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Research Article

The Product of Two Functions Using Positive Linear Operators

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ABSTRACT. In this paper, we estimate the speed of convergence of the difference $L_n(fg) - (L_n f) \cdot (L_n g)$ towards 0, where (L_n) are positive linear operators used in the approximation of continuous functions. We also study in what conditions the formula $L'_n(fg) - fL'_n g - gL'_n f \to 0$ holds true.

Keywords: Positive linear operators, exponential type operators, Voronovskaya formula, Chebyshev-Grüss functional, Baskakov operators, Jain operators, Balász operators.

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1. INTRODUCTION

In the last period of time, it was investigated the following difference

$$L_n(fg)(x) - (L_n f)(x) \cdot (L_n g)(x),$$

a generalization to positive linear operators of an expression appearing in the classical inequalities of Chebyshev [15] and Grüss [24]. Starting with the papers [10, 19] and [7, 33], these celebrated inequalities were extended to the case of positive linear functionals and positive linear operators. Bounds for this difference were given using different methods (see [22, 20, 21]). Asymptotic results of Voronovskaya type for this Chebyshev-Grüss quantity were obtained in [18, 6, 16, 35, 9, 30] for different operators.

In this paper, we give a quantitative result of Voronovskaya type for the Chebyshev-Grüss expression for a large class of positive linear operators and for a large class of continuous functions. Our results, presented in Section 3, do not need as in [9, 30] the hypothesis of the existence of the second derivatives of the functions involved.

In Section 4, we study in what conditions do the differentiation formula $L'_n(fg) - fL'_n g - gL'_n f$ converges to zero. We generalize the result of Impens and Gavrea [27], which was given for Bernstein type operators and for functions defined on a compact interval. Using another approach, we extend the result to larger class of positive linear operators and to a larger class of continuous functions, including bounded and unbounded functions. We also give a Voronovskaya type result for the differential formula just mentioned.

In Section 2, we present a class of positive linear operators which is defined using a Chebyshev-Grüss expression. This class, which was introduced in [26], contains the Bernstein type operators [31], but is much larger, including also positive linear operators which do not preserve linear functions. Some examples are given in the final section of the paper as applications of the results obtained.

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2. Properties of the operators defined by a Chebyshev-Grüss quantity

For the value of $L_n f$ in $x \in I$, we use the notations

$$(L_n f)(x) = L_n(f)(x) = L_n(f, x) = L_n(f(t), x)$$

interchangeably.

Consider a sequence of positive linear operators (L_n) which preserve the constants and which is defined by the following relation involving a Chebyshev-Grüss type expression

$$b_n \left[L_n(tf(t), x) - L_n(t, x) \cdot L_n(f(t), x) \right] = b(x) \left(L_n f \right)' (x) \tag{2.1}$$

for every $x \in I$, where $I \subset \mathbb{R}$ is an interval, for every $n \in \mathbb{N}$ and for every f for which $L_n f$ and $(L_n f)'$ exist, where $b(x)$ is a positive function which is differentiable on I and (b_n) is a sequence of positive real numbers such that $b_n \to \infty$. Concerning the domain of definition of the operators L_n, we will give explanations in the next section.

Remark 2.1. *If the operators L_n preserve the linear functions, then the condition (2.1) can be written*

$$\left(L_n f \right)' (x) = \frac{b_n}{b(x)} \cdot L_n((e_1 - x) f, x),$$

which is satisfied by the class of so called exponential operators (see [31] and [28]), in particular Bernstein polynomials, the operators of Szász-Mirakyan, Baskakov, Post-Widder and Gauss-Weierstrass. Condition (2.1) characterizes a more general class of operators, which do not necessarily preserve linear functions. Other examples will be given at the end of the paper. A relation equivalent with (2.1) is

$$b_n \left[L_n((t - x) f(t), x) - L_n(t - x, x) \cdot L_n(f(t), x) \right] = b(x) \left(L_n f \right)' (x),$$

a relation obtained in [36] for a particular kind of operators.

