

# Some Remarks on MCI Crossed Modules

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## Abstract

In an earlier work, it is proven that the category of crossed modules in a modified category of interest (MCI crossed modules) is finitely complete with a certain condition, in which all codomains are fixed. In this paper, we prove that this is also true without any restriction.

*Keywords:* Modified category of interest, crossed module, limit.

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## 1. Introduction

The notion of category of interest is introduced to unify various properties of algebraic structures. The main idea is due to Higgins [12], and the definition is improved by Orzech [14]. Many well-known algebraic categories are the examples of category of interest except the categories of  $\text{cat}^1$ -objects of Lie (associative, Leibniz, etc.) algebras. Then, to overcome this problem, the authors of [4] introduced a new type of this notion, called *modified category of interest* that satisfies all axioms of the former notion except one, which is replaced by a new and modified condition.

A *crossed module* of groups is a group homomorphism  $\partial: E \rightarrow G$ , together with a group action  $\triangleright$  of  $G$  on  $E$ , satisfying the following relations (for all  $e, f \in E$  and  $g \in G$ ):

$$\partial(g \triangleright e) = g \partial(e) g^{-1}, \quad \partial(e) \triangleright f = e f e^{-1}.$$

Crossed modules are introduced by Whitehead [16] as a model of homotopy 2-types and used to classify higher dimensional cohomology groups. See [5] for more details on crossed modules. Afterwards, crossed modules are also studied for various algebraic structures such as in the categories of (commutative) algebras, dialgebras, Lie and Leibniz algebras, etc. [6]. However, the current definition of crossed modules in modified categories of interest [4] unifies all of these definitions. Furthermore, there also exist some other generalizations of crossed modules of groups such as [13, 15, 17].

It is already proven that the category of crossed modules in a modified category of interest (MCI crossed modules) is finitely complete with a certain condition - i.e. all codomains are fixed [8]. Furthermore, the cocompleteness has been studied in [3]. In this paper, we prove that we also have the same completeness property without any restriction neither on domains nor on codomains. In conclusion, one can adapt this property to many different algebraic categories such as crossed modules of Lie algebras, Leibniz algebras, dialgebras, etc. and say that these categories are also finitely complete.

## 2. Preliminaries

In this section, we recall some notions from [4, 8] based on the modified category of interest.

## 2.1 Modified Category of Interest

**Definition 2.1.** Let  $\mathbb{C}$  be a category of groups with a set of operations  $\Omega$  and with a set of identities  $\mathbb{E}$ , such that  $\mathbb{E}$  includes the group identities and the following conditions hold.

If  $\Omega_i$  is the set of  $i$ -ary operations in  $\Omega$ , then:

- (a)  $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ ;
- (b) the group operations (written additively:  $0, -, +$ ) are elements of  $\Omega_0, \Omega_1$  and  $\Omega_2$  respectively. Let  $\Omega'_2 = \Omega_2 \setminus \{+\}$ ,  $\Omega'_1 = \Omega_1 \setminus \{-\}$ . Assume that if  $*$   $\in \Omega_2$ , then  $\Omega'_2$  contains  $*$  $^\circ$  defined by  $x *^\circ y = y * x$  and assume  $\Omega_0 = \{0\}$ ;
- (c) for each  $*$   $\in \Omega'_2$ ,  $\mathbb{E}$  includes the identity  $x * (y + z) = x * y + x * z$ ;
- (d) for each  $\omega \in \Omega'_1$  and  $*$   $\in \Omega'_2$ ,  $\mathbb{E}$  includes the identities  $\omega(x + y) = \omega(x) + \omega(y)$  and *either* the identity  $\omega(x * y) = \omega(x) * \omega(y)$  *or* the identity  $\omega(x * y) = \omega(x) * y$ .

Denote by  $\Omega'_{1S}$  the subset of those elements in  $\Omega'_1$ , which satisfy the identity  $\omega(x * y) = \omega(x) * y$ , and by  $\Omega'_1$  all other unary operations, i.e. those which satisfy the first identity from (d).

