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Numerical Analysis of the Ground-State Magnetic Moments of $^{143,145,147}\mathrm{Sm}$ Isotopes

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Abstract
The description of the ground-state magnetic properties of odd-mass nuclei is very informative in understanding of the complex structure of the deformed nuclei. The ground-state magnetic moments of most of the odd-A deformed nuclei have been measured by various experimental studies and there are numerous studies in the literature. However, many of the theoretical studies on magnetic moments and spin polarization effects affecting them are far from explaining these measured values. In this paper, the magnetic moments and effective spin g factors of $^{143,145,147}\mathrm{Sm}$ isotopes in the lanthanides region of the periodic table were investigated within the framework of the Quasiparticle-Phonon Nuclear Model (QPNM) based on the Lagrange Multiplier Method for the first time. Spin-spin interaction parameters (χ) were determined by comparing theoretical and experimental values of magnetic moments of the related isotopes and these interactions were found to have an isovector character $\mathrm{q} = -1$. It has been observed that the ground-state structures of the studied isotopes are weakly affected by quasiparticle$\otimes$phonon interactions and the contribution of these interactions ($G_{i\mu}^{K_0 v}$ values) to the ground-state wave functions is quite small (around 0.01%). Theoretical explanation of the renormalization of spin gyromagnetic factor is one of the most important problems of nuclear structure physics. The results obtained in this study for the effective spin gyromagnetic factor $g_s^{eff.}/g_s^{\tau}$ also agree with the phenological value $g_s^{eff.} = (0.5-0.7)g_s^{\tau}$.

Keywords: *Sm, Lagrange Multiplier Method, Magnetic Moment, Spin polarization*

AMS Subject Classification (2020): *17B81*

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1. Introduction

The theoretical investigation of the ground-state magnetic properties of deformed odd-A nuclei plays an important role in understanding the complex structure of these nuclei and in determining the success of existing nuclear models [1-8].
It shows systematically large deviations from the shell model calculations (Schmidt curves) of the experimentally observed magnetic moments. The theoretical explanation of these deviations in nuclear magnetic moments has been attempted by many nuclear physicists.
The interaction of the magnetic dipole (M1) excitations of the even-even core nuclei with the single remaining nucleon of the odd-A nuclei is considered in 1950s [9]. This approach is known as the core polarization phenomenon or Arima-Horie effect in the literature [10]. Besides, the effect of mesonic interactions on magnetic moments has been also discussed by many researchers, and this effect has been found to be important in the renormalization of spin orbital gyromagnetic factor [11,12]. Therefore, the mesonic effects are particularly important in the investigation of high spin levels where the main contribution to the magnetic moment comes from the orbital gyromagnetic part [13].
According to core polarization, the spin-dependent part of the interaction between the unpaired nucleon in the outer shell and the nucleons in the core tends to align the spins of the unlike nucleons parallel and the spins of the like

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nucleons antiparallel with the spin of the single-nucleon [14]. Studies made so far show that the interaction between the spin of the single nucleon and the spin of the nucleons in the core affects the magnetic properties of the odd-mass nuclei. The contribution of the spin part to the magnetic moment decreases due to core polarization, which explains the deviation of the theoretical predictions from the observed magnetic moment (μ) values [14-20]. Many theoretical studies for M1 transitions in deformed nuclei show that results consistent with the experiment are obtained by using the effective spin gs factors $\left(g_s^{eff.} = 0.6 - 0.7 g_s^{free}\right)$ instead of the free nucleon spin gymagnetic factor in calculations. This is one of the clearest evidence that spin polarization is important [11]. The first theoretical studies for the solution of this problem were carried out by Bochnacki and Ogaza using the perturbation method under the assumption that spin-spin interactions are responsible for the renormalization of single particle spin matrix elements [15,17]. However, since spin-spin interactions between quasiparticles are not weak, the perturbation method cannot properly explain quasiparticle interactions and magnetic moments [21]. In the 1970s, spin polarization effects were studied by Kuliev and Pyatov using the Nilsson potential in the framework of the Tamm-Dancoff approach (TDA). In their study, spin polarization effects in odd-A nuclei have been interpreted as a result of scattering of the unpaired nucleon in the outermost shell over 1^+ excitations in the even-even core nuclei [14, 19-21].

