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A Note on Generalized Sasakian Space Forms with Interpolating Sesqui-Harmonic Legendre Curves

Fatma Karaca*

Abstract
We classify the curvature of interpolating sesqui-harmonic Legendre curves in generalized Sasakian space forms. We investigate the necessary and sufficient conditions for these types of curves in nine cases to be interpolating sesqui-harmonic.

Keywords: *Generalized Sasakian space form; Legendre curve; interpolating sesqui-harmonic curve.*

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***Corresponding author**

1. Introduction

Biharmonic maps $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds are well known a natural generalization of the harmonic maps [7]. Biharmonic maps are a critical point of the *bienergy functional*

$$E_2(\varphi) = \int_\Omega \|\tau(\varphi)\|^2 \, d\nu_g,$$

where

$$\tau(\varphi) = tr(\nabla d\varphi) \tag{1.1}$$

is called the *tension field* of φ [7]. The Euler-Lagrange equation for $E_2(\varphi)$ is

$$\tau_2(\varphi) = tr(\nabla^\varphi \nabla^\varphi - \nabla^\varphi_\nabla)\tau(\varphi) - tr(R^N(d\varphi, \tau(\varphi))d\varphi), \tag{1.2}$$

which is the *bitension field* of φ [8]. The equation $\tau_2(\varphi) = 0$ is called biharmonic equation.

Interpolating sesqui-harmonic maps $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds are defined that interpolated between the actions for harmonic and biharmonic maps [6]. The map φ is called *interpolating sesqui-harmonic* if it is a critical point of $E_{\delta_1,\delta_2}(\varphi)$

$$E_{\delta_1,\delta_2}(\varphi) = \delta_1 \int_\Omega \|d\varphi\|^2 \, d\nu_g + \delta_2 \int_\Omega \|\tau(\varphi)\|^2 \, d\nu_g, \tag{1.3}$$

where $\delta_1, \delta_2 \in \mathbb{R}$ [6]. The interpolating sesqui-harmonic map equation is given by

$$\tau_{\delta_1,\delta_2}(\varphi) = \delta_2\tau_2(\varphi) - \delta_1\tau(\varphi) = 0 \tag{1.4}$$

for $\delta_1, \delta_2 \in \mathbb{R}$ [6]. If variations of the equation (1.3) that are normal to the image $\varphi(M) \subset N$ and $\delta_2 = 1$, $\delta_1 > 0$ then, an interpolating sesqui-harmonic map turns to biminimal [12].

In [6], Branding introduced an action functional for maps between Riemannian manifolds that interpolate between the actions for harmonic and biharmonic maps and studied interpolating sesqui-harmonic curves in a

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3-dimensional sphere. In [5], the same author studied a conservation law and used it to show the smoothness of weak solutions for a spherical target and found several classification results for interpolating sesqui-harmonic maps. In [10], the author, Özgür and De studied interpolating sesqui-harmonic Legendre curves in Sasakian space forms. In [16], Özgür and Güvenç studied biharmonic Legendre curves in generalized Sasakian space forms. Motivated by the above studies, in the present paper, we investigate interpolating sesqui-harmonic Legendre curves in generalized Sasakian space forms. We find the necessary and sufficient conditions for these types of curves in nine cases to be interpolating sesqui-harmonic.

2. Preliminaries

Let $N^{2n+1} = (N^{2n+1}, \phi, \xi, \eta, g)$ be an almost contact metric manifold with almost contact metric structure (ϕ, ξ, η, g). A manifold $(N^{2n+1}, \phi, \xi, \eta, g)$ is called a *Sasakian manifold* if it is normal, that is,

$$N_\phi = -2d\eta \otimes \xi$$

where N_ϕ is the Nijenhuis tensor field of ϕ [4]. An almost contact metric manifold N^{2n+1} is called a *Kenmotsu manifold* [9] if

$$(\nabla_X \phi)\, Y = g(\phi X, Y)\xi - \eta(Y)X$$

where ∇ is the Levi-Civita connection. An almost contact metric manifold N^{2n+1} is called a *cosymplectic manifold* if $\nabla\phi = 0$, which implies that $\nabla\xi = 0$ [13].

The sectional curvature of a ϕ-section is called a *ϕ-sectional curvature*. When the ϕ-sectional curvature is constant, the manifold is called a *space form* (*Sasakian,Kenmotsu, cosymplectic*) (see [4], [9], [13]). The manifold N^{2n+1} $= (N^{2n+1}, \varphi, \xi, \eta, g)$ is called a *generalized Sasakian space form* if its curvature tensor R is given by

$$R(X,Y)Z = f_1 \{g(Y,Z)X - g(X,Z)Y\}$$

$$+f_2 \{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}$$

$$+ f_3 \{\eta(X)\eta(Z)Y - \eta(X)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \tag{2.1}$$

for certain differentiable functions f_1, f_2 and f_3 on N^{2n+1} [1]. If N^{2n+1} is a Sasakian space form then $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$ [4], if N^{2n+1} is a Kenmotsu space form then $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$ [9], if N^{2n+1} is a cosymplectic space form then $f_1 = f_2 = f_3 = \frac{c}{4}$ [13].

A submanifold of a Sasakian manifold is called an *integral submanifold* if $\eta(X) = 0$, for every tangent vector X. An integral curve of a Sasakian manifold $(N^{2n+1}, \phi, \xi, \eta, g)$ is called a *Legendre curve* [4]. Thus, a curve $\gamma : I \longrightarrow (N^{2n+1}, \phi, \xi, \eta, g)$ is called a Legendre curve if $\eta(T) = 0$, where T is the tangent vector field of γ.

In [15], the notion of trans-Sasakian manifolds is introduced by Oubiña. An almost contact metric manifold N is said to be a *trans-Sasakian manifold* if there exist two functions α and β on N such that

$$(\nabla_X \phi)\, Y = \alpha \left[g(X,Y)\xi - \eta\,(Y)\,X\right] + \beta \left[g(\phi X, Y)\xi - \eta\,(Y)\,\phi X\right], \tag{2.2}$$

for any vector fields X, Y on N. From (2.2), it is easy to see that

$$\nabla_X \xi = -\alpha\phi X + \beta \left[X - \eta\,(X)\,\xi\right]. \tag{2.3}$$

If we have $\beta = 0$ (resp. $\alpha = 0$), then N is called an *$\alpha-$Sasakian manifold* (resp. *β-Kenmotsu manifold*). Another kind of trans-Sasakian manifolds is that of cosymplectic manifolds, obtained for $\alpha = \beta = 0$. By the use of the equation (2.3), we have

$$\nabla_X \xi = 0,$$

which means that ξ is a Killing vector field for a cosymplectic manifold [3].

3. Interpolating sesqui-harmonic Legendre curves in generalized Sasakian space forms

Let (N,g) be an n-dimensional Riemannian manifold and $\gamma : I \longrightarrow (N,g)$ a unit-speed curve in (N,g). γ is called a Frenet curve of osculating order r, $1 \leq r \leq n$, if there exists orthonormal vector fields $\{E_i\}_{i=1,2,...n}$ along γ satisfying Frenet equations given by

$$E_1 = T = \gamma',$$

$$\nabla_T E_1 = k_1 E_2,$$

$$\nabla_T E_i = -k_{i-1}E_{i-1} + k_i E_{i+1}, \ \ 2 \leq i \leq n-1, \tag{3.1}$$

$$\nabla_T E_n = -k_{n-1}E_{n-1},$$

where the function $\{k_1 = k, k_2 = \tau, k_3, ..., k_{n-1}\}$ are called the curvatures of γ [11].