Let  $C$  be an object of  $\mathbb{C}$  and  $x_1, x_2, x_3 \in C$ :

- (e)  $x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1$ , for each  $*$   $\in \Omega'_2$ .
- (f) For each ordered pair  $(*, \bar{*}) \in \Omega'_2 \times \Omega'_2$  there is a word  $W$  such that:

$$(x_1 * x_2) \bar{*} x_3 = W(x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1, (x_3x_2)x_1, x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2),$$

where each juxtaposition represents an operation in  $\Omega'_2$ .

A category of groups with operations  $\mathbb{C}$  satisfying conditions (a)-(f) is called a "modified category of interest", or "MCI" for short.

As indicated in [4], the difference between this definition and that of the original "category of interest" is the modification of the second identity in (d). According to this definition every category of interest is also a modified category of interest.

**Definition 2.2.** Let  $A, B$  be two objects of  $\mathbb{C}$ . A map  $f: A \rightarrow B$  is called a morphism of  $\mathbb{C}$  if it satisfies:

$$\begin{aligned} f(a + a') &= f(a) + f(a'), \\ f(a * a') &= f(a) * f(a'), \end{aligned}$$

for all  $a, a' \in A$ ,  $*$   $\in \Omega'_2$  and also commutes with all  $w \in \Omega'_1$ .

**Example 2.1.** The categories of groups, algebras, commutative algebras, Lie algebras, Leibniz algebras, dialgebras are all (modified) categories of interest.

**Example 2.2.** The categories  $\text{Cat}^1\text{Ass}$ ,  $\text{Cat}^1\text{Lie}$ ,  $\text{Cat}^1\text{Leibniz}$ , i.e. the categories of  $\text{cat}^1$ -associative algebras,  $\text{cat}^1$ -Lie algebras and  $\text{cat}^1$ -Leibniz algebras are the examples of modified categories of interest, which are not categories of interest (see [4] for details).

**Notation.** From now on,  $\mathbb{C}$  will denote an arbitrary but fixed modified category of interest.

**Definition 2.3.** Let  $A, B \in \mathbb{C}$ . An extension of  $B$  by  $A$  is a sequence:

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} B \longrightarrow 0 \tag{2.1}$$

where  $p$  is surjective and  $i$  is the kernel of  $p$ . We say that an extension is split if there exists a morphism  $s: B \rightarrow E$  such that  $ps = 1_B$ .

**Definition 2.4.** The split extension (2.1) induces an action of  $B$  on  $A$  corresponding to the operations of  $\mathbb{C}$  with:

$$\begin{aligned} b \cdot a &= s(b) + a - s(b), \\ b * a &= s(b) * a, \end{aligned}$$

for all  $b \in B, a \in A$  and  $*$   $\in \Omega'_2$ .

Actions defined by the previous equations are called *derived actions* of  $B$  on  $A$ . Remark that we use the notation "\*" to denote both the star operation and the star action.

Given an action of  $B$  on  $A$ , a semi-direct product  $A \rtimes B$  is a universal algebra, whose underlying set is  $A \times B$  and the operations are defined by:

$$\begin{aligned} \omega(a, b) &= (\omega(a), \omega(b)), \\ (a', b') + (a, b) &= (a' + b' \cdot a, b' + b), \\ (a', b') * (a, b) &= (a' * a + a' * b + b' * a, b' * b), \end{aligned}$$

for all  $a, a' \in A, b, b' \in B, *$   $\in \Omega'_2$ . An action of  $B$  on  $A$  is a derived action if and only if  $A \rtimes B$  is an object of  $\mathbb{C}$ .

