MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES
8 (1) 46-50 (2020) ©MSAEN
HTTPS://DOI.ORG/10.36753/MATHENOT.630031

On M-injective and M-projective Modules

Yusuf Alagöz*

Abstract
A left R-module M is called max-injective (or m-injective for short) if for any maximal left ideal I, any homomorphism $f : I \to M$ can be extended to $g : R \to M$, if and only if $Ext^1_R(R/I, M) = 0$ for any maximal left ideal I. A left R-module M is called max-projective (or m-projective for short) if $Ext^1_R(M, N) = 0$ for any max-injective left R-module N. We prove that every left R-module has a special m-projective precover and a special m-injective preenvelope. We characterize C-rings, SF rings and max-hereditary rings using m-projective and m-injective modules.

Keywords: *M-injective modules; m-projective modules; max-hereditary rings.*

AMS Subject Classification (2020): *Primary: 16D40 ; Secondary: 18G25.*

**Corresponding author*

Throughout, R will denote an associative ring with identity, and modules will be unital R-modules, unless otherwise stated. As usual, we denote by Mod$-R$ the category of right R-modules. For a module M, $E(M)$, M^+ denote the the injective hull and the character module $Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M, respectively.

Given a class $\mathfrak{C}$ of right R-modules, we will denote by $\mathfrak{C}^{\perp} = \{X : Ext^1_R(C, X) = 0$ for all $C \in \mathfrak{C}\}$ the right orthogonal class of $\mathfrak{C}$, and by ${}^{\perp}\mathfrak{C} = \{X : Ext^1_R(X, C) = 0$ for all $C \in \mathfrak{C}\}$ the left orthogonal class of $\mathfrak{C}$. Let M be a right R-module. A homomorphism $\phi : M \to F$ with $F \in \mathfrak{C}$ is called a $\mathfrak{C}$-preenvelope of M ([2]) if for any homomorphism $f : M \to G$ with $G \in \mathfrak{C}$, there is a homomorphism $g : F \to G$ such that $g\phi = f$. Moreover, if the only such g are automorphisms of F when $F = G$ and $f = \phi$, the $\mathfrak{C}$-preenvelope is called a $\mathfrak{C}$-envelope of M. Following [2], a monomorphism $\alpha : M \to C$ with $C \in \mathfrak{C}$ is said to be a special $\mathfrak{C}$-preenvelope of M if $coker(\alpha) \in {}^{\perp}\mathfrak{C}$. Dually, we have the definitions of a (special) $\mathfrak{C}$-precover and a $\mathfrak{C}$-cover. $\mathfrak{C}$-envelopes ($\mathfrak{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism. A pair $(\mathfrak{F}, \mathfrak{C})$ of classes of left R-modules is called a cotorsion theory ([2]) if $\mathfrak{F}^{\perp} = \mathfrak{C}$ and ${}^{\perp}\mathfrak{C} = \mathfrak{F}$. A cotorsion theory $(\mathfrak{F}, \mathfrak{C})$ is called perfect (complete) if every left R-module has a $\mathfrak{C}$-envelope and an $\mathfrak{F}$-cover (a special $\mathfrak{C}$-preenvelope and a special $\mathfrak{F}$-precover). A cotorsion theory $(\mathfrak{F}, \mathfrak{C})$ is said to be hereditary ([3]) if whenever $0 \to L' \to L \to L'' \to 0$ is exact with $L, L'' \in \mathfrak{F}$, then L' is also in $\mathfrak{F}$. By [3], $(\mathfrak{F}, \mathfrak{C})$ is hereditary if and only if $0 \to C' \to C \to C'' \to 0$ is exact with $C, C' \in \mathfrak{C}$, then C'' is also in $\mathfrak{C}$.