Remark 2.2. *If we consider a function $f = g(t, x)$ which depends on x and t and which has a partial derivative with respect to x in every point (t, x), then, condition (2.1) can be written*

$$\begin{aligned} & b_n \left[L_n((t - x) g(t, x), x) - L_n(t - x, x) \cdot L_n(g(t, x), x) \right] \\ & \quad = b(x) \left[\left(L_n(g(t, x), x) \right)' - L_n \left(\frac{\partial g}{\partial x}(t, x), x \right) \right], \end{aligned}$$

where the operator L_n acts on the variable t. In particular, for $f = (t - x)^k$, $k \geq 1$ we obtain

$$b_n \cdot \left[\mu_{n,k+1}(x) - \mu_{n,1}(x) \mu_{n,k}(x) \right] = b(x) \left[\mu'_{n,k}(x) + k \mu_{n,k-1}(x) \right], \tag{2.2}$$

where $\mu_{n,k}(x) = L_n((t - x)^k, x)$ are the central moments of order k for the operator L_n. This recurrence expresses all the central moments in terms of only one function, namely $\mu_{n,1}$, since the value of $\mu_{n,0}$ is known: $\mu_{n,0}(x) = 1$.

Let us suppose that

$$\lim_{n \to \infty} a_n \cdot \frac{d^i}{dx^i} \mu_{n,1}(x) = a^{(i)}(x) \tag{2.3}$$

is true for every $x \in I$, and $i = 0, 1, 2, \ldots$, where $a(x)$ is an infinitely differentiable function on I and (a_n) is an increasing and unbounded sequence of positive real numbers.

Lemma 2.1. *If the sequence (b_n/a_n) converges to the real number $c \geq 0$, then for every integer $\ell \geq 0$ we have*

$$\lim_{n\to\infty} b_n^{\ell} \cdot \mu_{n,2\ell}(x) = (b(x))^{\ell} \cdot (2\ell-1)!! \tag{2.4}$$

$$\lim_{n\to\infty} b_n^{\ell+1} \cdot \mu_{n,2\ell+1}(x) = (b(x))^{\ell}\,(2\ell)!! \sum_{i=0}^{\ell} \frac{(2i-1)!!}{(2i)!!}[ib'(x) + c\cdot a(x)] \tag{2.5}$$

uniformly for x in a compact interval contained in I. We have used the notations

$$(2\ell-1)!! = 1\cdot 3\ldots(2\ell-1) \text{ and } (2\ell)!! = 2\cdot 4\cdots(2\ell), \quad \ell \geq 1$$

and for $\ell = 0$ the value is 1.

Proof. The proof will be omitted since it is similar to the one found in Lemma 2 [26]. □

3. Quantitative Voronovskaya-type result for Chebyshev-Grüss expression

In this section, we are concerned with the asymptotic behaviour of the Grüss-Chebyshev expression, which will be denoted

$$T_n(f,g)(x) = L_n(fg)(x) - (L_n f)(x)\cdot(L_n g)(x).$$

We will prove that $b_n \cdot T_n(f,g)(x) \to b(x)f'(x)g'(x)$ and we will estimate the speed of this convergence. We show that such a result is valid for unbounded functions, too. In order to do this, let us introduce some notations.

Let $\theta : [0,\infty) \to \mathbb{R}$ be a uniformly continuous and monotonic function, let I be an interval $I \subset \mathbb{R}$ and let $\alpha \geq 0$ be a real number. We denote by $C_{\theta,\alpha}$ the space of continuous functions $f \in C(I)$ with the property that exists $M > 0$ such that $|f(x)| \leq Me^{\alpha\theta(|x|)}$, for every $x \in I$. Because of the symmetry and to simplify the notation, we consider in the following that $I \subset [0,\infty)$. This space $C_{\theta,\alpha}$ can be endowed with the norm

$$\|f\|_{\theta,\alpha} = \sup_{x\in I} e^{-\alpha\theta(x)}|f(x)|.$$

Lemma 3.2. *Consider a sequence of positive linear operators $L_n : C_{\theta,\alpha} \to C(I)$ such that*

$$\lim_{n\to\infty} L_n(e^{\alpha\theta(t)}, x) = e^{\alpha\theta(x)}. \tag{3.6}$$

Then, there is a positive function $M_\alpha(x)$ not depending on n such that

$$L_n\left(\max\left(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}\right), x\right) \leq M_\alpha(x), \quad n \geq n_0.$$

Proof. For $x \in I$, there is n_0 such that $\left|L_n(e^{\alpha\theta(t)}, x) - e^{\alpha\theta(x)}\right| \leq 1$ for every $n \geq n_0$. We obtain

$$L_n(\max(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}), x) \leq L_n(e^{\alpha\theta(t)} + e^{\alpha\theta(x)}, x) \leq 1 + 2e^{\alpha\theta(x)},$$

for every $n \geq n_0$. □

We will use the following weighted modulus

$$\omega_{\theta,\alpha}(f,\delta) = \sup_{\substack{x,t\in I\\ |t-x|\leq\delta}} \frac{|f(t)-f(x)|}{\max\left(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}\right)},$$

which is suitable for functions from the space $C_{\theta,\alpha}$ (see [25]).