Now, we can state the following theorem:

Theorem 3.1. *Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a generalized Sasakian space form with constant ϕ-sectional curvature c and $\gamma : I \subset \mathbb{R} \longrightarrow (N^{2n+1}, \phi, \xi, \eta, g)$ be a Legendre curve of osculating order r and $m = min\{r, 4\}$. Then γ is interpolating sesqui-harmonic if and only if there exists real numbers δ_1, δ_2 such that*
(1) $f_2 = 0$ or $\phi T \perp E_2$ or $\phi T \in \{E_2, ..., E_m\}$; and
(2) $f_3 = 0$ or $\xi \perp E_2$ or $\xi \in \{E_2, ..., E_m\}$; and
(3) the first m of the following equations are satisfied:

$$-3\delta_2 k_1 k_1' = 0, \tag{3.2}$$

$$\delta_2 \left(k_1'' - k_1^3 - k_1 k_2^2 - k_1 f_1\right) - \delta_1 k_1 + 3\delta_2 f_2 k_1 \left[g(\phi T, E_2)\right]^2 - \delta_2 f_3 k_1 \left[\eta\left(E_2\right)\right]^2 = 0, \tag{3.3}$$

$$\delta_2 \left(2k_1' k_2 + k_1 k_2'\right) + 3\delta_2 f_2 k_1 g(\phi T, E_2) g(\phi T, E_3) - \delta_2 f_3 k_1 \eta\left(E_2\right) \eta\left(E_3\right) = 0, \tag{3.4}$$

$$\delta_2 \left(k_1 k_2 k_3\right) + 3\delta_2 f_2 k_1 g(\phi T, E_2) g(\phi T, E_4) - \delta_2 f_3 k_1 \eta\left(E_2\right) \eta\left(E_4\right) = 0. \tag{3.5}$$

Proof. Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a generalized Sasakian space form and $\gamma : I \longrightarrow N^{2n+1}$ be a Legendre curve of osculating order r. By the use of (1.1) and (3.1), we can write

$$\tau(\gamma) = k_1 E_2.$$

Using the equations (3.1), we find

$$\nabla_T \nabla_T T = -k_1^2 E_1 + k_1' E_2 + k_1 k_2 E_3, \tag{3.6}$$

$$\begin{aligned} \nabla_T \nabla_T \nabla_T T = & -3k_1 k_1' E_1 + \left(k_1'' - k_1^3 - k_1 k_2^2\right) E_2 \\ & + \left(2k_1' k_2 + k_1 k_2'\right) E_3 + \left(k_1 k_2 k_3\right) E_4, \end{aligned} \tag{3.7}$$

$$R(T, \nabla_T T)T = -k_1 f_1 E_2 - 3f_2 k_1 g(\phi T, E_2)\phi T + f_3 k_1 \eta\left(E_2\right) \xi. \tag{3.8}$$

By the use of the equations (3.6), (3.7) and (3.8) into (4.1) in [6], we obtain

$$\begin{aligned} \tau_{\delta_1,\delta_2}(\gamma) = & \left(-3\delta_2 k_1 k_1'\right) E_1 + \left[\delta_2 \left(k_1'' - k_1^3 - k_1 k_2^2 + f_1 k_1\right) - \delta_1 k_1\right] E_2 \\ & + \delta_2 \left(2k_1' k_2 + k_1 k_2'\right) E_3 + \delta_2 \left(k_1 k_2 k_3\right) E_4 + 3\delta_2 f_2 k_1 g(\phi T, E_2)\phi T - \delta_2 f_3 k_1 \eta\left(E_2\right) \xi. \end{aligned} \tag{3.9}$$

Then taking the scalar product of (3.9) with E_2, E_3 and E_4 respectively, we obtain the desired results. □

Now, we give the interpretions of Theorem 3.1:

Case I. $f_2 = f_3 = 0$.

From Theorem 3.1, we obtain following theorem:

Theorem 3.2. *Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a generalized Sasakian space form with $f_2 = f_3 = 0$ and $\gamma : I \subset \mathbb{R} \longrightarrow N^{2n+1}(1)$ be a Legendre curve of osculating order r.*

(1) γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if it is a Legendre circle with $k_1 = \sqrt{f_1 - \frac{\delta_1}{\delta_2}}$ where $f_1 > \frac{\delta_1}{\delta_2}$ is a constant;

(2) γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if it is a Legendre helix with $k_1^2 + k_2^2 = f_1 - \frac{\delta_1}{\delta_2}$ where $f_1 > \frac{\delta_1}{\delta_2}$, δ_1, δ_2 is a constant.

In both cases, if $f_1 \leq \frac{\delta_1}{\delta_2}$, then such an interpolating sesqui-harmonic Legendre curve does not exist.

Proof. Let $\gamma : I \longrightarrow N^{2n+1}$ be an interpolating sesqui-harmonic curve. From Theorem 3.1, if we take $r = 2$, then γ is a circle with $k_1 = \sqrt{f_1 - \frac{\delta_1}{\delta_2}}$ where $f_1 > \frac{\delta_1}{\delta_2}$ is a constant. Similarly, if we take $r = 3$, then we obtain that k_2 is a non-zero constant. Thus, γ is a helix with $k_1^2 + k_2^2 = f_1 - \frac{\delta_1}{\delta_2}$ where $f_1 > \frac{\delta_1}{\delta_2}$ is a constant. On the contrary, let γ be a Legendre circle with $k_1 = \sqrt{f_1 - \frac{\delta_1}{\delta_2}}$ or a Legendre helix with $k_1^2 + k_2^2 = f_1 - \frac{\delta_1}{\delta_2}$ where $f_1 > \frac{\delta_1}{\delta_2}$ is a constant. It is clear that γ satisfies Theorem 3.1, respectively. Thus, we obtain the desired result. □

Case II. $f_2 = 0$, $f_3 \neq 0$ and $\xi \perp E_2$.
We can state:

Theorem 3.3. *Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a generalized Sasakian space form with $f_2 = 0$, $f_3 \neq 0$, $\xi \perp E_2$ and $\gamma : I \subset \mathbb{R} \longrightarrow N^{2n+1}(c)$ a Legendre curve of osculating order r.*

(1) γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if it is a Legendre circle with $k_1 = \sqrt{f_1 - \frac{\delta_1}{\delta_2}}$ where $f_1 > \frac{\delta_1}{\delta_2}$ is a constant; or

(2) γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if it is a Legendre helix with $k_1^2 + k_2^2 = f_1 - \frac{\delta_1}{\delta_2}$ where $f_1 > \frac{\delta_1}{\delta_2}$, δ_1, δ_2 is a constant.

If $r > 3$ or $f_1 \leq \frac{\delta_1}{\delta_2}$, then an interpolating sesqui-harmonic Legendre curve does not exist.