Denote a general category of groups with operations of a modified category of interest  $\mathbb{C}$  by  $\mathbb{C}_G$ . A set of actions of  $B$  on  $A$  in  $\mathbb{C}_G$  is a set of derived actions if and only if it satisfies the following conditions:

1.  $0 \cdot a = a$ ,
2.  $b \cdot (a_1 + a_2) = b \cdot a_1 + b \cdot a_2$ ,
3.  $(b_1 + b_2) \cdot a = b_1 \cdot (b_2 \cdot a)$ ,
4.  $b * (a_1 + a_2) = b * a_1 + b * a_2$ ,
5.  $(b_1 + b_2) * a = b_1 * a + b_2 * a$ ,
6.  $(b_1 * b_2) \cdot (a_1 * a_2) = a_1 * a_2$ ,
7.  $(b_1 * b_2) \cdot (a * b) = a * b$ ,
8.  $a_1 * (b \cdot a_2) = a_1 * a_2$ ,
9.  $b * (b_1 \cdot a) = b * a$ ,
10.  $\omega(b \cdot a) = \omega(b) \cdot \omega(a)$ ,
11.  $\omega(a * b) = \omega(a) * b = a * \omega(b)$  for any  $\omega \in \Omega'_{1S}$ , and  $\omega(a * b) = \omega(a) * \omega(b)$  for any  $\omega \in \Omega''_1$ ,
12.  $x * y + z * t = z * t + x * y$ ,

for each  $\omega \in \Omega'_1, *$   $\in \Omega'_2, b, b_1, b_2 \in B, a, a_1, a_2 \in A$ ; and for  $x, y, z, t \in A \cup B$  whenever both sides of the last condition are defined.

## 2.2 Limits in MCI

The usual cartesian product  $P \times R$  is the product object of  $P$  and  $R$  in  $\mathbb{C}$ , with the projection morphisms satisfying the universal property.

Suppose that  $\alpha : P \rightarrow S$  and  $\beta : R \rightarrow S$  are two morphisms in  $\mathbb{C}$ . Then the subobject of the cartesian product:

$$P \times_S R = \{(p, r) \mid \alpha(p) = \beta(r)\}, \quad (2.2)$$

called *fiber product*, defines the pullback of  $\alpha, \beta$ .

Therefore a modified category of interest  $\mathbb{C}$  has products and pullbacks which guarantees the existence of equalizer objects. Briefly, suppose that we have two parallel morphisms  $f, g : P \rightarrow R$ . Their equalizer is defined as  $\text{Eq}(f, g) = \{x \in P \mid f(x) = g(x)\}$ .

Consequently, we can say that  $\mathbb{C}$  has all finite limits since it has both products and equalizers. Thus,  $\mathbb{C}$  is finitely complete.

### 2.3 Crossed Modules

**Definition 2.5.** A crossed module  $(C_1, C_0, \partial)$  in  $\mathbb{C}$  is (namely a MCI crossed module) is given by a morphism  $\partial: C_1 \rightarrow C_0$  with a derived action of  $C_0$  on  $C_1$  such that:

$$\text{XM1) } \begin{aligned} \partial(c_0 \cdot c_1) &= c_0 + \partial(c_1) - c_0 \\ \partial(c_0 * c_1) &= c_0 * \partial(c_1) \end{aligned}$$

$$\text{XM2) } \begin{aligned} \partial(c_1) \cdot c'_1 &= c_1 + c'_1 - c_1 \\ \partial(c_1) * c'_1 &= c_1 * c'_1 \end{aligned}$$

for all  $c_0 \in C_0, c_1, c'_1 \in C_1, * \in \Omega'_2$ .

