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Some Remarks on a Direct Calculation of Probabilities in Urn Schemes

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Abstract
The paper considers urn schemes in which several urns can be involved. Simplified formulas are proposed that allow direct calculation of probabilities without the use of elements of combinatorics.

Keywords: *Urn; urn schemes; binomial coefficients.*

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Introduction

The urn schemes is one of the simplest models used in the elementary probability theory. Using urn schemes, it is convenient to calculate some basic probabilities through conditional probabilities. In most cases, when solving various problems, a model with a single urn is considered. For various values of the parameters of the scheme, many well-known schemes of probability theory are obtained, in particular, a random choice scheme with return as Bernoulli tests, a random choice without return scheme, Ehrenfest diffusion model [1], and Pólya urn models. These schemes serve as models of many real phenomena, as well as methods for their investigation; see, for example, [2], [3] and [4].

One of the simplest models of urn schemes is the following model. Consider two urns, which we denote by the symbols Π_0 and Π respectively. In both urns there are a lot of white and black balls. The following operation is allowed. The random number ξ of balls are taken out from the urn Π_0 in a random order, and they are transferred to the urn Π.

Scheme $[\mathbf{A}]$. Let urn Π_0 contains a white and b black balls, and there are c white and d black balls in the urn Π. We consider special case $\xi = 1$: one ball is taken from urn Π_0 at random and transferred to urn Π. Henceforth we denote by A the event that the taken out ball to be white. We illustrate Scheme $[\mathbf{A}]$ in Figure 1.

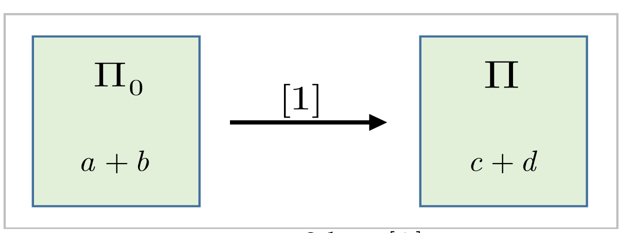

Figure 1. *Scheme* $[\mathbf{A}]$

The probability

$$\mathrm{P}_{\Pi}\left(A \middle| \xi = 1\right)$$

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of that the ball randomly taken out from urn Π, under condition $\xi = 1$, to be white, according to the classical scheme, is calculated by the following formula:

$$\begin{aligned} \mathrm{P}_{\Pi}\left(A \mid \xi = 1\right) &= \frac{1}{\mathrm{C}_{a+b}^{1}}\left(\frac{c+1}{c+d+1}\mathrm{C}_{a}^{1} + \frac{c}{c+d+1}\mathrm{C}_{b}^{1}\right) \\ &= \frac{a+(a+b)c}{(a+b)(c+d+1)} = \frac{c}{c+d+1} + \frac{a}{a+b}\frac{1}{c+d+1}. \end{aligned} \tag{1}$$

where $\mathrm{C}_{n}^{m} = \frac{n!}{m!(n-m)!}$ – Newton's binomial coefficients. In fact, this is a well-known elementary formula for calculating probability in scheme [**A**]. But its generalization can serve as a starting point for the emergence of new formulas for the calculation and geometric interpretation of probabilities in some urn schemes.

In this note we consider some modifications of scheme [**A**] in which a sequence of urns can be involved, and we propose formulas that simplify the method of directly calculating probabilities without using elements of combinatorics.

Modifications of scheme [A]

In this section, we consider some schemes that be generalization of scheme [**A**] .

First we need the following assertion.