Proof. Assume that $\gamma : I \longrightarrow N^{2n+1}$ be an interpolating sesqui-harmonic curve. From Theorem 3.1 and $\eta(E_2) = 0$, we have

$$k_1 = constant > 0,$$

$$k_1^2 + k_2^2 = f_1 - \frac{\delta_1}{\delta_2},$$

$$k_2' = 0,$$

$$k_2 k_3 = 0.$$

Using the above equations, we obtain the desired results. □

Case III. $f_2 = 0$, $f_3 \neq 0$, $\xi \in span\{E_2, ..., E_m\}$ and $\eta(E_2) \neq 0$.

Theorem 3.4. *Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a generalized Sasakian space form with $f_2 = 0$, $f_3 \neq 0$, $\xi \in span\{E_2, ..., E_m\}$, $\eta(E_2) \neq 0$ and $\gamma : I \subset \mathbb{R} \longrightarrow N^{2n+1}(c)$ a Legendre curve of osculating order r.*

(1) If $r \geq 4$, then γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if

$$k_1 = constant > 0, \tag{3.10}$$

$$k_1^2 + k_2^2 = f_1 - \frac{\delta_1}{\delta_2} - f_3 \cos^2 u_1, \tag{3.11}$$

$$k_2' - f_3 \cos u_1 \sin u_1 \cos u_2 = 0, \tag{3.12}$$

$$k_2 k_3 - f_3 \cos u_1 \sin u_1 \sin u_2 = 0.$$

(2) If we take $r = 3$, the equations (3.10), (3.11) and (3.12) are satisfied, taking $u_2 = 0$. If we take $r = 2$, then the equations (3.10) and (3.11) are satisfied, taking $u_1 = 0, \pi$.

Proof. Assume that $r \geq 4$. Thus, we can write

$$\xi = \cos u_1 E_2 + \sin u_1 \cos u_2 E_3 + \sin u_1 \sin u_2 E_4 \tag{3.13}$$

where $u_1, u_2 : I \rightarrow \mathbb{R}$ are the angle functions between ξ and E_2; E_3 and the orthogonal projection of ξ onto span$\{E_3, E_4\}$, respectively. From the equation (3.13), we have

$$\eta(E_2) = \cos u_1,$$

$$\eta(E_3) = \sin u_1 \cos u_2,$$

$$\eta(E_4) = \sin u_1 \sin u_2. \tag{3.14}$$

Assume that $r = 3$. We can write

$$\xi = \cos u_1 E_2 + \sin u_1 E_3 \tag{3.15}$$

where $u_1 : I \rightarrow \mathbb{R}$ is the angle function between ξ and E_2. The equation (3.15) can be found taking $u_2 = 0$ in (3.13). Finally, let $r = 2$. We can write

$$\xi = \mp E_2. \tag{3.16}$$

We obtain (3.16) from (3.13), taking $u_1 = 0, \pi$ and $u_2 = 0$. Using Theorem 3.1 and the equations (3.13), (3.15) and (3.16), we obtain the desired results. □

Now, let $\gamma : I \subset \mathbb{R} \longrightarrow N^{2n+1}$ be a Legendre curve of osculating order r in trans-Sasakian generalized Sasakian space form $(N^{2n+1}, \phi, \xi, \eta, g)$. Since γ is a Legendre curve, $\eta(T) = 0$. Then, we have

$$\nabla_T \xi = -\alpha \phi T + \beta T \tag{3.17}$$

which gives us

$$g(\nabla_T \xi, T) = \beta. \tag{3.18}$$

Differentiating $\eta(T) = 0$ along γ , if we use (3.1) and (3.18), we get

$$k_1 \eta(E_2) = -\beta. \tag{3.19}$$

Corollary 3.1. *Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a trans-Sasakian generalized Sasakian space form with f_1 =constant, $f_2 = 0$, f_3 and β are non-zero constants, $\xi \in span\{E_2, ..., E_m\}$, $\eta(E_2) \neq 0$ and $\gamma : I \subset \mathbb{R} \longrightarrow N^{2n+1}$ a Legendre curve of osculating order r.*

(1) *γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} < 0$ if and only if γ is a circle with $k_1 = \sqrt{f_1 - \frac{\delta_1}{\delta_2} - f_3}$ where $f_1 - \frac{\delta_1}{\delta_2} - f_3 > 0$ is a constant, $0 < \beta^2 < -\frac{\delta_1}{\delta_2}$, $\xi \parallel E_2$ and $\alpha = 0$, or*

(2) *γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if γ is a helix with $k_1 = \mp\beta > 0, k_2 = \sqrt{f_1 - \frac{\delta_1}{\delta_2} - f_3 - \beta^2}$, where $f_1 - \frac{\delta_1}{\delta_2} - f_3 - \beta^2 > 0$ is a constant, $\xi \parallel E_2$, $\alpha \neq 0$ is a constant, $\phi T \parallel E_3$ and $dim\ N = 3$.*

Proof. Let N^{2n+1} be a trans-Sasakian generalized Sasakian space form, then we have

$$\nabla_T \eta(E_1) = k_1 \eta(E_2) + \beta = 0, \tag{3.20}$$

$$\nabla_T \eta(E_2) = k_2 \eta(E_3) - \alpha g(\phi T, E_2), \tag{3.21}$$

$$\nabla_T \eta(E_3) = -k_2 \eta(E_2) + k_3 \eta(E_4) - \alpha g(\phi T, E_3),$$

$$\nabla_T \eta(E4) = -k_3 \eta(E_3) + k_4 \eta(E_5) - \alpha g(\phi T, E_4).$$

Let γ be interpolating sesqui-harmonic.

1. If $r = 2$, from Theorem 3.1, we have

$$k_1 = \text{constant} > 0,$$

$$k_1^2 = f_1 - \frac{\delta_1}{\delta_2} - f_3[\eta(E_2)]^2$$

and $\xi \in span\{E_2\}$. Hence we obtain $\eta(E_2) = \pm 1$. So γ is a circle with $k_1 = \sqrt{f_1 - \frac{\delta_1}{\delta_2} - f_3}$, where $f_1 - \frac{\delta_1}{\delta_2} - f_3 > 0$ is a constant and $\xi \parallel E_2$. Differentiating $\xi = \pm E_2$ along γ, we get $\alpha = 0$ and $k_1 = \pm\beta$. Since $\alpha = 0$, N is a β-Kenmotsu generalized Sasakian space form. Then β-Kenmotsu generalized Sasakian space forms satisfy

$$f_1 - f_3 + \beta^2 = 0.$$

Then, we have $0 < \beta^2 < -\frac{\delta_1}{\delta_2}$ with $\frac{\delta_1}{\delta_2} < 0$.

2. If we take $r = 3$, using Theorem 3.1, we have

$$k_1 = constant > 0, \tag{3.22}$$

$$k_1^2 + k_2^2 = f_1 - \frac{\delta_1}{\delta_2} - f_3[\eta(E_2)]^2, \tag{3.23}$$

$$k_2' - f_3\eta(E_2)\eta(E_4) = 0 \tag{3.24}$$

and $\xi \in span\{E_2, E_4\}$. Differentiating the equation (3.23) and using (3.21), (3.24), we get

$$2k_2\eta(E_3) = \alpha g(\phi T, E_2). \tag{3.25}$$

From the equation (3.20), we obtain that $\eta(E_2)$ is a constant, since $\beta \neq 0$ is a constant. Using (3.21), we find

$$k_2\eta(E_3) = \alpha g(\phi T, E_2). \tag{3.26}$$

By the use of (3.25) and (3.26), we obtain $\eta(E_3) = 0$. Since $\xi \in span\{E_2, E_3\}$ and $\eta(E_3) = 0$, we find $\xi \parallel E_2$. From the equations (3.20), (3.22), (3.23) and (3.24), we obtain that γ is a helix with $k_1 = \pm\beta > 0$, $k_2 = \sqrt{f_1 - \frac{\delta_1}{\delta_2} - f_3 - \beta^2}$, where $f_1 - \frac{\delta_1}{\delta_2} - f_3 - \beta^2 > 0$ is a constant and $\xi \parallel E_2$. Differentiating $\xi = \pm E_2$ along γ, we get $\alpha \neq 0$. From [14], we obtain $dim\ N = 3$.