A morphism between two crossed modules  $(C_1, C_0, \partial) \rightarrow (C'_1, C'_0, \partial')$  is a pair  $(\mu_1, \mu_0)$  of morphisms  $\mu_0: C_0 \rightarrow C'_0, \mu_1: C_1 \rightarrow C'_1$ , such that the diagram

$$\begin{array}{ccc} C_1 & \xrightarrow{\partial} & C_0 \\ \mu_1 \downarrow & & \downarrow \mu_0 \\ C'_1 & \xrightarrow{\partial'} & C'_0 \end{array}$$

commutes and

$$\begin{aligned} \mu_1(c_0 \cdot c_1) &= \mu_0(c_0) \cdot \mu_1(c_1), \\ \mu_1(c_0 * c_1) &= \mu_0(c_0) * \mu_1(c_1), \end{aligned}$$

for all  $c_0 \in C_0, c_1 \in C_1$  and  $* \in \Omega'_2$ .

Thus, crossed modules and their morphisms form a category in  $\mathbb{C}$ .

*Remark 2.1.* The following well-known definitions are the examples of MCI crossed modules.

**Example 2.3.** A crossed module of groups [10] is given by a group homomorphism  $\partial: E \rightarrow G$ , together with an action  $\triangleright$  of  $G$  on  $E$  such that (for all  $e, f \in E$  and  $g \in G$ ):

- $\partial(g \triangleright e) = g \partial(e) g^{-1}$ ,
- $\partial(e) \triangleright f = e f e^{-1}$ .

**Example 2.4.** A crossed module of Lie algebras [2] is given by a Lie algebra homomorphism  $\partial: \mathfrak{e} \rightarrow \mathfrak{g}$ , together with an action  $\triangleright$  of  $\mathfrak{g}$  on  $\mathfrak{e}$  such that (for all  $e, f \in \mathfrak{e}$  and  $g \in \mathfrak{g}$ ):

- $\partial(g \triangleright e) = [g, \partial(e)]$ ,
- $\partial(e) \triangleright f = [e, f]$ .

Note that  $\triangleright$  denotes the group action and the Lie algebra action respectively in the previous examples.

**Notation.** From now on, any crossed module in a modified category of interest  $\mathbb{C}$  will be shortly called "MCI crossed module" for the sake of simplicity.

## 3. The Completeness

In this section, we deal with limits in the category of MCI crossed modules – without any restriction.

*Remark 3.1.* Consider we have two MCI crossed modules  $(C_1, P_1, \partial_1)$  and  $(C_2, P_2, \partial_2)$ . There exists an action of  $P_1 \times P_2$  on  $C_1 \times C_2$  which is

$$\begin{aligned} (p_1, p_2) \cdot (c_1, c_2) &= (p_1 \cdot c_1, p_2 \cdot c_2), \\ (p_1, p_2) * (c_1, c_2) &= (p_1 * c_1, p_2 * c_2). \end{aligned}$$

for all  $(p_1, p_2) \in P_1 \times P_2$  and  $(c_1, c_2) \in C_1 \times C_2$ .

**Lemma 3.1.** *Considering two crossed modules given above, we obtain a MCI crossed module structure*

$$(C_1 \times C_2, P_1 \times P_2, \partial), \quad (3.1)$$

given by

$$\partial(c_1, c_2) = (\partial_1(c_1), \partial_2(c_2)), \quad (3.2)$$

for all  $(c_1, c_2) \in C_1 \times C_2$ .

*Proof.* Crossed module conditions are satisfied since

XM1)

$$\begin{aligned} \partial((p_1, p_2) \cdot (c_1, c_2)) &= \partial(p_1 \cdot c_1, p_2 \cdot c_2) \\ &= (\partial_1(p_1 \cdot c_1), \partial_2(p_2 \cdot c_2)) \\ &= (p_1 + \partial_1(c_1) - p_1, p_2 + \partial_2(c_2) - p_2) \\ &= (p_1, p_2) + (\partial_1(c_1), \partial_2(c_2)) - (p_1, p_2) \\ &= (p_1, p_2) + \partial(c_1, c_2) - (p_1, p_2), \end{aligned}$$

and

$$\begin{aligned} \partial((p_1, p_2) * (c_1, c_2)) &= \partial(p_1 * c_1, p_2 * c_2) \\ &= (\partial_1(p_1 * c_1), \partial_2(p_2 * c_2)) \\ &= (p_1 * \partial_1(c_1), p_2 * \partial_2(c_2)) \\ &= (p_1, p_2) * (\partial_1(c_1), \partial_2(c_2)) \\ &= (p_1, p_2) * \partial(c_1, c_2), \end{aligned}$$