Theorem 3.1. *Let $f, g \in C_{\theta,\alpha}$ be continuously differentiable functions such that $f'(x)e^{-\alpha\theta(x)}$ and $g'(x)e^{-\alpha\theta(x)}$ are uniformly continuous on I. Let $L_n : C_{\theta,\alpha} \to C^1(I)$ be a sequence of positive linear operators preserving constant functions and having the properties (2.1), (2.3) and (3.6). Then, for some $n_0 \in \mathbb{N}$ and for every $n \geq n_0$ and $x \in I$, we have*

$$\begin{aligned}
&|b_n[L_n(fg)(x) - (L_nf)(x)\cdot(L_ng)(x)] - b(x)f'(x)g'(x)|\\
&\quad\leq \frac{M(x)}{a_n}|b(x)f'(x)g'(x)| + M(x)\,\omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right)\omega_{\theta,\alpha}\left(g', \frac{1}{\sqrt{b_n}}\right)\\
&\quad\quad + M(x)\left(|f'(x)|\,\omega_{\theta,\alpha}\left(g', \frac{1}{\sqrt{b_n}}\right) + |g'(x)|\,\omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right)\right),
\end{aligned}$$

where $M(x) > 0$ does not depend on n and f.

Proof. Using the Taylor formula of the first order with Lagrange remainder, we obtain

$$\begin{aligned}
f(t) &= f(x) + f'(x)\cdot(t-x) + R_f, \quad R_f = (f'(c_1) - f'(x))\cdot(t-x)\\
g(t) &= g(x) + g'(x)\cdot(t-x) + R_g, \quad R_g = (g'(c_2) - g'(x))\cdot(t-x),
\end{aligned}$$

with c_1, c_2 between t and x. We multiply the relations and we apply the operators L_n. We get

$$\begin{aligned}
L_n(fg)(x) &= f(x)g(x)\mu_{n,0}(x) + [f(x)g'(x) + g(x)f'(x)]\mu_{n,1}(x) + L_n(R_fR_g)(x)\\
&\quad + f'(x)g'(x)\mu_{n,2}(x) + f(x)L_n(R_g)(x) + g(x)L_n(R_f)(x)\\
&\quad + f'(x)L_n((e_1 - xe_0)R_g)(x) + g'(x)L_n((e_1 - xe_0)R_f)(x).
\end{aligned}$$

We also have

$$\begin{aligned}
L_nf(x) &= f(x)\mu_{n,0}(x) + f'(x)\mu_{n,1}(x) + L_n(R_f)(x),\\
L_ng(x) &= g(x)\mu_{n,0}(x) + g'(x)\mu_{n,1}(x) + L_n(R_g)(x).
\end{aligned}$$

We get

$$\begin{aligned}
L_n(fg)(x) - (L_nf)(x)\cdot(L_ng)(x) &= f'(x)g'(x)[\mu_{n,2}(x) - \mu_{n,1}^2(x)]\\
&\quad + f'(x)[L_n((t-x)R_g)(x) - \mu_{n,1}(x)L_n(R_g)(x)]\\
&\quad + g'(x)[L_n((t-x)R_f)(x) - \mu_{n,1}(x)L_n(R_f)(x)]\\
&\quad + L_n(R_fR_g)(x) - L_n(R_f)(x)\cdot L_n(R_g)(x).
\end{aligned}$$

Because $b_n[\mu_{n,2}(x) - \mu_{n,1}^2(x)] = b(x)[1 + \mu'_{n,1}(x)]$, we have

$$\left|b_n[\mu_{n,2}(x) - \mu_{n,1}^2(x)] - b(x)\right| = \left|b(x)\mu'_{n,1}(x)\right| = \frac{\left|b(x)a_n\mu'_{n,1}(x)\right|}{a_n}.$$

We evaluate now the remainder from the Taylor formula using the modulus of continuity $\omega_{\theta,\alpha}$. From

$$\begin{aligned}
|R_f| &= |t-x|\cdot|f'(c_1) - f'(x)|\\
&\leq \max(e^{\alpha\theta(t)}, e^{\alpha\theta(x)})\,|t-x|\left(1 + \frac{|t-x|}{\delta}\right)\omega_{\theta,\alpha}(f', \delta),
\end{aligned}$$

we obtain

$$|(L_nR_f)(x)| \leq \left(A_{n,1}(x) + \sqrt{b_n}A_{n,2}(x)\right)\omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right),$$

where

$$A_{n,k}(x) = L_n\left(\max\left(e^{\alpha t}, e^{\alpha x}\right)|t-x|^k, x\right). \tag{3.7}$$