3. If we take $r \geq 4$, then $dim\ N \geq 5$. Since $\beta \neq 0$ is a constant, from [14], we have $\alpha = 0$. Thus we obtain that N is a β-Kenmotsu generalized Sasakian space form and $dim\ N \geq 5$. Using [1], we find $f_3 = 0$, which is a contradiction.

On the contrary, let γ be the given curve. It is easily seen that the first three of the equations in Theorem 3.1 are satisfied (replacing $k_m = 0$). So γ is interpolating sesqui-harmonic. □

Case IV. $f_2 \neq 0$, $f_3 = 0$ and $\phi T \perp E_2$.

In this case, we have $g(\phi T, E_2) = 0$. From Theorem 3.1, we have

Theorem 3.5. *Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a generalized Sasakian space form with $f_2 \neq 0$, $f_3 = 0$, $\phi T \perp E_2$ and $\gamma : I \subset \mathbb{R} \longrightarrow N^{2n+1}(c)$ a Legendre curve of osculating order r.*

(1) γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if it is a Legendre circle with $k_1 = \sqrt{f_1 - \frac{\delta_1}{\delta_2}}$ where $f_1 > \frac{\delta_1}{\delta_2}$ is a constant; or

(2) γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if it is a Legendre helix with $k_1^2 + k_2^2 = f_1 - \frac{\delta_1}{\delta_2}$ where $f_1 > \frac{\delta_1}{\delta_2}$, δ_1, δ_2 is a constant.

If $f_1 \leq \frac{\delta_1}{\delta_2}$, then an interpolating sesqui-harmonic Legendre curve does not exist.

Case V. $f_2 \neq 0$, $f_3 = 0$, $\phi T \in span\{E_2, E_3, E_4\}$ and $g(\phi T, E_2) \neq 0$.

Theorem 3.6. *Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a generalized Sasakian space form with $f_2 \neq 0$, $f_3 = 0$, $\phi T \in span\{E_2, E_3, E_4\}$, $g(\phi T, E_2) \neq 0$ and $\gamma : I \subset \mathbb{R} \longrightarrow N^{2n+1}(c)$ a Legendre curve of osculating order r.*

(1) If $r \geq 4$, then γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if

$$k_1 = constant > 0, \tag{3.27}$$

$$k_1^2 + k_2^2 = f_1 - \frac{\delta_1}{\delta_2} + 3f_2\cos^2 w_1, \tag{3.28}$$

$$k_2' + 3f_2 \cos w_1 \sin w_1 \cos w_2 = 0, \tag{3.29}$$

$$k_2k_3 + 3f_2 \cos w_1 \sin w_1 \sin w_2 = 0.$$

(2) If we take $r = 3$, the equations (3.27), (3.28) and (3.29) are satisfied, taking $w_2 = 0$. If we take $r = 2$, then the equations (3.27) and (3.28) are satisfied, taking $w_1 = 0, \pi$.

Proof. Assume that $r \geq 4$. Thus, we can write

$$\phi T = \cos w_1 E_2 + \sin w_1 \cos w_2 E_3 + \sin w_1 \sin w_2 E_4 \tag{3.30}$$

where $w_1, w_2 : I \to \mathbb{R}$ are the angle functions between ϕT and E_2; E_3 and the orthogonal projection of ϕT onto span$\{E_3, E_4\}$, respectively. From the equation (3.30), we can write

$$g(\phi T, E_2) = \cos w_1,$$

$$g(\phi T, E_3) = \sin w_1 \cos w_2,$$

$$g(\phi T, E_4) = \sin w_1 \sin w_2. \tag{3.31}$$

Let $r = 3$. We can write

$$\phi T = \cos w_1 E_2 + \sin w_1 E_3 \tag{3.32}$$

where $w_1 : I \to \mathbb{R}$ is the angle function between ϕT and E_2. The equation (3.32) can be found taking $w_2 = 0$ in (3.30). Finally, let $r = 2$. We can write

$$\phi T = \mp E_2. \tag{3.33}$$

We obtain (3.33) from (3.30), taking $w_1 = 0, \pi$ and $w_2 = 0$. Using Theorem 3.1 and the equations (3.30), (3.32) and (3.33), we obtain the desired results. □

Using the same method of Corollary 3.2 in [16], we have the following corollary:

Corollary 3.2. *Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a connected trans-Sasakian generalized Sasakian space form with f_1 =constant, $f_2 \neq 0$ is a constant, $f_3 = 0$, $\phi T \in span\{E_2, ..., E_m\}$, $g(\phi T, E_2) \neq 0$ and $\gamma : I \subset \mathbb{R} \longrightarrow N^{2n+1}$ a Legendre curve of osculating order r. Then γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if γ is a Frenet curve of order $r \geq 4$ with*

$$k_1 = \frac{-\beta}{\eta(E_2)} = constant > 0,$$

$$k_2 = \sqrt{f_1 - \frac{\delta_1}{\delta_2} + 3f_2 [g(\phi T, E_2)]^2 - \frac{\beta^2}{[\eta(E_2)]^2}} > 0,$$

$$k_3 = \frac{-3g(\phi T, E_2)\, g(\phi T, E_4)}{\sqrt{f_1 - \frac{\delta_1}{\delta_2} + 3f_2 [g(\phi T, E_2)]^2 - \frac{\beta^2}{[\eta(E_2)]^2}}} > 0,$$

$$k_4 = \frac{-\beta g(\phi E_2, E_5)}{\eta(E_2)\, g(\phi T, E_4)} > 0, if r \geq 5$$

where $f_1 - \frac{\delta_1}{\delta_2} + 3f_2 [g(\phi T, E_2)]^2 - \frac{\beta^2}{[\eta(E_2)]^2}$ is a constant, $g(\phi T, E_3) = 0$, $\alpha = 0$, $g(\phi T, E_2) \neq 0$ and $g(\phi T, E_4) \neq 0$ are constants, $\beta \neq 0$ and $\eta(E_2) \neq 0$.