A number of experimental studies on ground-state magnetic moments for $^{143,145,147}\mathrm{Sm}$ isotopes have been reported in the past. The comparison of our theoretical results with the experimental data is very important to confirm the accuracy of our theoretical predictions. In 1990, England et al. measured the magnetic moments of $^{143,145,147}\mathrm{Sm}$ nuclei by the cross beam laser fluorescence method [22]. In 1992, measurements for three Sm isotopes were made using the resonance ionization spectroscopy method by Letokhov et al. [23].

In addition to these, Kaplan et al., Woodgate and Childs et al. observed the magnetic moments of $^{145}\mathrm{Sm}$ and $^{147}\mathrm{Sm}$in experiments [24-26].

Since the spin-spin interaction potential is commutative with the total angular momentum operatör, the interaction strength parameter can not be determined based on the relation between rotation and spin-spin operators In many studies using the spin-spin interactions, the interaction strength parameter (χ) is determined by fitting with the experiment [27-33]. However, the validity of this application is controversial. A practical method based on QPNM, which takes into account the quasiparticle⊗phonon interactions, has been developed to determine the spin-spin interaction power parameter by Yakut [21]. This method also successfully explains the spin polarization phenomenon observed in odd-mass nuclei. In this method, the spin-spin interaction parameter is obtained by comparing the calculated ground-state intrinsic magnetic moment (gK) with the experimental value of the odd-mass nucleus. Method were successfully applied to explain the ground state magnetic properties of $^{157-167}\mathrm{Er}, ^{165}\mathrm{Dy}, ^{165-179}\mathrm{Hf}, ^{137-145}\mathrm{Ce}, \mathrm{Ho}, ^{239}\mathrm{Pu}, ^{183,185}\mathrm{W}, ^{185,187}\mathrm{Re}$ and $^{187,189}\mathrm{Os}$ [1-8].

In the present study, the ground state magnetic properties of odd-A $^{143-147}\mathrm{Sm}$ isotopes were firstly investigated using the method developed in the framework of the QPNM method. The results obtained from the theoretical calculations were compared with the experimental values and it was seen that they were in agreement with each other.

In the second part of this paper, the theoretical expressions used in the calculations are given. In the last part, numerical results and discussions are presented.

2. Method (Theory)

For a system where nucleons interact with the pairing and spin-spin forces in the axially symmetric deformed mean field, the QPNM hamilton is in the following form:

$$H = H_{sqp} + H_{coll.} + H_{int.} \tag{2.1}$$

here;

$$H_{sqp} = \sum_{s,\tau} \varepsilon_s(\tau) \alpha_{s\rho}^{+} \alpha_{s\rho} \tag{2.2}$$

$$H_{coll.} = \frac{1}{2} \sum_{\tau,\tau'} \chi_{\tau\tau'} \sum_{ss'} \sigma_{ss'}^{(\mu)} L_{ss'} g_{ss'}^{i} \left(Q_i^{+} + Q_i\right) \sum_{mm'} \sigma_{mm'}^{(\mu)} L_{mm'} g_{mm'}^{i} \left(Q_i^{+} + Q_i\right) \tag{2.3}$$

$$H_{\text{int.}} = \sum_{\tau,\tau'} \chi_{\tau\tau'} \sum_{mm'} \sum_{ss'} \left\{ \sigma_{ss'}^{(\mu)} M_{ss'} \sigma_{mm'}^{(\mu)} L_{mm'} g_{mm'}^{i} D_{ss'}(\tau) \left(Q_i^{+} + Q_i\right) + + \sigma_{ss'}^{(\mu)} L_{ss'} \sigma_{mm'}^{(\mu)} M_{mm'} g_{ss'}^{i} \left(Q_i^{+} + Q_i\right) D_{mm'}(\tau') \right\} \tag{2.4}$$

The first term H_{sqp} refers to the single-quasparticle motion in the nucleus and includes the axially symmetrically deformed Woods-Saxon potential. While $H_{coll.}$ represents phonon excitations in double-even core, the term $H_{\text{int.}}$ refers to the relationship between one-quasiparticle and collective motions.