Lemma. *The following equality is fair:*

$$\frac{a+b}{a}\frac{1}{\mathrm{C}_{a+b}^{k}}\sum_{i=1}^{k} i\mathrm{C}_{a}^{i}\mathrm{C}_{b}^{k-i} = k. \tag{2}$$

Proof. Initially we are convincing that according to the properties of binomial coefficients, the following relations are true:

$$\frac{k}{a+b}\mathrm{C}_{a+b}^{k} = \mathrm{C}_{a+b-1}^{k-1}$$

and

$$\sum_{i=0}^{k}\mathrm{C}_{a}^{i}\mathrm{C}_{b}^{k-i} = \mathrm{C}_{a+b}^{k}. \tag{3}$$

Using these relations, we have

$$\frac{1}{\frac{k}{a+b}\mathrm{C}_{a+b}^{k}}\sum_{i=1}^{k} i\mathrm{C}_{a}^{i}\mathrm{C}_{b}^{k-i} = \frac{1}{\mathrm{C}_{a+b-1}^{k-1}}\sum_{i=1}^{k} i\mathrm{C}_{a}^{i}\mathrm{C}_{b}^{k-i} = \frac{\sum_{i=1}^{k} i\mathrm{C}_{a}^{i}\mathrm{C}_{b}^{k-i}}{\sum_{i=0}^{k-1}\mathrm{C}_{a-1}^{i}\mathrm{C}_{b}^{k-i-1}}.$$

On the other hand

$$\sum_{i=1}^{k} i\mathrm{C}_{a}^{i}\mathrm{C}_{b}^{k-i} = a\sum_{i=1}^{k}\mathrm{C}_{a-1}^{i-1}\mathrm{C}_{b}^{k-i} = a\sum_{i=0}^{k-1}\mathrm{C}_{a-1}^{i}\mathrm{C}_{b}^{k-i-1}.$$

Therefore

$$\frac{1}{\frac{k}{a+b}\mathrm{C}_{a+b}^{k}}\sum_{i=1}^{k} i\mathrm{C}_{a}^{i}\mathrm{C}_{b}^{k-i} = a.$$

The last equality is equivalent to (2). □

Scheme [**A**(k)]. Consider two urns with the compositions from scheme [**A**]. From urn Π_0 randomly taken out $\xi = k$ number of balls, and they are transferred to urn Π, here k is natural number is such that $k \leq a+b$. In this case, formula (1) is generalized in the following theorem. Denote

$$\alpha := \frac{c}{c+d+k} \quad \text{and} \quad \beta := \frac{c+k}{c+d+k}.$$

Theorem 1. *For scheme $[\boldsymbol{A}(k)]$, the probability $P_{\Pi}\,(A|\,\xi=k)$ that the ball, randomly taken out from urn Π, turns out to be white, is calculated by the following formula:*

$$P_{\Pi}\,(A|\,\xi=k)=\alpha+(\beta-\alpha)\,\theta, \tag{4}$$

where $\theta=P_{\Pi_0}(A)$ is the probability of an appearance of the white ball in the urn Π_0.

Remark. *The expression on the right-hand side of formula (4) resembles intermediate points (values) in the Lagrange formula in mathematical analysis. It's obvious that*

$$\alpha\le P_{\Pi}\,(A|\,\xi=k)\le\beta$$

and the numbers α and β are nothing more than the smallest and greatest values of all possible probabilities of the appearance of white ball in urn Π, provided that $\xi=k$ balls are randomly transferred from urn Π_0 to urn Π. It can be seen from the formula (4) that probability $P_{\Pi}\,(A|\,\xi=k)$ is in such internal point of the "value interval" $[\alpha,\beta]$ that it divides this interval proportionally to θ; see Figure 2 below.

Proof of Theorem 1. We first consider a case $\xi=2$. In this case by direct calculation we find

$$\begin{aligned}\mathrm{P}_{\Pi}\,(A|\,\xi=2) &= \frac{(c+2)\mathrm{C}_a^2+(c+1)\mathrm{C}_a^1\mathrm{C}_b^1+c\mathrm{C}_b^2}{(c+d+2)\mathrm{C}_{a+b}^2}\\ &= \frac{2a+(a+b)c}{(a+b)(c+d+2)}=\frac{c}{c+d+2}+\frac{a}{a+b}\frac{2}{c+d+2}.\end{aligned}$$

For arbitrary $\xi=k$, according to standard reasoning

$$\mathrm{P}_{\Pi}\,(A|\,\xi=2)=\frac{1}{(c+d+k)\mathrm{C}_{a+b}^k}\sum_{i=0}^{k}(c+i)\mathrm{C}_a^i\mathrm{C}_b^{k-i}. \tag{5}$$