A left R-module M is called max-injective (or m-injective for short) if for any maximal left ideal I, any homomorphism $f : I \to M$ can be extended to $g : R \to M$, if and only if $Ext^1_R(R/I, M) = 0$ for any maximal left ideal I. A ring R is said to be left m-injective if R is m-injective as a left R-module [10]. m-injective modules have been studied further in [11]. In this paper, the concept of max-projective (or m-projective for short) modules is introduced. A left R-module M is said to be m-projective if $Ext^1_R(M, N) = 0$ for any m-injective left R-module N. In what follows, $\mathfrak{m} - \mathfrak{pr}$ (resp. $\mathfrak{m} - \mathfrak{in}$) stands for the class of all m-projective (resp. all m-injective) left R-modules. We prove that $(\mathfrak{m} - \mathfrak{pr}, \mathfrak{m} - \mathfrak{in})$ is a complete cotorsion theory. We prove that R is a left C-ring (i.e. for every essential left ideal I of R, R/I has a simple submodule) if and only if every cyclic left R-module is m-projective if and only if every m-injective left R-module is injective if and only if $(\mathfrak{m} - \mathfrak{pr}, \mathfrak{m} - \mathfrak{in})$ is hereditary, and every m-injective left R-module is m-projective. We also prove that R is left max-hereditary (i.e. if every maximal left ideal is projective) if and only if every quotient of an m-injective left R-module is m-injective if and only if every m-projective left R-module has projective dimension at most 1. It is also shown that, R is a left SF-ring if and only if every left R-module is m-injective if and only if every cotorsion left R-module is m-injective if and only if $(\mathfrak{m} - \mathfrak{pr}, \mathfrak{m} - \mathfrak{in})$ is hereditary and

Received : *07–10–2019,* ***Accepted :*** *14–02–2020*

every m-projective left R-module is m-injective.

1. M-projective Modules

In this section several properties and characterization of m-projective modules are given. First we recall the definition.

Definition 1.1. A left R-module M is said to be *m-projective* if $Ext^1_R(M,N)=0$ for any m-injective left R-module N. The right version can be defined similarly.

Recall that a ring R is called a *left coherent* ring if every finitely generated left ideal of R is finitely presented. Following [11], a ring R is said to be *left max-coherent* if every maximal left ideal is finitely presented. Obviously, Noetherian rings are max-coherent. But, left coherent rings need not be max-coherent.

Example 1.1. Obviously, any projective module is m-projective. By the definition, any simple left R-module is m-projective. Since over a left max-coherent ring, FP-injective modules are m-injective, m-projective modules are FP-projective. (a left R-module M is called FP-injective (resp. FP-projective) provided that $Ext^1_R(F,M)=0$ (resp. $Ext^1_R(M,F)=0$) for any finitely presented (resp. FP-injective) left R-module F).

Proposition 1.1. *The following are equivalent for a left m-injective left max-coherent ring R and a left R-module M.*

1. *M is m-projective.*
2. *M is projective with respect to every exact sequence $0\to K\to T\to L\to 0$ with K is m-injective.*
3. *For every exact sequence $0\to A\to E\to M\to 0$, where E is m-projective, $A\to E$ is an m-injective preenvelope of A.*
4. *M is a cokernel of an m-injective preenvelope $A\to E$ with E projective.*

Proof. $(1)\Rightarrow(2)$ and $(1)\Rightarrow(3)$ are trivial.

$(2)\Rightarrow(1)$ Let N be an m-injective left R-module. There exists an exact sequence $0\to N\to E\to L\to 0$ with E injective. This induces an exact sequence $Hom(M,E)\to Hom(M,L)\to Ext^1_R(M,N)\to 0$. Since $Hom(M,E)\to Hom(M,L)$ is exact by (2), $Ext^1_R(M,N)$. So M is m-projective.

$(3)\Rightarrow(4)$ Let $0\to A\to E\to M\to 0$ be an exact sequence with E projective. Since R is left max-coherent left m-injective ring, E is m-injective by [11, Proposition 2.4(2)]. Thus, $A\to E$ is an m-injective preenvelope.