The wave function for K>1/2 projection of the total angular momentum on the symmetry axis in a odd-A nuclei can be written as follows [1-8, 21];

$$\psi_K^j(\tau)=\left\{\sum_q N_{\varsigma_q}^j(\tau)\alpha_{\varsigma_q}^+(\tau)+\sum_{i\mu}\sum_\nu G_j^{i\mu\nu}\alpha_\nu^+(\tau)Q_{i\mu}^+\right\}|\psi_0\rangle\,;\ \mu=\pm 1 \tag{2.5}$$

Here, N_K^j and $G_{ij}^{KK\nu}$ are the amplitudes of single-quasiparticle and quasiparticle⊗fonon states, respectively. ψ_0 is the phonon vacuum representing the ground-state of the even-even core. This wave function should provide the normalization condition as given below.

$$\left\langle \psi_K^j(\tau)\,\middle|\,\psi_K^j(\tau)\right\rangle=\sum_q\left(N_{\varsigma_q}^j\right)^2(\tau)+\sum_{i\mu}\sum_\nu\left(G_j^{i\mu\nu}\right)^2=1. \tag{2.6}$$

Using Lagrange Multipliers Method (Variation Method) [34,35] given by the expression (2.7),

$$\delta\left\{\begin{array}{l}\left\langle\psi_K^j(\tau)\right|H_{inv.}\left|\psi_K^j(\tau)\right\rangle-\left\langle\psi_{K_0}(\tau)\right|H_{inv.}\left|\psi_{K_0}(\tau)\right\rangle\\ -\eta_K^j\left[\sum\limits_q\left(N_{\varsigma_q}^j\right)^2+\sum\limits_{i\mu}\sum\limits_\nu\left(G_j^{i\mu\nu}\right)^2-1\right]\end{array}\right\}=0 \tag{2.7}$$

the following secular equation is obtained when the equations obtained for N_K^j and $G_{ij}^{KK\nu}$ amplitudes are solved [21]:

$$P(\eta_K^j)\equiv\varepsilon_K-\eta_K^j-\sum_i\sum_\nu\frac{1}{(\chi F_n)^2 Z(\omega_i)}\frac{M_{KK_\nu}^2\sigma_{KK_\nu}^2}{\varepsilon_{K_\nu}+\omega_i-\eta_K^j}=0. \tag{2.8}$$

Here, the η_K^j roots of the equation (2.8) are energies of the odd-A nucleus. The $Z(\omega_i)$, $Y_\tau(\omega_i)$ and $F_\tau(\omega_i)$ abbreviations in Eq.(2.8) are also given below;

$$\begin{array}{l}Z(\omega_i)=\frac{1}{(-\chi F_n)^2}Y_n(\omega_i)+\frac{q^2}{(1+\chi F_p)^2}Y_p(\omega_i)\\ Y_\tau(\omega_i)=4\omega_i\sum\limits_{ss'}\frac{\varepsilon_{ss'}\sigma_{ss'}^2L_{ss'}^2}{(\varepsilon_{ss'}^2-\omega_i^2)^2}\,,\,F_\tau(\omega_i)=2\sum\limits_{ss'}\frac{\varepsilon_{ss'}\sigma_{ss'}^2L_{ss'}^2}{\varepsilon_{ss'}^2-\omega_i^2}.\end{array} \tag{2.9}$$

As known, the intrinsic magnetic moment of the odd-A nucleus is the expected value of the z component of the magnetic dipole operator [1-8]. The value of gK was obtained as follows after long and complex operations within the framework of QPNM.

$$\begin{array}{l}\mu_{K_0}=g_{K_0}K_0=\left\langle\psi_{K_0}^j(\tau)\right|\mu_z\left|\psi_{K_0}^j(\tau)\right\rangle=\left\{g_s^\tau\left(1-2N_{K_0\sigma_0}^2\sum\limits_{i,\nu}\frac{\chi M_{K_0\sigma_0 v}R_q^i(\tau,\tau')R_\tau^i}{(\varepsilon_v+\omega_i-\eta_{K_0}^j)}\right)-\right.\\ \left.-2\,(g_s^{\tau'}-g_\ell^{\tau'})N_{K_0\sigma_0}^2\sum\limits_{i,\nu}\frac{\chi M_{K_0\sigma_0 v}R_q^i(\tau,\tau')R_{\tau'}^i}{(\varepsilon_v+\omega_i-\eta_{K_0}^j)}\right\}\frac{\sigma_{K_0\sigma_0 v}^{(0)}}{2}+g_\ell^{\tau'}K_0\end{array} \tag{2.10}$$