Using formula (2) and equality (3) we transform the expression on the right-hand side of equality (5) to the form

$$\begin{aligned}\frac{1}{\mathrm{C}_{a+b}^k}\sum_{i=0}^{k}(c+i)\mathrm{C}_a^i\mathrm{C}_b^{k-i} &= \frac{c}{\mathrm{C}_{a+b}^k}\sum_{i=0}^{k}\mathrm{C}_a^i\mathrm{C}_b^{k-i}+\frac{1}{\mathrm{C}_{a+b}^k}\sum_{i=0}^{k}i\mathrm{C}_a^i\mathrm{C}_b^{k-i}\\ &= c+\frac{ak}{a+b}.\end{aligned}$$

Applying last equality in (5), we have

$$\mathrm{P}_{\Pi}\,(A|\,\xi=k)=\frac{c}{c+d+k}+\frac{a}{a+b}\frac{k}{c+d+k}. \tag{6}$$

Since

$$\mathrm{P}_{\Pi_0}(A)=\frac{a}{a+b}$$

according to our notation, the equality (6) is equivalent to (4).

Theorem 1 is proved. □

Continuing the discussion of Scheme $[\mathbf{A}(k)]$, we now set eyes on the geometric interpretation of Theorem 1. In the Figure 2, in the orthogonal coordinate system we put all possible values of probability $\theta=\mathrm{P}_{\Pi_0}(A)$ along the vertical axis. On the horizontal axis we place the "value interval" $[\alpha,\beta]$ of probability $\mathrm{P}_{\Pi}\,(A|\,\xi=k)$. Draw the straight line l_α through points α and β'. Denote P the intersection point of line l_α with the horizontal one l_θ passing through the point θ.

According to the similarity criterion of triangles, the following equalities are true:

$$\frac{\mathrm{P}_{\Pi}-\alpha}{\beta-\alpha}=\frac{|\alpha\mathrm{P}|}{|\alpha\beta'|}=\frac{|\mathrm{P}_{\Pi}\mathrm{P}|}{|\mathrm{P}_{\Pi}\mathrm{P}'|}=\theta.$$

This relation immediately implies formula (4). The conclusion made corresponds with the geometric definition of probability, since

$$\mathrm{P}_{\Pi}\,(A|\,\xi=k)=\frac{area\ of\ quadrangle\ S_{\mathrm{P}}}{area\ of\ unitary\ square\ S}.$$

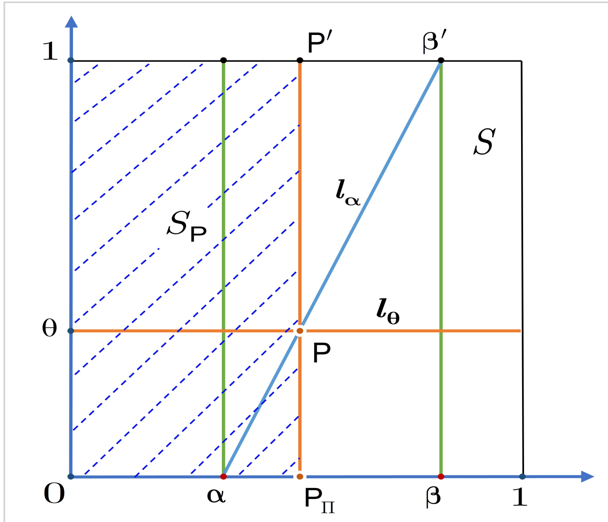

Figure 2. *Geometric interpretation of* $\mathrm{P}_\Pi\,(\,A|\,\xi = k)$

Now in Scheme $[\mathbf{A}(k)]$ we introduce the following notation: let in urn Π_0 contains M balls, of which a are white, and in urn Π there are N balls, of which c are white; see Figure 3.

In these designations Theorem 1 can be reformulated as following.

Theorem 2. *In scheme $[\boldsymbol{A}(k)]$, the probability $P_\Pi\,(\,A|\,\xi = k)$ that the ball, randomly taken out from urn Π, turns out to be white, is calculated by the following formula:*

$$\mathrm{P}_\Pi\,(\,A|\,\xi = k) = \frac{\theta k + c}{N + k}, \tag{7}$$

where $\theta = a/M$.