XM2)

$$\begin{aligned} \partial(c_1, c_2) \cdot (c'_1, c'_2) &= (\partial_1(c_1), \partial_2(c_2)) \cdot (c'_1, c'_2) \\ &= (\partial_1(c_1) \cdot c'_1, \partial_2(c_2) \cdot c'_2) \\ &= (c_1 + c'_1 - c_1, c_2 + c'_2 - c_2) \\ &= (c_1, c_2) + (c'_1, c'_2) - (c_1, c_2), \end{aligned}$$

and

$$\begin{aligned} \partial(c_1, c_2) * (c'_1, c'_2) &= (\partial_1(c_1), \partial_2(c_2)) * (c'_1, c'_2) \\ &= (\partial_1(c_1) * c'_1, \partial_2(c_2) * c'_2) \\ &= (c_1 * c'_1, c_2 * c'_2) \\ &= (c_1, c_2) * (c'_1, c'_2), \end{aligned}$$

for all  $(c_1, c_2), (c'_1, c'_2) \in C_1 \times C_2$  and  $(p_1, p_2) \in (P_1, P_2)$ . □

**Theorem 3.1.** *We have the product object (3.1), in the category of MCI crossed modules.*

*Proof.* We only need to prove the universal property. Let  $(T, S, \delta)$  be our test object, i.e. a MCI crossed module with two crossed module morphisms

$$\begin{aligned} (\alpha, \alpha') &: (T, S, \delta) \rightarrow (C_1, P_1, \partial_1), \\ (\beta, \beta') &: (T, S, \delta) \rightarrow (C_2, P_2, \partial_2). \end{aligned}$$

Then there must be a unique crossed module morphism

$$(\varphi, \varphi') : (T, S, \delta) \rightarrow (C_1 \times C_2, P_1 \times P_2, \partial),$$

such that the following diagram commutes:

$$\begin{array}{ccccc}
 & & (T, S, \delta) & & \\
 & \swarrow^{(\alpha, \alpha')} & \vdots^{(\varphi, \varphi')} & \searrow^{(\beta, \beta')} & \\
 & & (C_1 \times C_2, P_1 \times P_2, \partial) & & \\
 \swarrow^{(\pi_1, \pi_1)} & & & & \searrow^{(\pi_2, \pi_2)} \\
 (C_1, P_1, \partial_1) & & & & (C_2, P_2, \partial_2)
 \end{array} \tag{3.3}$$

Remark that, the tuple of projections  $(\pi_i, \pi_i)$  define crossed module morphisms. So let us define

$$\begin{aligned}
 \varphi(t) &= (\alpha(t), \beta(t)), \\
 \varphi'(s) &= (\alpha'(s), \beta'(s)),
 \end{aligned}$$

for each  $t \in T$  and  $s \in S$ .

$(\varphi, \varphi')$  defines a crossed module morphism with the following diagram

$$\begin{array}{ccc}
 T & \xrightarrow{\delta} & S \\
 \varphi \downarrow & & \downarrow \varphi' \\
 C_1 \times C_2 & \xrightarrow{\partial} & P_1 \times P_2
 \end{array}$$

since we have

$$\begin{aligned}
 \varphi(s \cdot t) &= (\alpha(s \cdot t), \beta(s \cdot t)) \\
 &= (\alpha'(s) \cdot \alpha(t), \beta'(s) \cdot \beta(t)) \\
 &= (\alpha'(s), \beta'(s)) \cdot (\alpha(t), \beta(t)) \\
 &= \varphi'(s) \cdot \varphi(t),
 \end{aligned}$$