When this expression is compared with the Nilsson formula,

$$\mu_{K_0}=g_{K_0}K_0=\frac{1}{2}(g_s^\tau-g_\ell^\tau)\sigma_{K_0K_0}^{(0)}(\tau)+g_\ell^\tau K_0\,(\tau) \tag{2.11}$$

the following analytical expression is obtained for the effective spin gyromagnetic factor [1-3]:

$$\begin{array}{l}g_s^{eff}-g_\ell^\tau=(g_s^\tau-g_\ell^\tau)\left\{1-2N_{K_0}^2\sum\limits_{i,\nu}\frac{\chi M_{K_0\nu}R_q^i(\tau,\tau')R_\tau^i}{(\varepsilon_\nu+\omega_i-\eta_{K_0}^j)}\right\}-\\ \qquad\qquad -2(g_s^{\tau'}-g_\ell^{\tau'})N_{K_0}^2\sum\limits_{i,\nu}\frac{\chi M_{K_0\nu}R_q^i(\tau,\tau')R_{\tau'}^i}{(\varepsilon_\nu+\omega_i-\eta_{K_0}^j)}\Bigg\}\end{array} \tag{2.12}$$

Here, the unpaired nucleon in the outer shell of the od-mass nucleus causes polarization by scattering from 1^+ one-phonon states of the even-even nucleus. The second and third terms to the right of Eq.(2.12) refer to the coherent contribution arising from the quasiparticle-phonon interactions in polarized core. There is a significant reduction in the spin factor due to the contribution from spin polarization. Details of the theory and analytical expressions are available in Reference [1-8, 21].

3. Results and Discussion

In the study, the method developed by Yakut [21] and the analytical expressions given in section 2 were used in the calculations of the ground-state magnetic properties of $^{143,145,147}\mathrm{Sm}$ nuclei.

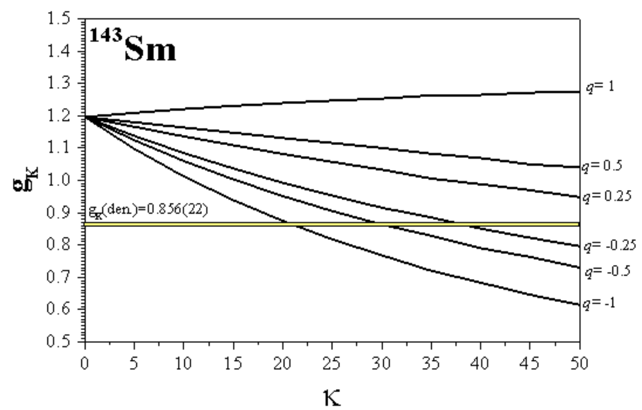
The $\mathrm{I}^{\pi}\mathrm{K}=1^{+}0$ phonons of even $^{142,144,146}\mathrm{Sm}$ for the ground-state calculations of $^{143,145,147}\mathrm{Sm}$ are used. The reason for this can be said that the ground-state and some low-lying excitations in odd-A nuclei are formed by the interaction of unparired nucleon and K = 0 phonons [36]. Quasiparticle energies are obtained by the solution of the deformed Woods-Saxon potential. Mean field deformation parameters (δ_2) are calculated using the (β_2) deformation parameters obtained from the experimental quadrupole moments in Ref.[37]. The calculation was carried out with $\delta_2 = 0,945\beta_2\left[1-2,56A^{-2/3}\right]+0,34\beta_2^2$ [38]. Pairing interaction constants have been taken from Ref. [31]. The pairing parameters $(\Delta \mathrm{and} \lambda))$, the mean field deformations (δ_2), the ground-state Nilsson configurations and the experimental ground-state magnetic moments [39] for $^{143,145,147}\mathrm{Sm}$ isotopes are given in Table 1.

Nucleus.	$\mathrm{I}^{\pi}\mathrm{K}\,[\mathrm{Nn_z}\Lambda]$	β_2	δ_2	Δ_n (MeV)	Δ_p (MeV)	λ_n	λ_p	$\mu_{\mathrm{exp.}}$
$^{143}\mathrm{Sm}$	$3/2^{+}[402]\downarrow$	0.354	0.306	1.078	1.322	-9.124	-4.778	1.01(2)
$^{143}\mathrm{Sm}$	$7/2^{-}[523]\uparrow$	0.144	0.124	1.046	1.318	-8.825	-5.071	-1.11(6)
$^{143}\mathrm{Sm}$	$7/2^{-}[503]\uparrow$	0.143	0.123	0.950	1.218	-7.809	-5.725	-0.81(2)