Because $L_n(e^{\alpha\theta(t)}, x)$ converges pointwise to $e^{\alpha\theta(x)}$ we have

$$L_n\left(\max\left(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}\right), x\right) \le M_\alpha(x), \quad n \ge n_0.$$

From Lemma 2.1, we have

$$L_n\left(|t-x|^{2k}, x\right) = \frac{1}{b_n^k} \cdot b_n^k \mu_{n,2k}(x) \le \frac{C_k(x)}{b_n^k}, \quad n \ge n_0.$$

Using the Cauchy-Schwarz inequality for positive linear operators we obtain for $k = 1, 2$

$$A_{n,k}(x) \le \sqrt{L_n\left(\max\left(e^{2\alpha t}, e^{2\alpha x}\right), x\right)} \cdot \sqrt{L_n\left(|t-x|^{2k}, x\right)} \le \frac{\sqrt{M_{2\alpha}(x) C_k(x)}}{\sqrt{b_n^k}}.$$

In conclusion,

$$|(L_n R_f)(x)| \le \frac{M_{0,2}(x)}{\sqrt{b_n}} \cdot \omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right).$$

Similarly,

$$|L_n((e_1 - xe_0)R_f, x)| \le L_n\left(|t-x|\,|R_f|, x\right) \le \frac{M_{0,3}(x)}{b_n} \cdot \omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right).$$

So, using $|\mu_{n,1}(x)| \le \sqrt{\mu_{n,2}(x)} \le \sqrt{C_2(x)}/\sqrt{b_n}$, we obtain

$$\begin{aligned}
&|L_n((e_1 - xe_0)R_f)(x) - \mu_{n,1}(x) L_n(R_f)(x)| \\
&\quad \le |L_n((e_1 - xe_0)R_f, x)| + |\mu_{n,1}(x)| \cdot |(L_n R_f)(x)| \\
&\quad \le \frac{M_{0,4}(x)}{b_n} \cdot \omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right).
\end{aligned}$$

Let us notice that if we replace f with g in the previous inequalities they hold true, too.

To evaluate the term $L_n(R_f R_g)(x) - L_n(R_f)(x) \cdot L_n(R_g)(x)$, let us observe that

$$\begin{aligned}
&|L_n(R_f R_g)(x)| \le L_n(|R_f|\,|R_g|, x) \\
&\le L_n\left(e^{2\alpha \max(\theta(t),\theta(x))} |t-x|^2 \left(1 + \frac{|t-x|}{\delta_n}\right)^2, x\right) \cdot \omega_{\theta,\alpha}(f', \delta_n) \cdot \omega_{\theta,\alpha}(g', \delta_n) \\
&\le 2\left[A_{n,2}(x) + b_n A_{n,4}(x)\right] \cdot \omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right) \omega_{\theta,\alpha}\left(g', \frac{1}{\sqrt{b_n}}\right) \\
&\le \frac{M_{0,5}(x)}{b_n} \cdot \omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right) \omega_{\theta,\alpha}\left(g', \frac{1}{\sqrt{b_n}}\right).
\end{aligned}$$

We have used the inequality $(1+u)^2 \le 2(1+u^2)$. We obtain

$$\begin{aligned}
&|L_n(R_f R_g)(x) - L_n(R_f)(x) \cdot L_n(R_g)(x)| \\
&\quad \le |L_n(R_f R_g)(x)| + |L_n(R_f)(x)| \cdot |L_n(R_g)(x)| \\
&\quad \le \frac{M_{0,5}(x) + M_{0,2}^2(x)}{b_n} \cdot \omega_{\theta,\alpha}\left(f', \frac{1}{\sqrt{b_n}}\right) \omega_{\theta,\alpha}\left(g', \frac{1}{\sqrt{b_n}}\right).
\end{aligned}$$

Choosing an appropriate expression $M(x) > 0$ not depending on n and f the proof is complete. □

Remark 3.3. *Because $1/a_n$ and $1/\sqrt{b_n}$ converge to zero when n tends to infinity and $f'(x)e^{-\alpha\theta(x)}$ and $g'(x)e^{-\alpha\theta(x)}$ are uniformly continuous on I, we have*

$$\lim_{n\to\infty} b_n[L_n(fg)(x) - (L_n f)(x) \cdot (L_n g)(x)] = b(x) f'(x) g'(x). \tag{3.8}$$