Proof of Theorem 2. Formula (7) is just a rewritten form of formula (6), taking into account the notation of the theorem. In fact, $a + b = M$ and $c + d = N$. Therefore, we obtain the following:

$$\mathrm{P}_\Pi\,(\,A|\,\xi = k) = \frac{c}{c+d+k} + \frac{a}{a+b}\frac{k}{c+d+k} = \frac{\theta k + c}{N + k}.$$

Theorem 2 is proved. □

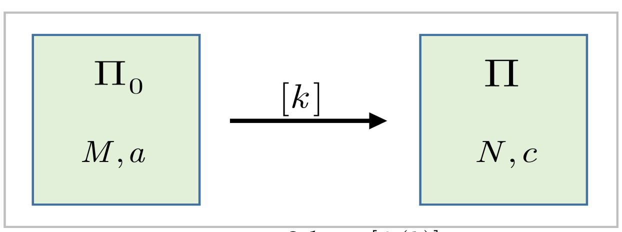

Figure 3. *Scheme* $[\mathbf{A}(k)]$

Exercise 1. *There are 15000 details in first warehouse Π_0, from them 11850 are standard. The second warehouse Π contain 17000 details, from them 15800 are standard. 10000 details were randomly taken from the warehouse Π_0 and transferred to the second one. Find the probability of the event that the detail taken randomly from the second warehouse Π, to be appeared standard.*

Solution. We find the desired probability by the formula (7) as follows:

$$\mathrm{P}_\Pi\,(\,A|\,\xi = 10000) = \frac{\frac{11850}{15000}10000 + 15800}{27000} = \frac{79}{90}$$

or the same as $P_{\Pi}(A|\xi = 10000) = 79/90 \approx 0.878$. □

Scheme $[\mathbf{A}_n(\mathbf{k})]$. There are $n+1$ urns: Π and $\Pi_1, \Pi_2, \ldots, \Pi_n$. In urn Π_1 there are M_1 balls, of which a_1 are white; in urn Π_2 there are M_2 balls, of which a_2 are white; and so forth, in urn Π_n there are M_n balls, of which a_n are white. In urn Π there are N balls, of which c are white. From all urns $\Pi_1, \Pi_2, \ldots \Pi_n$ randomly selected $k_1, k_2, \ldots, k_n$ balls respectively, and they are transferred to urn Π. Introduce the random vector $\mathbf{x} = (\xi_1, \xi_2, \ldots, \xi_n)$ characterizing the number of balls taken out of n numbered urns and transferred to urn Π. The event $\{\mathbf{x} = \mathbf{k}\}$ means that the number of balls transferred to urn Π is equal to $\sum_{i=1}^{n} k_i$, the sum of the components of the vector $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, where all $k_i \geq 0$. Scheme $[\mathbf{A}_n(\mathbf{k})]$ is a natural generalization of scheme $[\mathbf{A}(k)]$; see Figure 4.

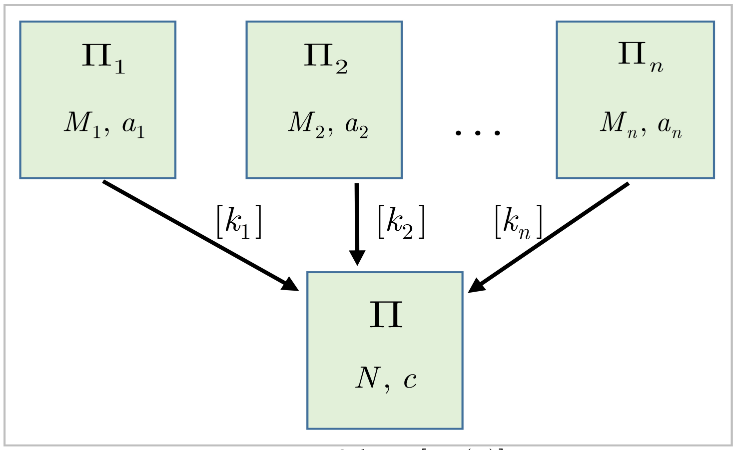

Figure 4. *Scheme* $[\mathbf{A}_n(\mathbf{k})]$

In this case we have the following statement generalizing of Theorem 2.