$$\begin{aligned}
 \varphi(s * t) &= (\alpha(s * t), \beta(s * t)) \\
 &= (\alpha'(s) * \alpha(t), \beta'(s) * \beta(t)) \\
 &= (\alpha'(s), \beta'(s)) * (\alpha(t), \beta(t)) \\
 &= \varphi'(s) * \varphi(t),
 \end{aligned}$$

and

$$\begin{aligned}
 \partial \varphi(t) &= \partial(\alpha(t), \beta(t)) \\
 &= (\partial_1 \alpha(t), \partial_2 \beta(t)) \\
 &= (\alpha' \delta(t), \beta' \delta(t)) \\
 &= \varphi' \delta(t),
 \end{aligned}$$

for all  $t \in T$  and  $s \in S$ .

On the other hand, we can easily get

$$\begin{aligned}
 (\pi_1, \pi_1)(\varphi, \varphi') &= (\alpha, \alpha'), \\
 (\pi_2, \pi_2)(\varphi, \varphi') &= (\beta, \beta'),
 \end{aligned}$$

and prove that the diagram (3.3) commutes.

Finally, consider  $(\nu, \nu')$  be a crossed module morphism with the same property as  $(\varphi, \varphi')$ . Define  $(c_1, c_2) \in C_1 \times C_2$  by  $\nu(t) = (c_1, c_2)$ , and also  $(p_1, p_2) \in P_1 \times P_2$  by  $\nu'(s) = (p_1, p_2)$ . Then we obtain

$$\begin{aligned}\pi_1 \nu(t) = \alpha(t) &\Leftrightarrow \pi_1(c_1, c_2) = \alpha(t) \Leftrightarrow c_1 = \alpha(t), \\ \pi_1 \nu'(s) = \alpha'(s) &\Leftrightarrow \pi_1(p_1, p_2) = \alpha'(s) \Leftrightarrow p_1 = \alpha'(s),\end{aligned}$$

and also

$$\begin{aligned}\pi_2 \nu(t) = \beta(t) &\Leftrightarrow \pi_2(c_1, c_2) = \beta(t) \Leftrightarrow c_2 = \beta(t), \\ \pi_2 \nu'(s) = \beta'(s) &\Leftrightarrow \pi_2(p_1, p_2) = \beta'(s) \Leftrightarrow p_2 = \beta'(s),\end{aligned}$$

for all  $t \in T, s \in S$  that proves the uniqueness of  $(\varphi, \varphi')$  by

$$\begin{aligned}\nu(t) = (c_1, c_2) &= (\alpha(t), \beta(t)) = \varphi(t), \\ \nu'(s) = (p_1, p_2) &= (\alpha'(s), \beta'(s)) = \varphi'(s).\end{aligned}$$

□

**Corollary 3.1.** Consider two MCI crossed module morphisms

$$\begin{aligned}(f_1, g_1): (C_1, P_1, \partial_1) &\rightarrow (C_3, P_3, \partial_3), \\ (f_2, g_2): (C_2, P_2, \partial_2) &\rightarrow (C_3, P_3, \partial_3).\end{aligned}$$

Recalling (2.2), define the fiber products  $C_1 \times_{C_3} C_2$  and  $P_1 \times_{P_3} P_2$  which are categorically pullbacks of  $(f_1, g_1)$  and  $(f_2, g_2)$ , respectively.

Then we obtain a MCI crossed module  $(C_1 \times_{C_3} C_2, P_1 \times_{P_3} P_2, \partial')$  namely

$$\partial': C_1 \times_{C_3} C_2 \rightarrow P_1 \times_{P_3} P_2, \quad (3.4)$$

where  $\partial'$  is the restriction of  $\partial$  given in (3.2).