$(4)\Rightarrow(1)$ Let M be a cokernel of an m-injective preenvelope $A\to E$ with E projective. Then, there is an exact $0\to A\to E\to M\to 0$. For each m-injective left R-module N, $Hom(E,N)\to Hom(A,N)\to Ext^1_R(M,N)\to 0$. Note that $Hom(E,N)\to Hom(A,N)$ is epic by (4). Thus $Ext^1_R(M,N)$, and so M is m-projective. □

Now we have the following Lemma.

Lemma 1.1. *Let R be a ring. $(\mathfrak{m}-\mathfrak{pr},\mathfrak{m}-\mathfrak{in})$ is a complete cotorsion theory.*

Proof. (3) Let $\mathfrak{C}$ be the set of representatives of simple left R-modules. Thus $\mathfrak{m}-\mathfrak{in}=\mathfrak{C}^{\perp}$. Since $\mathfrak{m}-\mathfrak{pr}={}^{\perp}(\mathfrak{C}^{\perp})$, the result follows from [2, Definition 7.1.5] and [4, Theorem 10]. □

A ring R is said to be a left C-ring if for every essential left ideal I of R, R/I has a simple submodule. Right perfect rings, left semiartinian rings are well known examples of left C-rings ([1, 10.10]).

Corollary 1.1. *Let R be a ring. Then the following are equivalent.*

1. *R is a left C-ring.*
2. *Every left R-module is m-projective.*
3. *Every cyclic left R-module is m-projective.*
4. *Every m-injective left R-module is injective.*
5. *$(\mathfrak{m}-\mathfrak{pr},\mathfrak{m}-\mathfrak{in})$ is hereditary, and every m-injective left R-module is m-projective.*

In this case, if R is left max-coherent, R is left Noetherian.

Proof. $(1) \Leftrightarrow (4)$ follows from [9, Lemma 4].

$(2) \Rightarrow (3)$ and $(2) \Rightarrow (5)$ are clear. $(4) \Rightarrow (2)$ holds by Lemma 1.1.

$(3) \Rightarrow (4)$ Let M be any m-injective left R-module and I any left ideal of R. Then $Ext^1_R(R/I, M) = 0$ by (3). Thus M is injective, as desired.

$(5) \Rightarrow (2)$ By Lemma 1.1, for any left R-module M, there is a short exact sequence $0 \to M \to F \to L \to 0$, where F is m-injective and L is m-projective. So (2) follows from (5).

In this case, if R is a left max-coherent ring, every FP-injective left R-module is m-injective, and so injective. This means R is left Noetherian by [6, Theorem 3]. □

Corollary 1.2. *Let R be a left max-coherent ring. Then the following are equivalent.*

1. *Every m-injective left R-module is FP-injective.*
2. *Every finitely presented left R-module is m-projective.*

In this case, R is a left coherent ring.

Proof. $(1) \Leftrightarrow (2)$ follow from Lemma 1.1, since every module has a special $\mathfrak{m}-\mathfrak{pr}$-precover and a special $\mathfrak{m}-\mathfrak{in}$-preenvelope.

To prove the last statement, let M be an FP-injective left R-module with N a pure submodule, then M/N is m-injective by [11, Proposition 2.6] since R is left max-coherent. Therefore M/N is FP-injective by (1), and hence R is a left coherent ring by [7, Theorem 3.7]. □

A ring R will be called left max-hereditary if every maximal left ideal is projective. Recall that a ring R is said to be left PP if every principal left ideal of R is projective. Then any left PP-ring with every maximal left ideal principal is left max-hereditary. Now we have the following characterizations of left max-hereditary rings.