Proof. Assume that N^{2n+1} is a trans-Sasakian generalized Sasakian space form, then we have

$$\nabla_T \phi T = \alpha \xi + k_1 \phi E_2, \tag{3.34}$$

$$\nabla_T g(\phi T, E_2) = \alpha \eta(E_2) + k_2 g(\phi T, E_3), \tag{3.35}$$

$$\nabla_T g(\phi T, E_3) = \alpha \eta(E_3) + k_1 g(\phi E_2, E_3) - k_2 g(\phi T, E_2) + k_3 g(\phi T, E_4), \tag{3.36}$$

$$\nabla_T g(\phi T, E_4) = \alpha \eta(E_4) + k_2 g(\phi E_2, E_4) - k_3 g(\phi T, E_3).$$

Let γ be interpolating sesqui-harmonic.
1. If we take $r = 2$, using Theorem 3.1, we have

$$k_1 = \text{constant} > 0,$$

$$k_1^2 = f_1 - \frac{\delta_1}{\delta_2} + 3f_2 [g(\phi T, E_2)]^2$$

and $\phi T \in span\{E_2\}$. Thus, we can write $\phi T = \pm E_2$. Differentiating $\phi T = \pm E_2$, using (3.1) and (3.34), we have

$$\alpha\xi + k_1\phi E_2 = \mp k_1 T.$$

Hence we obtain $\alpha = 0$. From the equation (3.20), we get $\beta = 0$. So N is cosymplectic which requires $f_2 = f_3$. This is a contradiction.

2. If $r = 3$, from Theorem 3.1, we have

$$k_1 = constant > 0, \tag{3.37}$$

$$k_1^2 + k_2^2 = f_1 - \frac{\delta_1}{\delta_2} + 3f_2[g(\phi T, E_2)]^2, \tag{3.38}$$

$$k_2' + 3f_2 g(\phi T, E_2) g(\phi T, E_3) = 0 \tag{3.39}$$

and $\phi T \in span\{E_2, E_4\}$. Differentiating the equation (3.38) and using (3.35), (3.39), we obtain

$$-2k_2 g(\phi T, E_3) = \alpha\eta(E_2). \tag{3.40}$$

For $\phi T \in span\{E_2, E_4\}$, we can write

$$\phi T = g(\phi T, E_2)E_2 + g(\phi T, E_3)E_3. \tag{3.41}$$

It is clear that $g(\phi E_2, E_3) = 0$. Differentiating the equation (3.41) and using (3.1), (3.34), (3.35) and (3.36), we find

$$\alpha\xi + k_1\phi E_2 = -k_1 g(\phi T, E_2)T + \alpha\eta(E_2)E_2 + \alpha\eta(E_3)E_3. \tag{3.42}$$

Let $\alpha = 0$. From the equation (3.42), we have $g(\phi T, E_3) = 0$, that is, $\phi T = \pm E_2$. Using the equation (3.36), we find $k_2 = 0$. This is a contradiction. Thus $\alpha \neq 0$. From (3.42), we obtain

$$[\eta(E_2)]^2 + [\eta(E_3)]^2 = 1.$$

So $\xi \in span\{E_2, E_3\}$ and $\phi T = \pm E_2$. Hence $\xi = \pm E_3$. Using the equation (3.20), we find $\beta = 0$. Differentiating $\xi = \pm E_3$ and using the equations (3.1), (3.17), (3.37), (3.38) and (3.39), we obtain $k_1 = \sqrt{f_1 - \frac{\delta_1}{\delta_2} + 3f_2 - \alpha^2}$, $k_2 = \pm\alpha > 0$, where $f_1 - \frac{\delta_1}{\delta_2} + 3f_2 - \alpha^2 > 0$, $\alpha \neq 0$ is a constant. Thus N is a connected α-Sasakian generalized Sasakian space form. If $\dim N \geq 5$, using [2], we find $f_2 = f_3$, which is a contradiction. If $dim\ N = 3$, using [2], we have $f_2 = 0$, which is also a contradiction.

Assume that $r \geq 4$. From Theorem 3.1, we have

$$k_1 = constant > 0, \tag{3.43}$$

$$k_1^2 + k_2^2 = f_1 - \frac{\delta_1}{\delta_2} + 3f_2[g(\phi T, E_2)]^2, \tag{3.44}$$

$$k_2' + 3f_2 g(\phi T, E_2) g(\phi T, E_3) = 0, \tag{3.45}$$

$$k_2 k_3 + 3f_2 g(\phi T, E_2) g(\phi T, E_4) = 0 \tag{3.46}$$

and $\phi T \in span\{E_2, E_3, E_4\}$. Differentiating (3.44) and using (3.35), (3.45), we get

$$-2k_2 g(\phi T, E_3) = \alpha\eta(E_2). \tag{3.47}$$

3. Assume that $r \geq 4$ and $g(\phi T, E_3) = 0$. We obtain $\alpha = 0$. Since $g(\phi T, E_3) = 0$, we get $\phi T \in span\{E_2, E_4\}$. By the use of equation (3.35), $g(\phi T, E_2) \neq 0$ is a constant. So using $\phi T \in span\{E_2, E_4\}$ and (3.46), $g(\phi T, E_4) \neq 0$ is a constant. Using the equations (3.20), (3.43), (3.44), (3.45) and (3.46), we find $k_1 = \frac{-\beta}{\eta(E_2)}$ =constant>0, $k_2 = \sqrt{f_1 - \frac{\delta_1}{\delta_2} + 3f_2\left[g\left(\phi T, E_2\right)\right]^2 - \frac{\beta^2}{[\eta(E_2)]^2}} > 0$, $k_3 = \frac{-3g(\phi T, E_2)g(\phi T, E_4)}{\sqrt{f_1 - \frac{\delta_1}{\delta_2} + 3f_2[g(\phi T, E_2)]^2 - \frac{\beta^2}{[\eta(E_2)]^2}}} > 0$ where $f_1 - \frac{\delta_1}{\delta_2} + 3f_2\left[g\left(\phi T, E_2\right)\right]^2 - \frac{\beta^2}{[\eta(E_2)]^2}$ =constant > 0. If $r \geq 5$, differentiating $g(\phi T, E_5) = 0$ and using (3.34), we find $k_4 = \frac{-\beta g(\phi E_2, E_5)}{\eta(E_2) g(\phi T, E_4)}$.

4. If $r \geq 4$ and $g(\phi T, E_3) \neq 0$, then $\alpha \neq 0$ and $\eta(E_2) \neq 0$. Since $\dim N \geq 5$ and $\alpha \neq 0$, we find $\beta = 0$. This contradicts $\eta(E_2) \neq 0$.

On the contrary, let γ be the given curve. Using Theorem 3.1 , γ is interpolating sesqui-harmonic. $\square$

Case VI. $f_2 \neq 0,\, f_3 \neq 0,\, \phi T \perp E_2$ and $\xi \perp E_2$.
In this case, we have $g(\phi T, E_2) = 0$ and $\eta(E_2) = 0$. Using Theorem 3.1, we have

Theorem 3.7. *Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a generalized Sasakian space form with $f_2 \neq 0,\, f_3 \neq 0,\, \phi T \perp E_2,\, \xi \perp E_2$ and $\gamma : I \subset \mathbb{R} \longrightarrow N^{2n+1}(c)$ a Legendre curve of osculating order r.*

(1) γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if it is a Legendre circle with $k_1 = \sqrt{f_1 - \frac{\delta_1}{\delta_2}}$ where $f_1 > \frac{\delta_1}{\delta_2}$ is a constant; or

(2) γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if it is a Legendre helix with $k_1^2 + k_2^2 = f_1 - \frac{\delta_1}{\delta_2}$ where $f_1 > \frac{\delta_1}{\delta_2},\, \delta_1,\, \delta_2$ is a constant.

If $f_1 \leq \frac{\delta_1}{\delta_2}$, then an interpolating sesqui-harmonic Legendre curve does not exist.