Theorem 3. *In scheme* $[\mathbf{A}_n(\mathbf{k})]$*, the probability* $P_{\Pi}(A|\mathbf{x} = \mathbf{k})$ *that the ball, randomly taken out from urn* Π*, turns out to be white, is calculated by the following formula:*

$$P_{\Pi}(A|\mathbf{x} = \mathbf{k}) = \frac{\theta_1 k_1 + \theta_2 k_2 + \cdots + \theta_n k_n + c}{N + k_1 + k_2 + \cdots + k_n}, \tag{8}$$

where $\theta_i = a_i/M_i$.

Proof. The proof may be omitted since it repeats the same arguments as in the proof of Theorem 2. □

Exercise 2. *Three urns contained lots of balls of different colors. Of all balls in these urns there were: 10 white from all 20 balls in urn* Π_1*, 15 white from all 25 balls in urn* Π_2 *and 20 white from all 30 balls in urn* Π_3*. From these urns 5, 10 and 15 balls, respectively, were randomly taken out, and then they were transferred to the empty urn* Π*. Next, all balls in urn* Π *randomly and in the appropriate amount were divided into three parts, and they all returned back to the primary urns. Eventually the amounts of the balls in numbered urns remained the same. We seek out the probability that the ball taken randomly from any urn will turn out to be white. For instance we will find the probability for urn* Π_2*.*

Solution. First using formula (8) for the case of $n = 3$, $N = c = 0$ and $\mathbf{k} = (5, 10, 15)$ we find the probability $P_{\Pi}(A|\mathbf{x} = \mathbf{k})$ that the ball, randomly taken out from urn Π, turns out to be white:

$$P_{\Pi}(A|\mathbf{x} = \mathbf{k}) = \frac{\frac{10}{20}5 + \frac{15}{25}10 + \frac{20}{30}15}{5 + 10 + 15} = \frac{37}{60}.$$

To find the sought-for probability $P_{\Pi_2}(A|\xi = 10)$, we just use formula (7) for the cases of $N = 15$, $k = 10$ and $\theta = 37/60$. Note that in the initial state $P_{\Pi_2}(A) = 15/25$. After transfer 15 balls are left in the urn Π_2 so that $15/25 = c/15$. Hence $c = \frac{15}{25}15$. Thus we have

$$P_{\Pi_2}(A|\xi = 10) = \frac{\frac{37}{60}10 + \frac{15}{25}15}{15 + 10} = \frac{182}{300}$$

or the same as $P_{\Pi_2}(A|\xi = 10) = 182/300 \approx 0.607$. □

Scheme $[\mathbf{A}_n(\boldsymbol{a}_s, \mathbf{K})]$. There is a sequence of urns $\{\Pi_i, i = 0, 1, 2, \dots\}$. Each urn contains particles of s types T_1, $T_2, \dots, T_s$. The state of the ith urn is determined by the s dimensional vector $\boldsymbol{a}_s^i = \left(a_1^i, a_2^i, \dots, a_s^i\right)$, where a_j^i is the number of particles of the type of T_j in the ith urn. We observe the operation $\mathbf{K} := [k_0, k_1, \dots]$ which is as follows: k_0 particles are randomly taken out from urn Π_0 and transferred to urn Π_1; after k_1 particles are randomly taken out from urn Π_1 and transferred to urn Π_2 and so forth. We use the symbol $\mathbf{K}_m := [k_0, k_1, \dots, k_{m-1}]$ in the case when the operation $\mathbf{K}$ is observed in the mth step, where $m = 1, 2, \dots$.