Similar results were obtained in [18, 6, 16, 35, 9, 30] *for functions for which the second derivative exists. But, there is no need to suppose the existence of the second derivative of f and g.*

Remark 3.4. *Theorem 3.1 is true even for operators L_n for which (2.1) cannot be proved. Indeed, it is only necessary that the following limits exist for a fixed x*

$$\lim_{n\to\infty} a_n \left(b_n[\mu_{n,2}(x) - \mu_{n,1}^2(x)] - b(x)\right) \text{ and } \lim_{n\to\infty} b_n^\ell \cdot \mu_{n,2\ell}(x),\ \ell = 1, 2, 3, 4,$$

where $b(x)$ is the limit of $b_n \cdot \mu_{n,2}(x)$.

For example, let us consider the Jain operators [29]

$$P_n^{\beta_n}(f, x) = \sum_{k=0}^{\infty} \frac{nx(nx + k\beta_n)^{k-1}}{k!} e^{-nx-k\beta_n} \cdot f\left(\frac{k}{n}\right),$$

where (β_n) is a sequence of positive real numbers from $[0, 1)$ converging to zero. It is known [17] *that*

$$P_n^{\beta_n}(t - x, x) = \frac{x}{1 - \beta_n} - x = \frac{x\beta_n}{1 - \beta_n},$$

so we choose $a_n = 1/\beta_n$ and condition (2.3) is satisfied with $a(x) = x$. We also have

$$P_n^{\beta_n}((t - x)^2, x) = \frac{x^2\beta_n^2}{(1 - \beta_n)^2} + \frac{x}{n(1 - \beta_n)^3}.$$

Choosing $b_n = n$ and supposing that $b_n/a_n = n\beta_n$ is convergent to the real number $c \geq 0$, we obtain

$$b(x) = \lim_{n\to\infty} nP_n^{\beta_n}((t - x)^2, x) = x.$$

After some computations, we obtain

$$\lim_{n\to\infty} a_n \left(b_n[\mu_{n,2}(x) - \mu_{n,1}^2(x)] - b(x)\right) = x.$$

The central moment of order 4 is (see [17]*)*

$$\begin{aligned} P_n^{\beta_n}((t - x)^4, x) &= \frac{x^4\beta_n^4}{(1 - \beta_n)^4} + \frac{6x^3\beta_n^2}{n(1 - \beta_n)^5} \\ &+ \frac{x^2(-24\beta_n^5 + 12\beta_n + 48\beta_n^3 - 28\beta_n^2 + 4\beta_n + 3)}{n^2(1 - \beta_n)^6} \\ &+ \frac{x(105\beta_n^5 - 14\beta_n^4 - 2\beta_n^3 + 12\beta_n^2 + 8\beta_n + 1)}{n^3(1 - \beta_n)^7}. \end{aligned}$$

We obtain $n^2P_n^{\beta_n}((t - x)^4, x) \to 3x^2$. For the central moments of order 6 and 8, we consider the significant terms from the formulas given in [23] *and obtain*

$$\lim_{n\to\infty} n^3P_n^{\beta_n}((t - x)^6, x) = 15x^3 \text{ and } \lim_{n\to\infty} n^4P_n^{\beta_n}((t - x)^8, x) = 105x^4.$$

The result of Theorem 3.1 is valid for $P_n^{\beta_n}$ in polynomial weighted space $C_{\theta,\alpha}$ with $\theta(x) = \ln x$, $x \in I = (0, \infty)$ (see [2]*).*

4. Voronovskaya-type result for a differentiation formula for positive linear operators

In [27], it is proved that the expression $L_n'(fg) - fL_n'g - gL_n'f$ converges to zero for exponential type operators under suitable conditions for the functions f and g. The result was proved for those operators $L_n : C(I) \to C(J)$, where I, J are compact intervals. We extend the result to noncompact intervals and to unbounded functions.

