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Some Properties on Sums of Element Orders in Finite p-groups

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Abstract
In literature, there are many papers on the sum of element orders of a finite group. In this study, in particular, we deal with the cases in finite p-groups. Our main aim is to investigate the sums of element orders in finite p-groups and to give some properties of such sums. Let $\psi(G)$ denote the sum of element orders of a finite group G. As an immediate consequence, we proved that $\psi(G) < \frac{3}{4}\psi(C)$ and $\psi(G) < \frac{1}{p-1}\psi(C)$, where G is a non-cyclic finite p-group of order p^r and C is a cyclic group of order p^r for some prime p.

Keywords: *Finite p-group; group element orders; sum of element orders.*

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1. Introduction

Our main starting point is given by the papers (see H. Amiri [2], H. Amiri and S.M.J. Amiri [1], Herzog et al. [5]) which studied on the sums of element orders in finite groups. Given a finite group G, we denote the sum of element orders in G by $\psi(G)$. Historically, the most enlightening in this area is due [5], who introduced the function $\psi(G)$ for a finite group G in [2] and proved that $\psi(G) < \psi(C)$, where C denotes a cyclic group of the same order with the order of G. Then, in [5], by improving the results obtained by S.M. Jafarian Amiri and M. Amiri in [4] and by R. Shen, G. Chen and C. Wu in [10], M. Herzog, P. Longobardi and M. Maj found an exact upper bound for sums of element orders in non-cyclic finite groups. In [8], N. Mansuroğlu derived the exact formula for the sums of element orders in symmetric groups.
Throughout this paper, we assume that G is a finite p-group of order p^r for a prime p. In this note we will focus on the study of $\psi(G)$. Our main aim is to investigate the sum of element orders in finite p-groups and to give some properties of the sum of element orders in finite p-groups. We investigate to find an exact upper bound for sums of element orders in non-cyclic p-finite groups.

2. Preliminaries

This section contains necessary preliminary results and notation. We use standard notation. We define the function $\psi(G) = \sum_{x \in G} o(x)$, where as usual, $o(x)$ is the order of the element x. Basic concepts and some results on group theory can be found in [6, 9]. Specially, more details for finite p-groups can be found in [7].
An important ingredient in our proofs is the following two lemmas which are particular case of Lemma 2.9 in the paper [5].

Lemma 2.1. *If C is a cyclic group of order p^r for some prime p, then*

$$\psi(C) = \frac{p^{2r+1}+1}{p+1}.$$

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Proof. Let $\varphi(p)$ be Euler's function.

$$\begin{aligned}\psi(C) &= 1 + p\varphi(p) + p^2\varphi(p^2) + \ldots + p^r\varphi(p^r) \\ &= 1 + pp(\frac{p-1}{p}) + p^2p^2(\frac{p-1}{p}) + \ldots + p^rp^r(\frac{p-1}{p}) = \frac{p^{2r+1}+1}{p+1}.\end{aligned}$$

□

Lemma 2.2. *If C is a cyclic group of order p^r for some prime p, then*

$$\psi(C) > p^{2r-1}(p-1).$$

Proof. By Lemma 2.1,

$$\begin{aligned}\psi(C) &= \frac{p^{2r+1}+1}{p+1} \\ &= p^{2r} - p^{2r-1} + p^{2r-2} - \ldots + 1 > p^{2r-1}(p-1).\end{aligned}$$

□

3. The main results

Now, in this section we give our main results.

Theorem 3.1. *Let G be a non-cyclic finite p-group of order p^r for some prime p. Then*

$$\psi(G) < p^{2r-1}.$$

Proof. Since G is a non-cyclic finite p-group of order p^r, for each element $x \in G$, $o(x) \le p^{r-1}$. But the order of the identity element 1 is 1, as a result of this, we have

$$\psi(G) \le (p^r - 1)p^{r-1} + 1 = \frac{(p^r-1)p^r}{p} + 1 < \frac{p^{2r}}{p} = p^{2r-1},$$

as required. □

Theorem 3.2. *Let G be a non-cyclic finite p-group of order p^r and C be a cyclic finite group of order p^r. Then*

$$\psi(G) < \frac{1}{p-1}\psi(C).$$

Proof. Suppose that $\psi(G) \ge \frac{1}{p-1}\psi(C)$. By Lemma 2.2,

$$\psi(G) \ge \frac{1}{p-1}\psi(C) > \frac{1}{p-1}p^{2r-1}(p-1) = p^{2r-1}.$$

This implies that there exists $x \in G$ with $o(x) > p^{r-1}$. Thus $|G : \langle x\rangle| < p$ and $\langle x\rangle$ is a p-group. As a consequence, $|G| = o(x)$, namely $G = \langle x\rangle$. But G is non-cyclic group, which is a contradiction. Hence, $\psi(G) < \frac{1}{p-1}\psi(C)$. □

We now provide a quantitative version of Theorem 3.2.