Table 1. Δ and λ pairing correlation quantities, mean field deformations (δ_2), Nilsson configurations and experimental ground state magnetic moments (μ_{exp}) for $^{143,\,145,\,147}\mathrm{Sm}$

As known from the literature, the isovector part of the magnetic dipole operator is more dominant than the isoscaler part (about 12 times) [32,38]. By comparing the theoretical and experimental magnetic moments of the 143,145,147Sm, the good results are obtained when q = -1. This overlaps with the presence of isovector character of magnetic dipole interactions [32]. The spin-spin interaction parameter χ, in which the theoretical and experimental magnetic moments are in agreement, has been found between the 20 and 30 MeV/A. As an example, the variation of the calculated gK values of $^{143}\mathrm{Sm}$ isotope according to "$\kappa=\chi.$A" and q parameters is presented in Figure 1.

Figure 1. Calculation of gK values for the $^{143}\mathrm{Sm}$ isotope according to "$\kappa=\chi.A$" and "q" parameters



The calculations made using Kuliev-Pyatov Method (KPM), Single-Particle Model (SPM) and QPNM method (QRPA and QTDA approaches) mentioned in the introduction section and their comparison with the experimental data are given in Table 2.

	$\mathrm{g_s}^{\mathrm{eff}}/\mathrm{g_s}^{\mathrm{n}}$			gk				
Nuc.	KPM	QTDA	QRPA	SPM	KPM	QTDA	QRPA	Deney
$^{143}\mathrm{Sm}$	0.693	0.694	0.731	1.196	0.829	0.830	0.874	0.856(22)
$^{143}\mathrm{Sm}$	0.657	0.657	0.705	-0.413	-0.271	-0.272	-0.291	-0.436(22)
$^{143}\mathrm{Sm}$	0.642	0.643	0.694	-0.320	-0.205	-0.205	-0.222	-0.327(7)

Table 2. Theoretical (KPM, SPM, QTDA, QRPA) results and experimental values for $^{143,\,145,\,147}\mathrm{Sm}$ isotopes

The results show that QPNM calculations on the basis of QRPA are more successful than other theoretical predictions in explaining exp. data. The difference between predictions of the single-particle model and the other theoretical models is due to neglecting the contributions from the quasiparticle-phonon interactions in the single-particle model [1-3]. Now, let us briefly remind you the basic principles of KPM method in order to compare QPNM calculations with QRPA and QTDA based in detail. Let us briefly remind you the basic principles of the KPM method in order to compare the KPM results with QPNM calculations on the basis of QRPA and QTDA in detail. The KPM method is based on the independent quasiparticles model and explains the ground-state magnetic properties of odd-mass nuclei within the framework of TDA, based on the assumption of the magnetic dipole vibrations in the core nuclei [14-16]. The wave function of the odd-A nucleus in KPM consists of single and three quasiparticle components, respectively. One of the reasons why the g_s^{eff}/g_s^{τ} and gK values calculated with KPM method differ from the QPNM (QRPA) results is that only quasiparticle interactions are taken into account in KPM.

Studies show that taking into account the interactions between the quasiparticle and phonons contributes significantly to the appropriate depiction of the level structure of odd-A nuclei [36,39]. Another reason for the difference of QRPA results from other estimates is the fundamental differences between their ground-states. As it is known, while the ground-state of QTDA is independent quasiparticle vacuum, QRPA also takes into account interactions between the quasiparticles. This situation causes asymmetric behavior of ground- and excited states in QTDA and KPM calculations. Considering base-state interactions in calculations is very importance in accurately depicting 1+ vibration levels. On the other hand, as expected, the results of QTDA and KPM are very close to each other. As mentioned just before, the BCS vacuum in both approaches is like the double-even core of odd-mass nucleus [1-3].

As can be seen from Table 2, QRPA results for all nuclei are in the range of fenemological g_s values $g_s^{eff.} = (0.5-0.7)g_s^{\tau}$. One of the most important problems of nuclear physics is the explanation of the renormalization of spin gyromagnetic factor. Spin gyromagnetic factor not only has an effect on the magnetic moments of nuclei, but also plays an active role in calculating the magnetic dipole excitations in odd-A and even N-even Z nuclei. In order to solve the renormalization problem in the microscopic approach presented here, the interaction between the unpaired nucleon and the double-even core phonons must be accurately determined.