Theorem 4.2. *Let $f, g \in C_{\theta,\alpha}$ such that*

$$\omega_{\theta,\alpha}(f,\delta)\cdot\omega_{\theta,\alpha}(g,\delta) = o(\delta) \quad (\delta \to 0+). \tag{4.9}$$

Let $L_n : C_{\theta,\alpha} \to C^1(I)$ be a sequence of positive linear operators preserving constant functions and having the properties (2.1), (2.3) and (3.6). Then, for every $x \in I$

$$L_n'(fg)(x) - f(x)(L_n g)'(x) - g(x)(L_n f)'(x) \to 0.$$

Proof. Let us denote

$$\Delta_n(x) = L_n'(fg)(x) - f(x)(L_n g)'(x) - g(x)(L_n f)'(x).$$

Using (2.1), we obtain the following relation

$$\begin{aligned}\Delta_n(x) = {} & \frac{b_n}{b(x)} \cdot L_n((t-x)(f(t)-f(x))(g(t)-g(x)), x) \\ & - \frac{b_n}{b(x)} \cdot L_n(t-x,x)\cdot L_n((f(t)-f(x))(g(t)-g(x)),x).\end{aligned} \tag{4.10}$$

As in the proof of Theorem 3.1, because

$$|f(t)-f(x)| \le \max\left(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}\right)\left(1+\frac{|t-x|}{\delta}\right)\omega_{\theta,\alpha}(f,\delta)$$

$$|g(t)-g(x)| \le \max\left(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}\right)\left(1+\frac{|t-x|}{\delta}\right)\omega_{\theta,\alpha}(g,\delta),$$

we have

$$\begin{aligned}&|L_n((t-x)(f(t)-f(x))(g(t)-g(x)),x)| \\ &\quad \le 2\left(A_{n,1}(x)+b_nA_{n,3}(x)\right)\omega_{\theta,\alpha}\left(f,\frac{1}{\sqrt{b_n}}\right)\omega_{\theta,\alpha}\left(g,\frac{1}{\sqrt{b_n}}\right) \\ &\quad \le \frac{M_{1,1}(x)}{\sqrt{b_n}}\cdot\omega_{\theta,\alpha}\left(f,\frac{1}{\sqrt{b_n}}\right)\omega_{\theta,\alpha}\left(g,\frac{1}{\sqrt{b_n}}\right)\end{aligned}$$

and

$$\begin{aligned}&|L_n((f(t)-f(x))(g(t)-g(x)),x)| \\ &\quad \le 2\left(A_{n,0}(x)+b_nA_{n,2}(x)\right)\omega_{\theta,\alpha}\left(f,\frac{1}{\sqrt{b_n}}\right)\omega_{\theta,\alpha}\left(g,\frac{1}{\sqrt{b_n}}\right) \\ &\quad \le M_{1,2}(x)\cdot\omega_{\theta,\alpha}\left(f,\frac{1}{\sqrt{b_n}}\right)\omega_{\theta,\alpha}\left(g,\frac{1}{\sqrt{b_n}}\right).\end{aligned}$$

Because $L_n(t-x,x) \le \sqrt{\mu_{n,2}(x)} \le \frac{\sqrt{C_2(x)}}{\sqrt{b_n}}$, we finally obtain

$$|\Delta_n(x)| \le M(x)\sqrt{b_n}\cdot\omega_{\theta,\alpha}\left(f,\frac{1}{\sqrt{b_n}}\right)\omega_{\theta,\alpha}\left(g,\frac{1}{\sqrt{b_n}}\right), \quad n \ge n_0,$$

for some $M(x)$ not depending on n and f. The condition (4.9) proves that Δ_n converges to zero for every $x \in I$. □

Remark 4.5. *We have the following evaluation of the modulus $\omega_{\theta,\alpha}$ (see relation (1) from [25])*

$$\omega_{\theta,\alpha}(f,\delta) \le (1-e^{-\alpha\omega(\theta,\delta)})\,\|f\|_{\theta,\alpha} + \omega(f/w,\delta) \le \alpha\omega(\theta,\delta)\,\|f\|_{\theta,\alpha} + \omega(f/w,\delta),$$

where $w(x) = e^{\alpha\theta(x)}$ and ω is the usual modulus of continuity (the modulus $\omega_{\theta,\alpha}$ for $\alpha = 0$).