**Theorem 3.2.** We have the pullback object (3.4), in the category of MCI crossed modules.

*Proof.* We only need to check the universal property. For this aim, let  $(T, S, \delta)$  be the test object with the following crossed module morphisms

$$\begin{aligned}(\alpha, \alpha'): (T, S, \delta) &\rightarrow (C_1, P_1, \partial_1), \\ (\beta, \beta'): (T, S, \delta) &\rightarrow (C_2, P_2, \partial_2),\end{aligned}$$

such that the following diagram commutes:

$$\begin{array}{ccc} (T, S, \delta) & \xrightarrow{(\beta, \beta')} & (C_2, P_2, \partial_2) \\ & \searrow^{(\alpha, \alpha')} & \downarrow (g_1, g_2) \\ & (C_1, P_1, \partial_1) & \xrightarrow{(f_1, f_2)} (C_3, P_3, \partial_3). \end{array}$$

Then there must be a unique crossed module morphism

$$(\varphi, \varphi'): (T, S, \delta) \rightarrow (C_1 \times_{C_3} C_2, P_1 \times_{P_3} P_2, \partial'),$$

such that the following diagram commutes.

$$\begin{array}{ccc}
 (T, S, \delta) & & \\
 \swarrow (\alpha, \alpha') & \searrow (\beta, \beta') & \\
 & (C_1 \times_{C_3} C_2, P_1 \times_{P_3} P_2, \partial') & \xrightarrow{(\pi_2, \pi_2)} & (C_2, P_2, \partial_2) \\
 & \downarrow (\pi_1, \pi_1) & & \downarrow (g_1, g_2) \\
 & (C_1, P_1, \partial_1) & \xrightarrow{(f_1, f_2)} & (C_3, P_3, \partial_3)
 \end{array}
 \tag{3.5}$$

Define

$$\begin{aligned}
 \varphi(t) &= (\alpha(t), \beta(t)), \\
 \varphi'(s) &= (\alpha'(s), \beta'(s)),
 \end{aligned}$$

for all  $t \in T$  and  $s \in S$ .

A direct calculation shows that  $(\varphi, \varphi')$  is a crossed module morphism. Furthermore, we get

$$(\pi_1, \pi_1)(\varphi, \varphi') = (\alpha, \alpha'), \quad (\pi_2, \pi_2)(\varphi, \varphi') = (\beta, \beta')$$

which proves the commutativity of diagram (3.5). The uniqueness of  $(\varphi, \varphi')$  can be shown analogous to the previous proof.  $\square$

**Corollary 3.2.** *Consequently, one can obtain equalizer object through product and pullback objects in the category of MCI crossed modules. More clearly, the equalizer of two parallel morphisms  $f, g: A \rightarrow B$  is the pullback of  $(1_A, f): A \rightarrow A \times B$  and  $(1_A, g): A \rightarrow A \times B$ .*

*Remark 3.2.* Furthermore, the category of MCI crossed modules has the zero object  $(0, 0, \text{id})$  where 0 denotes the zero object (trivial object with a single element) in a modified category of interest.

**Theorem 3.3.** *The category of MCI crossed modules is finitely complete.*

*Proof.* Follows directly from Theorem 3.1, Theorem 3.2 and Corollary 3.2.  $\square$

## 4. Conclusion

As we mentioned in the preliminaries section, the categories of crossed modules of groups, of (commutative) algebras, of Lie (and Leibniz) algebras, etc are particular examples of MCI crossed modules. Correspondingly, we have the following corollary of Theorem 3.3, as an application of modified categories of interest.

**Corollary 4.1.** *The categories of crossed modules of,*

- groups
- (associative) algebras
- commutative algebras
- Lie algebras
- Leibniz algebras
- dialgebras

*are finitely complete.*

*Remark 4.1.* However, there exist some other crossed module structures which are MCI crossed modules, such as crossed modules of racks [7], of Hopf algebras [10], of polygroups [1]. Therefore, such structures can not be included in the above theorem. In fact, some of them are already studied – for instance, see [9, 11] for the case of racks.



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