The one quasiparticle $\left(N_{K_0}\right)$ and quasiparticle⊗phonon $\left(G_{i\mu}^{K_0 v}\right)$ amplitudes in the wave functions of $^{143-147}$Sm isotopes, the ground-state nuclear structure and the $1^{+}0$ phonon energies of even-even core are given in Table 3. The quasiparticle-phonon and two quaziparticle amplitudes $\psi_{ss'}^{i}$ given in the table are only given for states whose contribution to the wave function is greater than 1%.

Nucleus	K^{π}	The structure of low-lying (ground) states		Two - quasiparticle characteristics of the even - even phonon core		
		Single-quasiparticle	Quasiparticle⊗phonon	ω_i (MeV)	$[\mathrm{Nn_z}\Lambda]\sum$	ψ_{ss}
^{143}Sm	$3/2^{+}$	99.80% $[402]\downarrow$	0.010% $[402]\downarrow\otimes Q_6$	5.089	pp420 $\uparrow -411\downarrow$	-0.700
			0.012% $[402]\downarrow\otimes Q_9$	5.874	nn411 $\uparrow -402\downarrow$	0.202
					pp550 $\uparrow -550\uparrow$	0.648
			-0.012% $[402]\downarrow\otimes Q_{20}$	7.477	nn523 $\downarrow -512\uparrow$	-0.196
					pp431 $\uparrow -411\uparrow$	0.439
			-0.011% $[402]\downarrow\otimes Q_{23}$	7.708	nn541 $\downarrow -521\downarrow$	0.650
					pp550 $\uparrow -530\uparrow$	-0.122
^{145}Sm	$7/2^{-}$	99.9% $[523]\uparrow$	-0.013% $[523]\uparrow\otimes Q_3$	3.905	pp420 $\uparrow -411\downarrow$	-0.705
			0.011% $[523]\uparrow\otimes Q_{20}$	7.071	nn532 $\uparrow -523\downarrow$	0.618
					pp413 $\uparrow -404\downarrow$	-0.148
			0.015% $[523]\uparrow\otimes Q_{35}$	8.421	nn541 $\downarrow -501\downarrow$	0.314
					pp530 $\uparrow -541\downarrow$	-0.465
^{147}Sm	$7/2^{-}$	99.9% $[503]\uparrow$	0.012% $[503]\uparrow\otimes Q_{39}$	8.326	nn550 $\uparrow -530\uparrow$	0.121
					pp431 $\uparrow -422\downarrow$	-0.471

Table 3. Ground-state structures calculated on QPNM(QRPA) of $^{143,\,145,\,147}$Sm nuclei

It is known that one-quasiparticle and qp-phonon components contribute to ground- and excited states as a result of qp⊗phonon interactions in odd-A nuclei [40]. However, it can be seen from the results in Table 3 that the ground-states of the nuclei are weakly affected by the qp-phonon interactions, and almost all contribution comes from the one- quasiparticle components. In addition, as emphasized in Ref. [41], the ground-and low-lying excitation- states of the nuclei are weakly affected by qp-phonon interactions. In general, the main contribution to $G_{i\mu}^{K_0 v}$ amplitude comes from $\mathrm{I}^{\pi}\mathrm{K} = 1^{+}0$ phonons of M1 dipole resonance in the 6–11 MeV energy range. Calculations show that N_{K_0} amplitude χ is very poorly dependent on the interaction parameter.

As a result, it is very important to choose suitable nuclear interactions in magnetic moment calculations. Since spin-spin residual interactions play an active role in renormalization of g_s factors, we take into account spin-dependent interactions in our calculations. The results obtained reveal that spin interactions allow to successfully explain the spin polarization phenomenon observed in odd-mass nuclei and its effects on ground-state magnetic properties [1-3]. The applications in physics have been mostly in the field of general and differential equation [42-44].

The importance of spin-spin forces is not only limited to the description of the spin polarization effect and ground-state magnetic properties in odd-A nuclei, it is also known that the spin-dependent part of neutron-proton interactions is responsible for the Gammow-Teller β transitions between the low energy levels of the nuclei [45, 46]. More importantly, the studies performed to now show that spin-spin forces have successfully explained 1+ excitations and scissor mode in even-even nuclei [47-56]. For these reasons, it is very important to determine the spin-spin interaction parameters that are the subject of this article.

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