If $\theta \in \mathrm{Lip}_a(I)$, $f/w \in \mathrm{Lip}_b(I)$ and $g/w \in \mathrm{Lip}_c(I)$ then, (4.9) is true if and only if

$$a+a>1, \quad a+b>1, \quad a+c>1 \quad \text{and} \quad b+c>1.$$

Indeed, a function h belongs to $\mathrm{Lip}_\alpha(I)$ if and only if there is a constant $C_h>0$ such that $\omega(f,\delta)\le C_h\delta^\alpha$. So,

$$\omega_{\theta,\alpha}(f,\delta)\cdot\omega_{\theta,\alpha}(g,\delta)\le(C_1\delta^a+C_2\delta^b)(C_1\delta^a+C_3\delta^c)=o(\delta)\quad(\delta\to0+).$$

Remark 4.6. *Theorem 4.2 remains true even if L_n does not satisfy a condition like (2.1). We only need that the sequence of functions $b_n^\ell\cdot\mu_{n,2\ell}(x)$ converges pointwise for $\ell=1,2$ and 3.*

Theorem 4.3. *Let $f,g\in C_{\theta,\alpha}$ be two twice differentiable functions such that $f''(x)e^{-\alpha\theta(x)}$ and $g''(x)e^{-\alpha\theta(x)}$ are uniformly continuous on I. We suppose that (b_n/a_n) is convergent to $c\ge0$. Let $L_n: C_{\theta,\alpha}\to C^1(I)$ be a sequence of positive linear operators preserving constant functions and having the properties (2.1), (2.3) and (3.6). Then, for every $x\in I$*

$$\begin{aligned}\lim_{n\to\infty} b_n\left[L_n'(fg)(x)-f(x)(L_ng)'(x)-g(x)(L_nf)'(x)\right]\\ =[b'(x)+2c\,a(x)]f'(x)g'(x)+\frac{3b(x)}{2}[f'(x)g''(x)+f''(x)g'(x)].\end{aligned}$$

Proof. We use Taylor's formula

$$h(t)=h(x)+h'(x)\cdot(t-x)+\frac{h''(x)}{2}\cdot(t-x)^2+R_h,$$

for the functions f and g, where $R_h=(h''(c)-h''(x))\cdot(t-x)^2/2$, with some c between t and x. We replace these formulas in the expression of Δ_n (see relation (4.10)) and after some computations, we obtain

$$\begin{aligned}b_n\Delta_n(x)&=f'(x)g'(x)\frac{b_n^2}{b(x)}[\mu_{n,3}(x)-\mu_{n,1}(x)\mu_{n,2}(x)]\\&\quad+[f'(x)g''(x)+f''(x)g'(x)]\cdot\frac{b_n^2}{2b(x)}[\mu_{n,4}(x)-\mu_{n,1}(x)\mu_{n,3}(x)]+\frac{b_n^2R}{b(x)},\end{aligned}$$

where

$$\begin{aligned}R&=\frac14f''(x)g''(x)[\mu_{n,5}(x)-\mu_{n,1}(x)\mu_{n,5}(x)]\\&\quad+f'(x)\cdot E_{n,1}(g)+\frac12f''(x)\cdot E_{n,2}(g)+g'(x)\cdot E_{n,1}(f)+\frac12g''(x)\cdot E_{n,2}(f)\\&\quad+L_n(R_f\cdot R_g\cdot(t-x),x)-\mu_{n,1}(x)\cdot L_n(R_f\cdot R_g,x)\end{aligned}$$

and

$$E_{n,k}(f)=L_n(R_f\cdot(t-x)^{k+1},x)-\mu_{n,1}(x)\cdot L_n(R_f\cdot(t-x)^k,x).$$

We have

$$\frac{b_n^2}{b(x)}[\mu_{n,3}(x)-\mu_{n,1}(x)\mu_{n,2}(x)]=b_n\mu_{n,2}'(x)+2b_n\mu_{n,1}(x)\to b'(x)+2ca(x)$$

and

$$\frac{b_n^2}{2b(x)}[\mu_{n,4}(x)-\mu_{n,1}(x)\mu_{n,3}(x)]=\frac{b_n}{2}\mu_{n,3}'(x)+\frac{3b_n}{2}\mu_{n,2}(x)\to0+\frac{3b(x)}{2}.$$

We also have $b_n^2R\to0$, but since the computations are similar to those in the proof of Theorem 3.1, we omit the details. □

Remark 4.7. *Let $f, g \in C_{\theta,\alpha}$ be two twice differentiable functions such that $f''(x)e^{-\alpha\theta(x)}$ and $g''(x)e^{-\alpha\theta(x)}$ are uniformly continuous on I. It can be proved in a similar way that*

$$\lim_{n\to\infty} b_n \left[L_n'(fg)(x) - (L_n f)(x)(L_n g)'(x) - (L_n g)(x)(L_n f)'(x)\right]$$
$$= b'(x)f'(x)g'(x) + b(x)[f'(x)g''(x) + f''(x)g'(x)].$$

This is just relation (3.8), where both terms have been differentiated.

