can be extended to $g : R \to M$, if and only if $\text{Ext}_R^1(R/I, M) = 0$ for any maximal left ideal $I$. A ring $R$ is said to be left m-injective if $R$ is m-injective as a left $R$-module [10]. m-injective modules have been studied further in [11]. In this paper, the concept of max-projective (or m-projective for short) modules is introduced. A left $R$-module $M$ is said to be m-projective if $\text{Ext}_R^1(M, N) = 0$ for any m-injective left $R$-module $N$. In what follows, $m - \text{pr}$ (resp. $m - \text{in}$) stands for the class of all m-projective (resp. all m-injective) left $R$-modules. We prove that $(m - \text{pr}, m - \text{in})$ is a complete cotorsion theory. We prove that $R$ is a left $C$-ring (i.e. for every essential left ideal $I$ of $R$, $R/I$ has a simple submodule) if and only if every cyclic left $R$-module is m-projective if and only if every m-injective left $R$-module is injective if and only if $(m - \text{pr}, m - \text{in})$ is hereditary, and every m-injective left $R$-module is m-projective. We also prove that $R$ is left max-hereditary (i.e. if every maximal left ideal is projective) if and only if every quotient of an m-injective left $R$-module is m-injective if and only if every m-projective left $R$-module has projective dimension at most 1. It is also shown that, $R$ is a left SF-ring if and only if every left $R$-module is m-injective if and only if every cotorsion left $R$-module is m-injective if and only if $(m - \text{pr}, m - \text{in})$ is hereditary and
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m-projective. Since over a left max-coherent ring, FP-injective modules are m-injective, m-projective modules are
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Example 1.1. Noetherian rings are max-coherent. But, left coherent rings need not be max-coherent.
Following [7], a ring R is said to be left max-coherent if every maximal left ideal is finitely presented. Obviously, Noetherian rings are max-coherent. But, left coherent rings need not be max-coherent.

Exercise 1.1. Obviously, any projective module is m-projective. By the definition, any simple left R-module is m-projective. Since over a left max-coherent ring, FP-injective modules are m-injective, m-projective modules are FP-projective. (a left R-module M is called FP-injective (resp. FP-projective) provided that Ext^1_R(F, M) = 0 (resp. Ext^1_R(M, F) = 0) for any finitely presented (resp. FP-injective) left R-module F).

Proposition 1.1. The following are equivalent for a left m-injective left max-coherent ring R and a left R-module M.

1. M is m-projective.
2. M is projective with respect to every exact sequence 0 → K → T → L → 0 with K is m-injective.
3. For every exact sequence 0 → A → E → M → 0, where E is m-projective, A → E is an m-injective preenvelope of A.
4. M is a cokernel of an m-injective preenvelope A → E with E projective.

Proof. (1) ⇒ (2) and (1) ⇒ (3) are trivial.
(2) ⇒ (1) Let N be an m-injective left R-module. There exists an exact sequence 0 → N → E → L → 0 with E injective. This induces an exact sequence Hom(M, E) → Hom(M, L) → Ext^1_R(M, N) → 0. Since Hom(M, E) → Hom(M, L) is exact by (2), Ext^1_R(M, N). So M is m-projective.
(3) ⇒ (4) Let 0 → A → E → M → 0 be an exact sequence with E projective. Since R is left max-coherent left m-projective ring, E is m-projective by [11, Proposition 2.4(2)]. Thus, A → E is an m-injective preenvelope.
(4) ⇒ (1) Let M be a cokernel of an m-injective preenvelope A → E with E projective. Then, there is an exact 0 → A → E → M → 0. For each m-injective left R-module N, Hom(E, N) → Hom(A, N) → Ext^1_R(M, N) → 0. Note that Hom(E, N) → Hom(A, N) is epic by (4). Thus Ext^1_R(M, N), and so M is m-projective.

Now we have the following Lemma.

Lemma 1.1. Let R be a ring. (m − pr, m − in) is a complete cotorsion theory.