Case VII. $f_2 \neq 0,\, f_3 \neq 0,\, \phi T \perp E_2,\, \xi \in span\{E_2, ..., E_m\}$ and $\eta(E_2) \neq 0$.
Since $g(\phi T, E_2) = 0$, using Theorem 3.1 and equations (3.13) and (3.14), we have

Theorem 3.8. *Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a generalized Sasakian space form with $f_2 \neq 0,\, f_3 \neq 0,\, \phi T \perp E_2,\, \xi \in span\{E_2, ..., E_m\},\, \eta(E_2) \neq 0$. and $\gamma : I \subset \mathbb{R} \longrightarrow N^{2n+1}(c)$ a Legendre curve of osculating order r.*

(1) If $r \geq 4$, then γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if

$$k_1 = \text{constant} > 0, \tag{3.48}$$

$$k_1^2 + k_2^2 = f_1 - \frac{\delta_1}{\delta_2} - f_3 \cos^2 u_1, \tag{3.49}$$

$$k_2' - f_3 \cos u_1 \sin u_1 \cos u_2 = 0, \tag{3.50}$$

$$k_2 k_3 - f_3 \cos u_1 \sin u_1 \sin u_2 = 0.$$

(2) If we take $r = 3$, the equations (3.48), (3.49) and (3.50) are satisfied, taking $u_2 = 0$. If we take $r = 2$, then the equations (3.48) and (3.49) are satisfied, taking $u_1 = 0, \pi$.

Corollary 3.3. *Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a trans-Sasakian generalized Sasakian space form with f_1 =constant, f_2 and f_3 are non-zero constants, $\phi T \perp E_2,\, \xi \in span\{E_2, ..., E_m\},\, \eta(E_2) \neq 0$ and $\gamma : I \subset \mathbb{R} \longrightarrow N^{2n+1}$ a Legendre curve of osculating order r. Then γ is interpolating sesqui-harmonic if and only if γ is a helix of order $r \geq 4$ with*

$$k_1 = \frac{-\beta}{\eta(E_2)} = \text{constant} > 0,$$

$$k_2 = \sqrt{f_1 - \frac{\delta_1}{\delta_2} - f_3 [\eta(E_2)]^2 - \frac{\beta^2}{[\eta(E_2)]^2}} > 0,$$

$$k_3 = \frac{f_3 \eta(E_2) \eta(E_4)}{\sqrt{f_1 - \frac{\delta_1}{\delta_2} - f_3 [\eta(E_2)]^2 - \frac{\beta^2}{[\eta(E_2)]^2}}} = \text{constant} > 0,$$

where $f_1 - \frac{\delta_1}{\delta_2} - f_3 [\eta(E_2)]^2 - \frac{\beta^2}{[\eta(E_2)]^2}$ is a positive constant, $\eta(E_3) = 0,\, \alpha = 0$.

Proof. The proof is similar to the proof of Corollary 3.1. □

Case VIII. $f_2 \neq 0,\, f_3 \neq 0,\, \phi T \perp E_2,\, \phi T \in span\{E_2, ..., E_m\},\, g(\phi T, E_2) \neq 0$ and $\xi \perp E_2$.
Since $\eta(E_2) = 0$, using Theorem 3.1 and equations (3.30) and (3.31), we obtain the following theorem:

Theorem 3.9. *Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a generalized Sasakian space form with $f_2 \neq 0,\, f_3 \neq 0,\, \phi T \perp E_2,\, \phi T \in span\{E_2, ..., E_m\},\, g(\phi T, E_2) \neq 0,\, \xi \perp E_2$ and $\gamma : I \subset \mathbb{R} \longrightarrow N^{2n+1}(c)$ a Legendre curve of osculating order r.*

(1) If $r \geq 4$, then γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if

$$k_1 = \text{constant} > 0, \tag{3.51}$$

$$k_1^2 + k_2^2 = f_1 - \frac{\delta_1}{\delta_2} + 3 f_2 \cos^2 w_1, \tag{3.52}$$

$$k_2' + 3 f_2 \cos w_1 \sin w_1 \cos w_2 = 0, \tag{3.53}$$

$$k_2 k_3 + 3 f_2 \cos w_1 \sin w_1 \sin w_2 = 0.$$

(2) If we take $r = 3$, the equations (3.51), (3.52) and (3.53) are satisfied, taking $w_2 = 0$. If we take $r = 2$, then the equations (3.51) and (3.52) are satisfied, taking $w_1 = 0, \pi$.

Corollary 3.4. *Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a trans-Sasakian generalized Sasakian space form with f_1 =constant, f_2 and f_3 are non-zero constants, $\phi T \in span\{E_2, ..., E_m\}$, $g(\phi T, E_2) \neq 0$, $\xi \perp E_2$ and $\gamma : I \subset \mathbb{R} \longrightarrow N^{2n+1}$ a Legendre curve of osculating order r. Then γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if*

(1) *γ is a circle with $k_1 = \sqrt{f_1 - \frac{\delta_1}{\delta_2} + 3f_2}$ where $\alpha = \beta = 0$, $\phi T \parallel E_2$ and $f_1 - \frac{\delta_1}{\delta_2} + 3f_2$ is a positive constant; or*

(2) *γ is a helix with $k_1 = \sqrt{f_1 - \frac{\delta_1}{\delta_2} + 3f_2 - \alpha^2}$, $k_1 = \mp\alpha > 0$ where $f_1 - \frac{\delta_1}{\delta_2} + 3f_2\beta - \alpha^2 > 0$, $\alpha \neq 0$ is a constant, $\beta = 0$, $\phi T \parallel E_2$ and $\xi \parallel E_3$; or*

(3) *γ is a Frenet curve of order $r \geq 4$ with*

$$k_1 = \lambda > 0,$$

$$k_2 = \sqrt{f_1 - \frac{\delta_1}{\delta_2} + 3f_2\left[g(\phi T, E_2)\right]^2 - \lambda^2} > 0,$$

$$k_3 = \frac{-3f_2 g(\phi T, E_2)\, g(\phi T, E_4)}{\sqrt{f_1 - \frac{\delta_1}{\delta_2} + 3f_2\left[g(\phi T, E_2)\right]^2 - \lambda^2}} > 0,$$

$$k_4 = \frac{\lambda g(\phi E_2, E_5)}{g(\phi T, E_4)} > 0, \text{if } r \geq 5$$

where $g(\phi T, E_3) = 0$, $g(\phi T, E_2) \neq 0$ and $g(\phi T, E_4) \neq 0$ are constants, $f_1 - \frac{\delta_1}{\delta_2} + 3f_2\left[g(\phi T, E_2)\right]^2 - \lambda^2 > 0$ and $\lambda > 0$ are constants.

Proof. The proof is similar to the proof of Corollary 3.2. □

Case IX. $f_2 \neq 0$, $f_3 \neq 0$, $\phi T \perp E_2$, $\phi T \in span\{E_2, ..., E_m\}$, $g(\phi T, E_2) \neq 0$ and $\xi \in span\{E_2, ..., E_m\}$ and $\eta(E_2) \neq 0$.

