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A Characteristic of Similarities by Use of Steinhaus' Problem on Partition of Triangles

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Abstract
H. Steinhaus [7] has asked whether inside each acute triangle there is a point from which perpendiculars to the sides divide the triangle into three parts of equal areas. In this paper we present a new characteristic of similarities by use of Steinhaus' Problem on partition of a triangle.

Keywords: *Möbius transformation; similarity; Steinhaus' problem.*

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1. Introduction

A Möbius transformation $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a mapping of the form $f(z) = (az + b)/(cz + d)$ satisfying $ad - bc \neq 0$, where $a, b, c, d \in \mathbb{C}$. Notice that

$$f(\infty) = \lim_{z\to+\infty} f(z) = \frac{a}{c} \;\; and \;\; f(-\frac{d}{c}) = \infty.$$

It is well known that the set of all Möbius transformations is a group with respect to the composition and that Möbius transformations have many beautiful properties. Some of these properties are as follows:

- Any Möbius transformation has at most two fixed points in $\mathbb{C} \cup \{\infty\}$.
- The cross-ratio $[z_1, z_2, z_3, z_4]$ of any four complex numbers, which is defined by

$$[z_1, z_2, z_3, z_4] = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3}$$

  is invariant under Möbius transformations, that is

$$[z_1, z_2, z_3, z_4] = [f(z_1), f(z_2), f(z_3), f(z_4)]$$

- Möbius transformations are conformal and continuous.
- Möbius transformations map circles to circles, where straight lines are considered to be circles through ∞.

Translations, rotations about origin, strech transformations (complex dilations), inversions and similarities are most familiar Möbius transformations, which are defined by $f(z) = z + b$, $g(z) = e^{i\theta}z$, $h(z) = az$ $(a \neq 0)$, $j(z) = \frac{1}{z}$, $m(z) = az + b$, respectively. It is well known that any Möbius transformation can be written as a composition of translations, complex dilations and inversions. In the literature there are many characterizations of Möbius transformations by use of some geometric objects such as Apollonius points of triangles [2], Apollonius quadrilaterals [3], Apollonius pentagons [1], Apollonius hexagons [4] and others. The aim of this paper is to present

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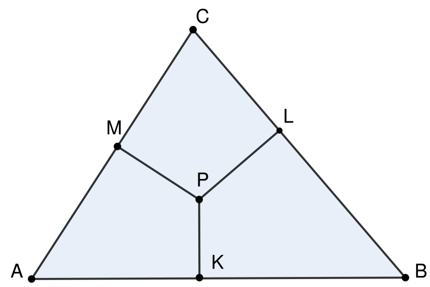

Figure 1. If P is a solution of Steinhaus' problem for an acute triangle ABC, then there exist corresponding points K, L, M on AB, BC and CA, respectively, such that $AB \perp PK, BC \perp PL, CA \perp PM$ satisfying $Area(AKPM) = Area(BLPK) = Area(CMPL) = \frac{Area(ABC)}{3}$.

a new characterization of similarities by use of Steinhaus' problem on partition of triangles. H. Steinhaus [7] has asked whether inside each acute triangle there is a point from which perpendiculars to the sides decide the triangle into three parts of equal areas, see *Fig*.1. For the solution of this problem, we refer the reader to [8].

Example 1.1. Let ABC be an arbitrary equilateral triangle in the Euclidean plane and let L be its center. Then

$$Area(AELD) = Area(BFLE) = Area(CDLF) = \frac{AreaABC}{3}$$

holds, where D, E, F are the midpoints of the sides AC, AB and BC, respectively.

2. Main Results

Lemma 2.1. *Let ABC be an equilateral triangle in the Euclidean plane and let L be its center. Denote the midpoints of the sides AC, AB, BC by D, E, F respectively. Then $AL \perp DE$.*

The proof is clear, so we omit it.

Throughout the paper we denote by X' the image of X under f, by AB the geodesic segment between points A and B, by $|AB|$ the distance between points A and B, by ABC the triangle with three ordered vertices A, B and C, and by $\angle BAC$ the angle between AB and AC. Unless otherwise stated, we consider $w = f(z)$ as a nonconstant meromorphic function of a complex variable z in the plane $|z| < +\infty$.

Now we consider *Property S*.