Proof. (3) Let C be the set of representatives of simple left R-modules. Thus m − in = C^⊥. Since m − pr = +(C⊥), the result follows from [2, Definition 7.1.5] and [4, Theorem 10].

A ring R is said to be a left C-ring if for every essential left ideal I of R, R/I has a simple submodule. Right perfect rings, left semiartinian rings are well known examples of left C-rings ([1, 10.10]).

Corollary 1.1. Let R be a ring. Then the following are equivalent.

1. R is a left C-ring.
2. Every left R-module is m-projective.
3. Every cyclic left R-module is m-projective.
4. Every m-injective left R-module is injective.
5. (m − pr, m − in) is hereditary, and every m-injective left R-module is m-projective.

In this case, if R is left max-coherent, R is left Noetherian.
we characterize left max-hereditary rings over a left max-coherent ring. Hence there is an epimorphism. By Lemma 1.1, for any left R-module M, there is a short exact sequence 0 \to M \to F \to L \to 0, where F is m-injective and L is m-projective. So (2) follows from (5).

In this case, if R is a left max-coherent ring, every FP-injective left R-module is m-injective, and so injective. This means R is left Noetherian by [6, Theorem 3].

**Corollary 1.2.** Let R be a left max-coherent ring. Then the following are equivalent.

1. Every m-injective left R-module is FP-injective.
2. Every finitely presented left R-module is m-projective.

In this case, R is a left coherent ring.

*Proof.* (1) ⇔ (2) follow from Lemma 1.1, since every module has a special m – pr-precover and a special m – in-preenvelope.

To prove the last statement, let M be an FP-injective left R-module with N a pure submodule, then M/N is m-injective by [11, Proposition 2.6] since R is left max-coherent. Therefore M/N is FP-injective by (1), and hence R is a left coherent ring by [7, Theorem 3.7].

A ring R will be called left max-hereditary if every maximal left ideal is projective. Recall that a ring R is said to be left PP if every principal left ideal of R is projective. Then any left PP-ring with every maximal left ideal principal is left max-hereditary. Now we have the following characterizations of left max-hereditary rings.

**Proposition 1.2.** Let R be a ring. The following are equivalent.

1. R is left max-hereditary.
2. Every quotient of an m-injective left R-module is m-injective.
3. Every m-projective left R-module has projective dimension at most 1.

*Proof.* (1) ⇒ (2) Let M be an m-injective left R-module and N a submodule of M. We shall show that M/N is m-injective. To this end, let I be a maximal left ideal of R and i : I \to R the inclusion and π : M \to M/N the canonical map. For any f : I \to M/N, then there exists g : I \to M such that πg = f since I is projective by (1). Hence there is h : R \to M such that hi = g since M is m-injective. It follows that (πh)i = f, and so M/N is m-injective.

(2) ⇒ (3) Let M be an m-projective left R-module and N a left R-module, then there is a short exact sequence 0 \to N \to E \to L \to 0 with E injective. Note that L is m-projective by (2), and so we have the exact sequence 0 = Ext_R^1(M, L) \to Ext_R^2(M, N) \to Ext_R^2(M, E) = 0. Thus Ext_R^2(M, N) = 0 and hence M has projective dimension at most 1.

(3) ⇒ (1) holds since every simple left R-module is m-projective.

A left R-module M is said to be MI-injective [11] if Ext_R^1(N, M) = 0 for any m-injective left R-module N. Next we characterize left max-hereditary rings over a left max-coherent ring.

**Theorem 1.1.** Let R be a left max-coherent ring. The following are equivalent.

1. R is left max-hereditary.
2. Every MI-injective left R-module is injective.
3. (m – pr, m – in) is hereditary, and every m-projective left R-module has a monic m-injective cover.

*Proof.* (1) ⇒ (2) is clear by [11, Proposition 3.4].