From Theorem 3.1 and equations (3.13), (3.14), (3.30) and (3.31), we have the following theorem:

Theorem 3.10. *Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a generalized Sasakian space form with $\phi T \in span\{E_2, ..., E_m\}$, $\xi \in span\{E_2, ..., E_m\}$ and $\gamma : I \subset \mathbb{R} \longrightarrow N^{2n+1}(c)$ a Legendre curve of osculating order r.*

(1) *If $r \geq 4$, then γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if*

$$k_1 = constant > 0, \tag{3.54}$$

$$k_1^2 + k_2^2 = f_1 - \frac{\delta_1}{\delta_2} + 3f_2\cos^2 w_1 - f_3\cos^2 u_1, \tag{3.55}$$

$$k_2' + 3f_2\cos w_1 \sin w_1 \cos w_2 - f_3 \cos u_1 \sin u_1 \cos u_2 = 0, \tag{3.56}$$

$$k_2 k_3 + 3f_2 \cos w_1 \sin w_1 \sin w_2 - f_3 \cos u_1 \sin u_1 \sin u_2 = 0.$$

(2) *If we take $r = 3$, the equations (3.54), (3.55) and (3.56) are satisfied, taking $w_2 = 0$ and $u_2 = 0$. If we take $r = 2$, then the equations (3.54) and (3.55) are satisfied, taking $w_1 = 0, \pi$ and $u_2 = 0, \pi.w_1 = 0, \pi$.*

Corollary 3.5. *Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a trans-Sasakian generalized Sasakian space form with f_1 =constant, f_2 and f_3 are non-zero constants, $\phi T \in span\{E_2, ..., E_m\}$, $g(\phi T, E_2) \neq 0$, $\xi \in span\{E_2, ..., E_m\}$, $\eta(E_2) \neq 0$ and $\gamma : I \subset \mathbb{R} \longrightarrow N^{2n+1}$ a Legendre curve of osculating order $r \geq 4$. Then γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if*

(1)

$$k_1 = \frac{-\beta}{\eta(E_2)} = constant > 0,$$

$$k_2 = \frac{\lambda}{2\mu} > 0,$$

$$k_3 = \frac{2\mu\left\{f_3\eta(E_2)\,\eta(E_4) - 3f_2 g(\phi T, E_2) g(\phi T, E_4)\right\}}{\lambda} > 0,$$

where $\lambda \neq 0$ and $\mu \neq 0$; or

(2)

$$k_1 = \frac{-\beta}{\eta(E_2)} = constant > 0,$$

$$k_2=\sqrt{f_1-\frac{\delta_1}{\delta_2}+3f_2\left[g\left(\phi T,E_2\right)\right]^2-f_3\left[\eta\left(E_2\right)\right]^2-\frac{\beta^2}{\left[\eta\left(E_2\right)\right]^2}}>0,$$

$$k_3=\frac{\left\{f_3\eta\left(E_2\right)\eta\left(E_4\right)-3f_2g(\phi T,E_2)g(\phi T,E_4)\right\}}{\sqrt{f_1-\frac{\delta_1}{\delta_2}+3f_2\left[g\left(\phi T,E_2\right)\right]^2-f_3\left[\eta\left(E_2\right)\right]^2-\frac{\beta^2}{\left[\eta\left(E_2\right)\right]^2}}}>0,$$

where $f_1-\frac{\delta_1}{\delta_2}+3f_2\left[g\left(\phi T,E_2\right)\right]^2-f_3\left[\eta\left(E_2\right)\right]^2-\frac{\beta^2}{\left[\eta(E_2)\right]^2}$ *is a positive constant,* $\lambda=\mu=0$, $\lambda=(3f_2-f_3)\alpha g(\phi T,E_2)\eta\left(E_2\right)$ *and* $\mu=f_3\eta\left(E_2\right)\eta\left(E_3\right)-3f_2g(\phi T,E_2)g(\phi T,E_3)$.

Proof. By the use of Theorem 3.1, we have

$$k_1=\text{constant}>0,$$

$$k_1^2+k_2^2=f_1-\frac{\delta_1}{\delta_2}+3f_2\left[g\left(\phi T,E_2\right)\right]^2-f_3\left[\eta\left(E_2\right)\right]^2, \tag{3.57}$$

$$k_2'-f_3\eta\left(E_2\right)\eta\left(E_3\right)+3f_2g(\phi T,E_2)g(\phi T,E_3)=0, \tag{3.58}$$

$$k_2k_3-f_3\eta\left(E_2\right)\eta\left(E_4\right)+3f_2g(\phi T,E_2)g(\phi T,E_4)=0. \tag{3.59}$$

Differentiating $\eta\left(T\right)=0$ along γ and using (3.1), we obtain $k_1\eta\left(E_2\right)=-\beta$, that is,

$$k_1=\frac{-\beta}{\eta\left(E_2\right)}.$$

Differentiating the equation (3.57) along γ, we find

$$k_2k_2'=3f_2g\left(\phi T,E_2\right)\nabla_Tg\left(\phi T,E_2\right)-f_3\eta\left(E_2\right)\nabla_T\eta\left(E_2\right). \tag{3.60}$$

Since N is a trans- Sasakian manifold, if we replace (3.21), (3.35), (3.58) in (3.60), we have

$$2k_2\mu=\lambda. \tag{3.61}$$

If $\lambda\neq0$ and $\mu\neq0$, then (3.61) gives us $k_2=\frac{\lambda}{2\mu}\neq0$. Thus, the equation (3.59) gives us k_3. If $\mu=0$, from the equation (3.58) that k_2 is a constant. Using the equation (3.57), we obtain $k_2=\sqrt{f_1-\frac{\delta_1}{\delta_2}+3f_2\left[g\left(\phi T,E_2\right)\right]^2-f_3\left[\eta\left(E_2\right)\right]^2-\frac{\beta^2}{\left[\eta(E_2)\right]^2}}$, where $f_1-\frac{\delta_1}{\delta_2}+3f_2\left[g\left(\phi T,E_2\right)\right]^2-f_3\left[\eta\left(E_2\right)\right]^2-\frac{\beta^2}{\left[\eta(E_2)\right]^2}>0$ is a constant. So the equation (3.59) gives us $k_3=\frac{\left\{f_3\eta(E_2)\eta(E_4)-3f_2g(\phi T,E_2)g(\phi T,E_4)\right\}}{\sqrt{f_1-\frac{\delta_1}{\delta_2}+3f_2[g(\phi T,E_2)]^2-f_3[\eta(E_2)]^2-\frac{\beta^2}{[\eta(E_2)]^2}}}>0$. □

4. Applications

Let $(N^{2n+1},\phi,\xi,\eta,g)$ be a Sasakian space form. Thus we have $\alpha=1,\ \beta=0,\ f_1=\frac{c+3}{4}$, $f_2=f_3=\frac{c-1}{4}$. In this case, equation (3.19) gives us $\eta(E_2)=0$, since $k_1>0$.

For Case I and Case VI, using Theorem 3.2 and Theorem 3.7, then we obtain the following result in [10]:

Theorem 4.1. *[10]Let* $(N^{2n+1},\phi,\xi,\eta,g)$ *be a Sasakian space form with* $c=1$ *or* $\phi T\perp E_2$ *and* $\gamma:I\subset\mathbb{R}\longrightarrow N^{2n+1}$ *a Legendre curve of osculating order* r.

(1) γ *is interpolating sesqui-harmonic with* $\frac{\delta_1}{\delta_2}\neq0$ *if and only if it is a Legendre circle with* $k_1=\sqrt{1-\frac{\delta_1}{\delta_2}}$ *where* $1>\frac{\delta_1}{\delta_2}$; *or*

(2) γ *is interpolating sesqui-harmonic with* $\frac{\delta_1}{\delta_2}\neq0$ *if and only if it is a Legendre helix with* $k_1^2+k_2^2=1-\frac{\delta_1}{\delta_2}$ *where* $1>\frac{\delta_1}{\delta_2}$.