***Property S*:** Suppose that $w = f(z)$ is an analytic and a univalent mapping in a nonempty domain R of the complex plane. Let ABC be an arbitrary triangle contained in R. If L is a solution of Steinhaus' problem for ABC, (that is there exist corresponding points D, E, F on the sides AC, AB, BC respectively, such that $LD \perp AC, LE \perp AB, LF \perp BC$ satisfying

$$Area(AELD) = Area(BFLE) = Area(CDLF) = \frac{Area(ABC)}{3}),$$

then L' is a solution of $A'B'C'$ (that is the points D', E', F' are on the sides $A'C', A'B', B'C'$, respectively, such that $L'D' \perp A'C', L'E' \perp A'B', L'F' \perp B'C'$ satisfying

$$Area(A'E'L'D') = Area(B'F'L'E') = Area(C'D'L'F') = \frac{Area(A'B'C')}{3}).$$

Lemma 2.2. *If $w = f(z)$ is analytic and univalent in a nonempty domain R, then $f'(z) \neq 0$ in R, see [6].*

Lemma 2.3. *Let $w = f(z)$ satisfy Property S. If l_1 and l_2 are two lines meeting perpendicularly, then $f(l_1)$ meets $f(l_2)$ perpendicularly.*

Proof. Let l_1 and l_2 be two lines meeting at a point, say B, perpendicularly. Let A be a point on l_1 and let C be a point on l_2 such that $\angle ACB = \frac{\pi}{6}$, $\angle CBA = \frac{\pi}{2}$, $\angle BAC = \frac{\pi}{3}$. It is enough to prove that $C'B' \perp A'B'$. Denote the reflection of C with respect to AB by D and denote the reflection of B with respect to AC by E. Let F be the symmetry of C with respect to E. Hence we construct an equilateral triangle FCD. Clearly A is the center of FCD. Since A is the solution of Steinhaus' problem for FCD, that is

$$Area(CBAE) = Area(BDGA) = Area(GFEA) = \frac{Area(FCD)}{3},$$

where G is the midpoint of the side DF. By *Property S*, we get

$$Area(C'B'A'E') = Area(B'D'G'A') = Area(G'F'E'A') = \frac{Area(F'C'D')}{3},$$

which implies that $C'B' \perp A'B'$. Therefore $f(l_1)$ meets $f(l_2)$ perpendicularly. □

Theorem 2.1. *$w = f(z)$ has Property S if and only if $w = f(z)$ is a similarity.*

Proof. Let f be a similarity defined by

$$f(z) = az + b$$

satisfying $a, b \in \mathbb{C}, a \neq 0$ and let ABC be an acute angled triangle. Clearly

$$|A'B'| = |a||AB|, \quad |A'C'| = |a||AC|, \quad |CB'| = |a||CB|.$$

By the side-side-side theorem, we get

$$Area(A'B'C') = |a|^2 Area(ABC).$$

Let L be a solution of Steinhaus' problem for ABC. Then one can easily see that there exist three points D, E, F on the sides AC, AB and BC, respectively such that

$$Area(AELD) = Area(BFLE) = Area(FCDL) = \frac{Area(ABC)}{3}.$$

Since f preserves the measures of the angles of triangles and preserves the collinearity property of points, we get

$$Area(AELD) = Area(ALD) + Area(ALE) = \frac{Area(A'L'D')}{|a|^2} + \frac{Area(A'L'E')}{|a|^2}$$

$$Area(BFLE) = Area(BFL) + Area(BEL) = \frac{Area(B'F'L')}{|a|^2} + \frac{Area(B'E'L')}{|a|^2}$$

$$Area(FCDL) = Area(CLF) + Area(CLD) = \frac{Area(C'L'F')}{|a|^2} + \frac{Area(C'L'D')}{|a|^2},$$

which implies that f has *Property S*.

Now assume that $w = f(z)$ has *Property S*. Because of the fact that $w = f(z)$ is analytic and univalent in the domain R, by *Lemma* 2.2,

$$f'(z) \neq 0 \tag{2.1}$$

holds in R. If x is an arbitrarily fixed point of R, then by (2.1) we get

$$f'(x) \neq 0. \tag{2.2}$$

Let L be the point represented by x. Because of $L \in R$, there exists a positive real number ϵ such that $V(L, \epsilon)$ is contained in R, where $V(L, \epsilon)$ is ϵ-closed circular neighborhood of L. Throughout the proof let ABC denote an arbitrary equilateral triangle which is contained in $V(L, \epsilon)$ and whose center is at L. Since ABC is an equilateral triangle contained in $V(L, \epsilon)$, we can represent the points A, B, C by complex numbers

$$A = x + y, \quad B = x + wy, \quad C = x + w^2 y,$$

where $w = \frac{-1+\sqrt{3}i}{2}$ and $|y| \le \epsilon$. Then the midpoints of the sides AC, AB and BC are

$$D = x + \frac{w^2+1}{2}y, \ \ E = x + \frac{w+1}{2}y, \ \ F = x + \frac{w^2+w}{2}y,$$

respectively. Since $w = f(z)$ is univalent in R, the points $A', B', C', D', E', F', L'$ are different points. Clearly, there exists some sufficiently small $\delta \in \mathbb{R}^+$ satisfying $\delta \le \epsilon$ such that A', B', C' are not collinear on the w-plane for all y satisfying $0 < |y| \le \epsilon$ by (2.2) and by the property of analytic functions, see [5]. By hypothesis, A', B', C' are not collinear and L' is a solution of Steinhaus' Problem for $A'B'C'$, that is

$$Area(A'E'L'D') = Area(B'F'L'E') = Area(F'C'D'L') = \frac{Area(A'B'C')}{3},$$

where

$$A' = f(x+y), \ \ B' = f(x+wy), \ \ C' = f(x+w^2y),$$

$$D' = f(x + \frac{w^2+1}{2}y), \ \ E' = f(x + \frac{w+1}{2}y), \ \ F' = f(x + \frac{w^2+w}{2}y).$$