(2) ⇒ (1) Let N be a quotient of an m-injective left R-module M. Suppose f : F \to N is a m-injective cover of N by [11, Remark 2.10(1)]. Then there exists a homomorphism h : M \to F such that fh = π, where π : M \to N. Hence f is an epimorphism. By [11, Remark 3.2(1)], ker(f) is MI-injective, and so it is injective by (2). So, N is m-injective.
Let \(0\) for any left \(R\)-module. We have to prove that \(M/N\) is \(m\)-injective. In fact, there exists an exact sequence \(0 \to N \to E \xrightarrow{\pi} L \to 0\) with \(E\) is \(m\)-injective and \(L\) is \(m\)-projective by Lemma 1.1. Since \(L\) has a monic \(m\)-injective cover \(\phi : F \to L\) by (3), there is \(\alpha : E \to F\) such that \(\pi = \phi \alpha\). Thus \(\phi\) is epic, and hence it is an isomorphism. So \(L\) is \(m\)-injective. For any simple left \(R\)-module \(S\), we have the exact sequence \(0 = \text{Ext}_R^1(S, L) \to \text{Ext}_R^1(S, N) \to \text{Ext}_R^1(S, E)\). Note that \(\text{Ext}_R^1(S, E) = 0\) by [3, Proposition 1.2] since \((m - \text{pr}, m - \text{in})\) is hereditary, and hence \(\text{Ext}_R^1(S, N) = 0\). On the other hand, the short exact sequence \(0 \to N \to M \to M/N \to 0\) induces the exactness of the sequence \(0 = \text{Ext}_R^1(S, M) \to \text{Ext}_R^1(S, M/N) \to \text{Ext}_R^1(S, N) = 0\). Therefore \(\text{Ext}_R^1(S, M/N) = 0\), as desired.

Finally, we give some new characterizations of left SF-rings. Recall that a ring \(R\) is called a left SF-ring [8] if each simple left \(R\)-module is flat.

**Theorem 1.2.** Let \(R\) be a ring. The following are equivalent.

1. \(R\) is a left SF-ring.
2. Every left \(R\)-module is \(m\)-injective.
3. Every \(m\)-projective left \(R\)-module is projective.
4. Every cotorsion left \(R\)-module is \(m\)-injective.
5. \((m - \text{pr}, m - \text{in})\) is hereditary and every \(m\)-projective left \(R\)-module is \(m\)-injective.

**Proof.** (2) \(\Rightarrow\) (4) and (2) \(\Rightarrow\) (5) are trivial.

(4) \(\Rightarrow\) (1) Let \(S\) be any simple left \(R\)-module and \(M\) be a right \(R\)-module. Since \(M^+\) is pure-injective and hence cotorsion, \(M^+\) is \(m\)-injective by (4). So \(0 = \text{Ext}_R^1(S, M^+) \cong \text{Tor}_R^1(M, S)^+\). Thus, \(\text{Tor}_R^1(M, S) = 0\) for any right \(R\)-module \(M\). Hence \(S\) is flat, and so \(R\) is a left SF-ring.

(1) \(\Rightarrow\) (2) Let \(M\) be a left \(R\)-module. Then for any simple left \(R\)-module \(S\), \(0 = \text{Tor}_R^1(M^+, S) \cong \text{Ext}_R^1(S, M)^+\). So \(\text{Ext}_R^1(S, M) = 0\). Hence \(M\) is \(m\)-injective.

(2) \(\Rightarrow\) (3) Let \(M\) be any \(m\)-projective left \(R\)-module. Since every left \(R\)-module is \(m\)-injective by (2), \(\text{Ext}_R^1(M, N) = 0\) for any left \(R\)-module \(N\). Hence \(M\) is projective.

(3) \(\Rightarrow\) (2) Let \(M\) be a left \(R\)-module. There exists an exact sequence \(0 \to M \to E \to P \to 0\) with \(E\) \(m\)-injective and \(P\) \(m\)-projective by Lemma 1.1. By (3), \(P\) is projective, and so \(M\) is \(m\)-injective.

(5) \(\Rightarrow\) (2) Let \(M\) be any left \(R\)-module. By Lemma 1.1, there is a short exact sequence \(0 \to K \to F \to M \to 0\) with \(F\) \(m\)-projective and \(K\) \(m\)-injective. Then \(F\) is \(m\)-injective and hence \(M\) is \(m\)-injective by (5).

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**References**


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