Proposition 1.2. *Let R be a ring. The following are equivalent.*

1. *R is left max-hereditary.*
2. *Every quotient of an m-injective left R-module is m-injective.*
3. *Every m-projective left R-module has projective dimension at most 1.*

Proof. $(1) \Rightarrow (2)$ Let M be an m-injective left R-module and N a submodule of M. We shall show that M/N is m-injective. To this end, let I be a maximal left ideal of R and $i : I \to R$ the inclusion and $\pi : M \to M/N$ the canonical map. For any $f : I \to M/N$, then there exists $g : I \to M$ such that $\pi g = f$ since I is projective by (1). Hence there is $h : R \to M$ such that $hi = g$ since M is m-injective. It follows that $(\pi h)i = f$, and so M/N is m-injective.

$(2) \Rightarrow (3)$ Let M be an m-projective left R-module and N a left R-module, then there is a short exact sequence $0 \to N \to E \to L \to 0$ with E injective. Note that L is m-injective by (2), and so we have the exact sequence $0 = Ext^1_R(M, L) \to Ext^2_R(M, N) \to Ext^2_R(M, E) = 0$. Thus $Ext^2_R(M, N) = 0$ and hence M has projective dimension at most 1.

$(3) \Rightarrow (1)$ holds since every simple left R-module is m-projective. □

A left R-module M is said to be MI-injective [11] if $Ext^1_R(N, M) = 0$ for any m-injective left R-module N. Next we characterize left max-hereditary rings over a left max-coherent ring.

Theorem 1.1. *Let R be a left max-coherent ring. The following are equivalent.*

1. *R is left max-hereditary.*
2. *Every MI-injective left R-module is injective.*
3. *($\mathfrak{m}-\mathfrak{pr}$,$\mathfrak{m}-\mathfrak{in}$) is hereditary, and every m-projective left R-module has a monic m-injective cover.*

Proof. $(1) \Rightarrow (2)$ is clear by [11, Proposition 3.4].

$(2) \Rightarrow (1)$ Let N be a quotient of an m-injective left R-module M. Suppose $f : F \to N$ is a m-injective cover of N by [11, Remark 2.10(1)]. Then there exists a homomorphism $h : M \to F$ such that $fh = \pi$, where $\pi : M \to N$. Hence f is an epimorphism. By [11, Remark 3.2(1)], $\ker(f)$ is MI-injective, and so it is injective by (2). So, N is m-injective.

$(1) \Rightarrow (3)$ holds by [5, Proposition 4] since the class of m-injective left R-modules is closed under direct sums over a left max-coherent ring by [11, Proposition 2.4(2)].

$(3) \Rightarrow (1)$ Let M be any m-injective left R-module and N any submodule of M. We have to prove that M/N is m-injective. In fact, there exists an exact sequence $0 \to N \to E \xrightarrow{\pi} L \to 0$ with E is m-injective and L is m-projective by Lemma 1.1. Since L has a monic m-injective cover $\phi : F \to L$ by (3), there is $\alpha : E \to F$ such that $\pi = \phi\alpha$. Thus ϕ is epic, and hence it is an isomorphism. So L is m-injective. For any simple left R-module S, we have the exact sequence $0 = Ext^1_R(S, L) \to Ext^2_R(S, N) \to Ext^2_R(S, E)$. Note that $Ext^2_R(S, E) = 0$ by [3, Proposition 1.2] since $(\mathfrak{m}-\mathfrak{pr}, \mathfrak{m}-\mathfrak{in})$ is hereditary, and hence $Ext^2_R(S, N) = 0$. On the other hand, the short exact sequence $0 \to N \to M \to M/N \to 0$ induces the exactness of the sequence $0 = Ext^1_R(S, M) \to Ext^1_R(S, M/N) \to Ext^2_R(S, N) = 0$. Therefore $Ext^1_R(S, M/N) = 0$, as desired. □

Finally, we give some new characterizations of left SF-rings. Recall that a ring R is called a left SF-ring [8] if each simple left R-module is flat.