Since

$$Area(A'E'L'D') = Area(B'F'L'E'),$$

it follows that

$$\frac{1}{2}|A'L'||D'E'|sin\alpha = \frac{1}{2}|B'L'||F'E'|sin\beta, \tag{2.3}$$

by the area formula, where α is the measure of the angle between $A'L'$ and $D'E'$, and β is the measure of the angle between $B'L'$ and $F'E'$. By *Lemma* 2.1, we get that $AL \perp DE$ and $BL \perp EF$. Since f preserves right angles by *Lemma* 2.3, we get $\alpha = \beta = \frac{\pi}{2}$. Then by (2.2), we obtain

$$|A'L'||D'E'| = |B'L'||F'E'|,$$

which implies

$$\left|(f(x+y) - f(x))(f(x + \frac{w+1}{2}y) - f(x + \frac{w^2+1}{2}y))\right| = \left|(f(x+wy) - f(x))(f(x + \frac{w^2+w}{2}y) - f(x + \frac{w+1}{2}y))\right|$$

and this yields

$$\left|\frac{(f(x+y) - f(x))(f(x + \frac{w+1}{2}y) - f(x + \frac{w^2+1}{2}y))}{(f(x+wy) - f(x))(f(x + \frac{w^2+w}{2}y) - f(x + \frac{w+1}{2}y))}\right| = 1.$$

If we set

$$g(y) = \frac{(f(x+y) - f(x))(f(x + \frac{w+1}{2}y) - f(x + \frac{w^2+1}{2}y))}{(f(x+wy) - f(x))(f(x + \frac{w^2+w}{2}y) - f(x + \frac{w+1}{2}y))}$$

then we get $|g(y)| = 1$ in the punctured closed disk $0 < |y| \le \delta$. Since the numerator and the denominator of $g(y)$ are analytic functions for all y satisfying $0 < |y| \le \delta$ and since, by the fact that $w = f(z)$ is univalent in R, the denominator of $g(y)$ never vanishes in $0 < |y| \le \delta$, $g(y)$ is analytic in $0 < |y| \le \delta$. Next we prove that $g(y)$ is also analytic at $y = 0$. As $y \to 0$, by L'Hopital's rule and by the fact that $f'(x) \ne 0$, we obtain

$$\frac{f(x+y) - f(x}{f(x+wy) - f(x)} \to \frac{f'(x)}{wf'(x)} = \frac{1}{w}$$

and

$$\frac{f(x + \frac{w+1}{2}y) - f(x + \frac{w^2+1}{2}y)}{f(x + \frac{w^2+w}{2}y) - f(x + \frac{w+1}{2}y)} \to \frac{-w}{1+w}$$

holds. Hence, for $y \to 0$, we immediately get

$$g(y) \to \frac{1}{w} \cdot \frac{-w}{1+w} = \frac{-1}{w+1}.$$

If we define

$$g(0) = \frac{-1}{w+1}$$

and by Riemann's theorem on removable singularities, the function $g(y)$ is analytic at $y = 0$. Furthermore, since $g(0) = \frac{-1}{w+1}$ holds, the equality $|g(y)| = 1$ still holds at $y = 0$. Therefore $g(y)$ is analytic in the closed disk $|y| \leq \delta$ and that $|g(y)| = 1$ holds for all y with $|y| \leq \delta$. By the maximum modulus principle for analytic functions we obtain

$$g(y) = K$$

in $|y| \leq \delta$, where K is a complex constant with modulus 1. Setting $y = 0$ in $g(y) = K$ and using $g(0) = \frac{-1}{w+1}$, we get

$$K = \frac{-1}{w+1}.$$

Thus we get

$$(w+1)(f(x+y)-f(x))(f(x+\frac{w+1}{2}y)-f(x+\frac{w^2+1}{2}y))+(f(x+wy)-f(x))(f(x+\frac{w^2+w}{2}y)-f(x+\frac{w+1}{2}y)) = 0 \quad (2.4)$$

Differentiating both sides of (2.4) three times with respect to y and setting $y = 0$, we get

$$f'(x)f''(x) = 0.$$

Since $f'(x) \neq 0$, we obtain that

$$f''(x) = 0,$$

which implies that f must be a similarity, that is it must be of the form

$$f(z) = az + b$$

for some $a, b \in \mathbb{C}$ with $a \neq 0$. □

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