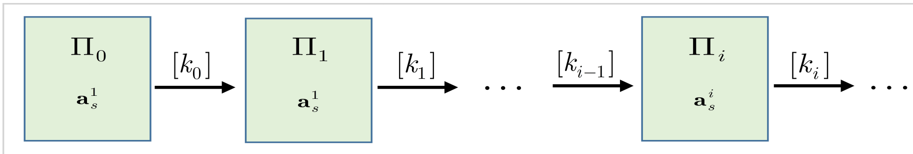

Figure 5. *Scheme* $[\mathbf{A}_n(\boldsymbol{a}_s, \mathbf{K})]$

Fix the type T_s, $1 \le \mathrm{s} \le s$. Denote by $\mathrm{P}_{\Pi_1}(T_\mathrm{s}|\,\mathbf{K}_1)$ the probability that the particle randomly taken from urn Π_1 provided the operation $\mathbf{K}_1$, will turn out to be a particle of the type T_s. According to Theorem 2, we have

$$\mathrm{P}_{\Pi_1}(T_\mathrm{s}|\,\mathbf{K}_1) = \alpha_{\mathrm{s}1} + (\beta_{\mathrm{s}1} - \alpha_{\mathrm{s}1})\,\theta_\mathrm{s}, \tag{9}$$

where $\theta_\mathrm{s} = \mathrm{P}_{\Pi_0}(T_\mathrm{s})$ is the probability of an appearance of the particle of type T_s in urn Π_0 and

$$\alpha_{\mathrm{s}1} = \frac{a_\mathrm{s}^1}{\sum_{j=1}^s a_j^1 + k_0} \quad \text{and} \quad \beta_{\mathrm{s}1} = \frac{a_\mathrm{s}^1 + k_0}{\sum_{j=1}^s a_j^1 + k_0}.$$

Continuing discussions, we can generalize the formula (9) for all $\mathrm{P}_{\Pi_m}(T_\mathrm{s}|\,\mathbf{K}_m)$, $m = 2, 3, \dots$, and obtain a recurrence formula that would allows us to find the probability of appearance of particle of the type T_s in arbitrary urn at the any step of operation $\mathbf{K}$. We have the following theorem.

Theorem 4. *In scheme* $[\boldsymbol{A}_n(\boldsymbol{a}_s, \mathbf{K})]$*, the probability* $P_{\Pi_m}(T_s|\,\boldsymbol{K}_m)$ *that the particle, randomly taken out from urn* Π_m*, provided that operation* $\boldsymbol{K}_m$*, turns out to be the particle of type* T_s*, is calculated by the following formula:*

$$P_{\Pi_m}(T_s|\,\boldsymbol{K}_m) = \alpha_{s1} + (\beta_{s1} - \alpha_{s1})\,P_{\Pi_{m-1}}(T_s|\,\boldsymbol{K}_{m-1}), \tag{10}$$

where $m = 2, 3, \dots$

$$\alpha_{s1} = \frac{a_s^m}{\sum_{j=1}^s a_j^m + k_{m-1}} \quad \textit{and} \quad \beta_{s1} = \frac{a_s^m + k_{m-1}}{\sum_{j=1}^s a_j^m + k_{m-1}},$$

here $P_{\Pi_1}(T_s|\,\boldsymbol{K}_1)$ *is in* (9).

Exercise 3. *Three urns contain balls of white, black and yellow colors. Urn* Π_0 *contains 310 white, 350 black and 370 yellow balls; urn* Π_1 *contains no white ball, 480 black and 530 yellow balls; urn* Π_2 *contains 600 white, 640 black and 670 yellow balls. 500 balls are randomly taken from urn* Π_0 *and they transferred to the urn* Π_1 *and then 600 balls are randomly taken from urn* Π_1 *and transferred to the urn* Π_2*; see Figure 6 below. We are looking for the probabilities that a ball randomly selected from urn* Π_2 *will turn out to be white, black, and yellow, respectively.*