If $1\leq\frac{\delta_1}{\delta_2}$, *then an interpolating sesqui-harmonic Legendre curve does not exist.*

For Case VIII, if we use Corollary 3.4, we obtain the following theorem:

Theorem 4.2. *Let $(N^{2n+1},\phi,\xi,\eta,g)$ be a Sasakian space form with $c\neq 1$, $\phi T\in\{E_2,...,E_m\}$, $g(\phi T,E_2)\neq 0$ and $\gamma: I\subset\mathbb{R}\longrightarrow N^{2n+1}$ a Legendre curve of osculating order r.*

(1) *γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2}\neq 0$ if and only if it is a helix with $k_1=\sqrt{c-1-\frac{\delta_1}{\delta_2}}$ and $k_2=1$ where $c>\frac{\delta_1}{\delta_2}+1$, $\phi T\parallel E_2$ and $\xi\parallel E_3$; or*

(2) *γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2}\neq 0$ if and only if it is a Frenet curve of order $r\geq 4$ with*

$$k_1=\lambda>0,$$

$$k_2=\sqrt{\frac{c+3}{4}-\frac{\delta_1}{\delta_2}+\frac{3(c-1)}{4}\left[g(\phi T,E_2)\right]^2-\lambda^2}>0,$$

$$k_3=\frac{-\frac{3(c-1)}{4}g(\phi T,E_2)\,g(\phi T,E_4)}{\sqrt{\frac{c+3}{4}-\frac{\delta_1}{\delta_2}+\frac{3(c-1)}{4}\left[g(\phi T,E_2)\right]^2-\lambda^2}}>0,$$

$$k_4=\frac{\lambda g(\phi E_2,E_5)}{g(\phi T,E_4)}>0, \text{if } r\geq 5$$

where $g(\phi T,E_3)=0$, $g(\phi T,E_2)\neq 0$ and $g(\phi T,E_4)\neq 0$ are constants, $\frac{c+3}{4}-\frac{\delta_1}{\delta_2}+\frac{3(c-1)}{4}\left[g(\phi T,E_2)\right]^2-\lambda^2>0$ and $\lambda>0$ are constants.

Proof. If we take $\alpha=1,\ \beta=0,\ f_1=\frac{c+3}{4}, f_2=f_3=\frac{c-1}{4}$ in Corollary 3.4, we obtain the desired results. □

Remark 4.1. k_4 does not need to be constant. So, there exists *interpolating sesqui-harmonic* curves which are not helices in a Sasakian space form with $\dim N\geq 5$.

Let $(N^{2n+1},\phi,\xi,\eta,g)$ be a Kenmotsu space form. Thus we have $\alpha=0,\ \beta=1,\ f_1=\frac{c-3}{4}, f_2=f_3=\frac{c+1}{4}$. From [1] and [2] we obtain $f_2=\frac{c+1}{4}=0$, that is $c=-1$.

By the use of Theorem 3.2, we obtain the following theorem:

Theorem 4.3. *Let $(N^{2n+1},\phi,\xi,\eta,g)$ be a cosymplectic space form with $c=-1$ and $\gamma: I\subset\mathbb{R}\longrightarrow N^{2n+1}(-1)$ be a Legendre curve of osculating order r.*

(1) *γ is interpolating sesqui-harmonic if and only if it is a circle with $k_1=\sqrt{-1-\frac{\delta_1}{\delta_2}}$ where $\frac{\delta_1}{\delta_2}<-1$ or;*

(2) *γ is interpolating sesqui-harmonic if and only if it is a helix with $k_1^2+k_2^2=-1-\frac{\delta_1}{\delta_2}$ where $\frac{\delta_1}{\delta_2}<-1$.*

Let $(N^{2n+1},\phi,\xi,\eta,g)$ be a cosymplectic space form. Hence $\alpha=\beta=0,\ f_1=f_2=f_3=\frac{c}{4}$. From the equation (3.19), we obtain $k_1\eta(E_2)=0$, that is $\eta(E_2)=0$, with $k_1>0$.

For the Case VI, using Theorem 3.7, we obtain the following theorem:

Theorem 4.4. *Let $(N^{2n+1},\phi,\xi,\eta,g)$ be a cosymplectic space form with $c\neq 0$, $\phi T\perp E_2$, $\xi\perp E_2$ and $\gamma: I\subset\mathbb{R}\longrightarrow N^{2n+1}(c)$ a Legendre curve of osculating order r.*

(1) *γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2}\neq 0$ if and only if it is a circle with $k_1=\sqrt{\frac{c}{4}-\frac{\delta_1}{\delta_2}}$ where $\frac{c}{4}>\frac{\delta_1}{\delta_2}$ or;*

(2) *γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2}\neq 0$ if and only if it is a helix with $k_1^2+k_2^2=\frac{c}{4}-\frac{\delta_1}{\delta_2}$ where $\frac{c}{4}>\frac{\delta_1}{\delta_2}$.*

For the Case VIII, from Corollary 3.4, we obtain the following theorem:

Theorem 4.5. *Let $(N^{2n+1},\phi,\xi,\eta,g)$ be a cosymplectic space form with $c\neq 0$, $\phi T\in\{E_2,E_3,E_4\}$, $g(\phi T,E_2)\neq 0$, $\xi\perp E_2$ and $\gamma: I\subset\mathbb{R}\longrightarrow N^{2n+1}$ a Legendre curve of osculating order r.*

(1) *γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2}\neq 0$ if and only if it is a circle with $k_1=\sqrt{c-\frac{\delta_1}{\delta_2}}$ where $c>\frac{\delta_1}{\delta_2}$, $\phi T\parallel E_2$; or*

(2) *γ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2}\neq 0$ if and only if it is a Frenet curve of order $r\geq 4$ with*

$$k_1=\lambda>0,$$

$$k_2=\sqrt{\frac{c}{4}-\frac{\delta_1}{\delta_2}+\frac{3c}{4}\left[g(\phi T,E_2)\right]^2-\lambda^2}>0,$$

$$k_3 = \frac{-\frac{3c}{4} g(\phi T, E_2) g(\phi T, E_4)}{\sqrt{\frac{c}{4} - \frac{\delta_1}{\delta_2} + \frac{3c}{4}[g(\phi T, E_2)]^2 - \lambda^2}} > 0,$$

$$k_4 = \frac{\lambda g(\phi E_2, E_5)}{g(\phi T, E_4)} > 0, \text{ if } r \geq 5$$

where $g(\phi T, E_3) = 0$, $g(\phi T, E_2) \neq 0$ *and* $g(\phi T, E_4) \neq 0$ *are constants,* $\frac{c}{4} - \frac{\delta_1}{\delta_2} + \frac{3c}{4}[g(\phi T, E_2)]^2 - \lambda^2 > 0$ *and* $\lambda > 0$ *are constants.*

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Affiliations

Fatma Karaca
Address: Beykent University, Dept. of Mathematics, 34550, Istanbul-Turkey.
e-mail: fatmagurlerr@gmail.com
ORCID ID: 0000-0002-0382-8028