Corollary 3.1. *Suppose that p is odd prime. Let G be a non-cyclic finite p-group of order p^r and C be a cyclic finite group of order p^r. Then*

$$\psi(G) < \frac{1}{2}\psi(C).$$

Proof. By Theorem 3.2, we have $\psi(G) < \frac{1}{p-1}\psi(C)$. Since $p-1 \ge 2$, we obtain

$$\psi(G) < \frac{1}{p-1}\psi(C) \le \frac{1}{2}\psi(C).$$

This competes the proof. □

The next result is the analogous of the result obtained by Theorem 1 in [5] for finite p-groups.

Theorem 3.3. *Let G be a non-cyclic finite p-group of order p^r and C be a cyclic finite group of order p^r. Then*

$$\psi(G) \leq \frac{7}{11}\psi(C).$$

Proof. Assume that G is a non-cyclic finite p-group of order p^r satisfying $\psi(G) > \frac{7}{11}\psi(C)$. By Lemma 2.1,

$$\psi(G) > \frac{7}{11}\psi(C) = \frac{7}{11}.\frac{p^{2r+1}+1}{p+1} > \frac{7}{11}.\frac{2p^{2r}}{p+1} = \frac{14}{11}.\frac{p^{2r}}{p+1}.$$

There exists $x \in G$ with $o(x) > \frac{14}{11(p+1)}$. It follows that

$$|G : \langle x \rangle| < \frac{11(p+1)}{14}.$$

Now first we suppose that $p = 2$, then G is 2-group. Therefore, we have

$$|G : \langle x \rangle| < \frac{33}{14}.$$

Thus $|G : \langle x \rangle| = 2, 2^r \geq 4, r \geq 2$. Let $C_{2^{r-1}}$ be a cyclic group of order 2^{r-1}. Hence

$$\begin{aligned} \psi(G) \leq \psi(C_{2^{r-1}}) + 2^{2(r-1)} &= \frac{2^{2r-1}+1}{3} + \frac{2^{2r}}{4} = \frac{5}{12}.2^{2r} + \frac{1}{3} \\ &\leq \frac{7}{11}.(\frac{2^{2r+1}+1}{3}) = \frac{7}{11}\psi(C), \end{aligned}$$

which contradicts to the fact that $\psi(G) > \frac{7}{11}\psi(C)$.
Now we investigate the case that $p \geq 3$. Since p is odd prime, it follows from Corollary 3.1 that

$$\psi(G) < \frac{1}{2}\psi(C) < \frac{7}{11}\psi(C),$$

which is a contradiction. This completes the proof. □

Our main result is the following theorem.

Theorem 3.4. *Let G be a non-cyclic finite p-group of order p^r and C be a cyclic finite group of order p^r. Then*

$$\psi(G) < \frac{3}{4}\psi(C).$$

Proof. Assume that G is a non-cyclic finite p-group of order p^r satisfying $\psi(G) > \frac{2}{3}\psi(C)$. Since G is a non-cyclic, for each element $x \in G$, $o(x) \leq p^{r-1}$. But the order of the identity element 1 is 1, as a result of this, we have

$$\psi(G) \leq (p^r - 1)p^{r-1} + 1.$$

Now first we suppose that $p = 2$, then G is 2-group. Therefore, we have

$$\psi(G) \leq (2^r - 1)2^{r-1} + 1 = 2^{2r-1} - 2^{r-1} + 1 < \frac{3}{4}(\frac{2}{3}2^{2r} + \frac{1}{3}) = \frac{3}{4}\psi(C).$$

Now we investigate the case that $p \geq 3$. Since p is odd prime, by Corollary 3.1, we have

$$\psi(G) < \frac{1}{2}\psi(C) < \frac{3}{4}\psi(C),$$

a contradiction. This completes the proof. □

The following examples hold all our main consequences.

Example 3.1. Let $G = \langle a, b | a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$ be a non-cyclic 2-group of order 8 and C be a cyclic group of order 8.

Example 3.2. Let $G = \langle a, b | a^8 = b^4 = 1, a^4 = b^2, b^{-1}ab = a^{-1} \rangle$. Then G is a generalized quarternion group of order 16. It is easy to see that G satisfies $\psi(G) < \frac{2}{3}\psi(C)$ where C is a cyclic group of order 16.

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