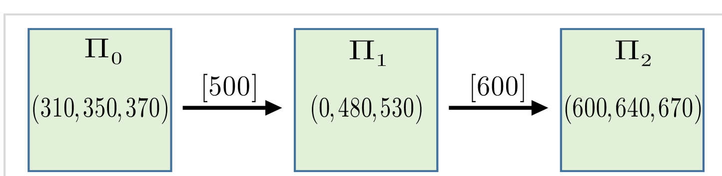

Figure 6. *Illustration of Exercise 3*

Solution. Let T_w be white, T_b – black and T_y – yellow types. It's obvious that

$$\mathrm{P}_{\Pi_0}(T_\mathrm{w}) = \frac{310}{1030} \quad \text{and} \quad \mathrm{P}_{\Pi_0}(T_\mathrm{b}) = \frac{350}{1030} \quad \text{and} \quad \mathrm{P}_{\Pi_0}(T_\mathrm{y}) = \frac{370}{1030}.$$

We can easily calculate, that

$$\alpha_{\mathrm{w}1} = \frac{0}{1510} \quad \text{and} \quad \beta_{\mathrm{w}1} = \frac{500}{1510};$$

and

$$\alpha_{\mathrm{b}1} = \frac{480}{1510} \quad \text{and} \quad \beta_{\mathrm{b}1} = \frac{980}{1510};$$

and

$$\alpha_{\mathrm{y}1} = \frac{530}{1510} \quad \text{and} \quad \beta_{\mathrm{y}1} = \frac{1030}{1510}.$$

Using the data above and formula (9) we obtain

$$\mathrm{P}_{\Pi_1}\left(T_{\mathrm{w}}\middle|\,\mathbf{K}_1\right) = 0 + \left(\frac{500}{1510} - 0\right)\frac{310}{1030} = \frac{1550}{15553},$$

$$\mathrm{P}_{\Pi_1}\left(T_{\mathrm{b}}\middle|\,\mathbf{K}_1\right) = \frac{480}{1510} + \left(\frac{980}{1510} - \frac{480}{1510}\right)\frac{350}{1030} = \frac{6694}{15553},$$

$$\mathrm{P}_{\Pi_1}\left(T_{\mathrm{y}}\middle|\,\mathbf{K}_1\right) = \frac{530}{1510} + \left(\frac{1030}{1510} - \frac{530}{1510}\right)\frac{370}{1030} = \frac{7309}{15553},$$

By the same way and using the formula (10) we can find sought probabilities as follows:

$$\mathrm{P}_{\Pi_2}\left(T_{\mathrm{w}}\middle|\,\mathbf{K}_2\right) = \frac{600}{2510} + \left(\frac{1200}{2510} - \frac{600}{2510}\right)\frac{1550}{15553} = \frac{1026180}{3903803}\,(\approx 0.263)\,,$$

$$\mathrm{P}_{\Pi_2}\left(T_{\mathrm{b}}\middle|\,\mathbf{K}_2\right) = \frac{640}{2510} + \left(\frac{1240}{2510} - \frac{640}{2510}\right)\frac{6694}{15553} = \frac{1397032}{3903803}\,(\approx 0.358)\,,$$

$$\mathrm{P}_{\Pi_2}\left(T_{\mathrm{y}}\middle|\,\mathbf{K}_2\right) = \frac{670}{2510} + \left(\frac{1270}{2510} - \frac{670}{2510}\right)\frac{7309}{15553} = \frac{1480591}{3903803}\,(\approx 0.379)\,.$$

It can be checked that $\mathrm{P}_{\Pi_2}\left(T_{\mathrm{w}}\middle|\,\mathbf{K}_2\right) + \mathrm{P}_{\Pi_2}\left(T_{\mathrm{b}}\middle|\,\mathbf{K}_2\right) + \mathrm{P}_{\Pi_2}\left(T_{\mathrm{y}}\middle|\,\mathbf{K}_2\right) = 1.$ □

Conclusion

The paper proposes several modified formulas for the direct calculation of probabilities in classical urn schemes. One of the principal advantages of the formulas proposed in the paper from the classical formulas is the absence of binomial coefficients in them. In the framework of our goal, the result of Theorem 1 looks convex. Here the observed probability is interpreted from a geometric point of view. In addition, the formula (4), in appearance and in fact, is the formula for the value of the intermediate point in the Lagrange formula from classical analysis. In the subsequent Theorems, both the considered scheme and the corresponding probability calculation formulas are modified. In understanding the goals of the authors, the fact that for each theorem exercises are given and the corresponding schemes are illustrated is useful.

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