Theorem 1.2. *Let R be a ring. The following are equivalent.*

1. *R is a left SF-ring.*
2. *Every left R-module is m-injective.*
3. *Every m-projective left R-module is projective.*
4. *Every cotorsion left R-module is m-injective.*
5. *$(\mathfrak{m}-\mathfrak{pr}, \mathfrak{m}-\mathfrak{in})$ is hereditary and every m-projective left R-module is m-injective.*

Proof. $(2) \Rightarrow (4)$ and $(2) \Rightarrow (5)$ are trivial.

$(4) \Rightarrow (1)$ Let S be any simple left R-module and M be a right R-module. Since M^+ is pure-injective and hence cotorsion, M^+ is m-injective by (4). So $0 = Ext^1_R(S, M^+) \cong Tor^R_1(M, S)^+$. Thus, $Tor^R_1(M, S) = 0$ for any right R-module M. Hence S is flat, and so R is a left SF-ring.

$(1) \Rightarrow (2)$ Let M be a left R-module. Then for any simple left R-module S, $0 = Tor^1_R(M^+, S) \cong Ext^R_1(S, M)^+$. So $Ext^R_1(S, M) = 0$. Hence M is m-injective.

$(2) \Rightarrow (3)$ Let M be any m-projective left R-module. Since every left R-module is m-injective by (2), $Ext^1_R(M, N) = 0$ for any left R-module N. Hence M is projective.

$(3) \Rightarrow (2)$ Let M be a left R-module. There exists an exact sequence $0 \to M \to E \to P \to 0$ with E m-injective and P m-projective by Lemma 1.1. By (3), P is projective, and so M is m-injective.

$(5) \Rightarrow (2)$ Let M be any left R-module. By Lemma 1.1, there is a short exact sequence $0 \to K \to F \to M \to 0$ with F m-projective and K m-injective. Then F is m-injective and hence M is m-injective by (5). □

References

[1] Clark, J., Lomp, C., Vanaja, N., Wisbauer, R.: Lifting modules. Frontiers in Mathematics, Birkhaauser Verlag, Basel (2006).

[2] Enochs, E.E., Jenda, O.M.G.: Relative homological algebra. Berlin: Walter de Gruyter (2000).

[3] Enochs, E.E., Jenda, O.M.G., Lopez-Ramos, J.A.: *The existence of Gorenstein flat covers.* Math. Scand. **94**(1), 46-62 (2004).

[4] Eklof, P.C., Trlifaj, J.: *How to make Ext vanish*, Bull. London Math. Soc. **33**(12), 41-51 (2001).

[5] Garcia Rozas, J.R., Torrecillas, B.: *Relative injective covers*, Comm. Algebra, **22**(8), 2925-2940 (1994).

[6] Megibben, C.: *Absolutely pure modules*. Proc. Amer. Math. Soc. **18**, 155-158 (1967).

[7] Moradzadeh-Dehkordi, A., Shojaee, S.H.: *Rings in which every ideal is pure-projective or FP-projective.* J. Algebra, **478**, 419-436 (2017).

[8] Ramamurthi, V.S.: *On the injectivity and flatness of certain cyclic modules.* Proc. Amer. Math. Soc. **48**, 21-25 (1975).

[9] Smith, P.F.: *Injective modules and prime ideals*. Comm. Algebra, **9**(9), 989-999 (1981).

[10] Wang, M.Y., Zhao, G.: *On maximal injectivity*. Acta Math. Sin. **21**(6), 1451-1458 (2005).

[11] Xiang, Y.: *Max-injective, max-flat modules and max-coherent rings.* Bull. Korean Math. Soc. **47**(3), 611-622 (2010).

Affiliations

YUSUF ALAGÖZ
ADDRESS: Siirt University, Dept. of Mathematics, Siirt, Turkey.
E-MAIL: yusuf.alagoz@siirt.edu.tr
ORCID ID: 0000-0002-2535-4679