5. Applications

We give a couple of examples of applications.

Example 5.1. *Consider the following Baskakov operators of Stancu type*

$$(L_n^{[\alpha,\beta,c]} f)(x) = \sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) \cdot f\left(\frac{k+\alpha}{n+\beta}\right), \quad n \geq 1$$

where α and β are real numbers such that $0 \leq \alpha \leq \beta$ and

$$p_{n,k}^{[c]}(x) = (-1)^k \binom{-n/c}{k} (cx)^k (1+cx)^{-\frac{n}{c}-k}, \quad c \neq 0$$

$$p_{n,k}^{[0]}(x) = \lim_{c\to 0} p_{n,k}^{[c]}(x) = \frac{(nx)^k}{k!} e^{-nx},$$

where $x \in [0,\infty)$ for $c \geq 0$ and $x \in [0, -1/c]$ for $c < 0$.

These operators are a particular example of the more general operators considered in [11]*. For $\alpha = \beta = 0$, some properties of the operators were given in* [1, 14] *(see also* [5, 32] *and the references therein).*

These operators preserve the constants and

$$L_n^{[\alpha,\beta,c]}(t,x) = \sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) \cdot \frac{k+\alpha}{n+\beta} = \frac{nx+\alpha}{n+\beta}.$$

We deduce that

$$(n+\beta) \cdot \left(L_n^{[\alpha,\beta,c]}(t,x) - x\right) = \alpha - \beta x,$$

which proves (2.3) for $a_n = n+\beta$ and $a(x) = \alpha - \beta x$.

We also have

$$\frac{\mathrm{d}}{\mathrm{d}x} p_{n,k}^{[c]}(x) = p_{n,k}^{[c]}(x) \cdot \frac{k-nx}{x(1+cx)} = \frac{n+\beta}{x(1+cx)} p_{n,k}^{[c]}(x) \cdot \left(\frac{k+\alpha}{n+\beta} - L_n^{[\alpha,\beta,c]}(t,x)\right).$$

Multiplying this equality with $f((k+\alpha)/(n+\beta))$ and summing up for k from 0 to infinity, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(L_n^{[\alpha,\beta,c]} f)(x) = \frac{n+\beta}{x(1+cx)} \left[L_n^{[\alpha,\beta,c]}(tf(t),x) - L_n^{[\alpha,\beta,c]}(t,x) \cdot L_n^{[\alpha,\beta,c]}(f,x)\right],$$

which is (2.1) for $b_n = n+\beta$ and $b(x) = x(1+cx)$.

The results of Theorems 1,2 and 3 are valid for functions in the exponential space $C_{\theta,\alpha}$ for $\theta(x) = x$, because

$$L_n^{[\alpha,\beta,c]}(e^{\alpha t},x) = \left(1 + cx - cxe^{\frac{\alpha}{n}}\right)^{-\frac{n}{c}} \to e^{\alpha x} \quad (n\to\infty).$$

Example 5.2. *Consider the Balász operators*

$$R_n(f,x) = \frac{1}{(1+\alpha_n x)^n} \sum_{k=0}^{n} \binom{n}{k} (\alpha_n x)^k \cdot f\left(\frac{k}{\beta_n}\right), n \geq 1$$

introduced in [12] *and studied in* [13, 34, 4, 8, 3] *for some particular cases of the sequences (α_n) and (β_n) of positive real numbers.*

The operators R_n preserve the constants and

$$R_n(t,x) = \frac{n\alpha_n x}{\beta_n(1+\alpha_n x)} \text{ and } R_n(t^2,x) = \frac{n\alpha_n x + n^2\alpha_n^2 x^2}{\beta_n^2(1+\alpha_n x)^2}.$$

We must have $R_n(t,x) \to x$ and $R_n(t^2,x) \to x^2$, so we choose (α_n) and (β_n) such that $\alpha_n \to 0$ and the sequence (β_n) such that $\beta_n \to \infty$ and $n\alpha_n/\beta_n \to 1$. The central moment of order 1 is

$$R_n(t-x,x) = \frac{(n\alpha_n - \beta_n)x - \alpha_n\beta_n x^2}{\beta_n(1+\alpha_n x)}.$$

We further impose that $n\alpha_n - \beta_n \to 0$ and $\alpha_n\beta_n \to c$, $c \geq 0$. With these conditions, we can choose $a_n = \beta_n$ and obtain $\beta_n R_n(t-x,x) = -cx^2$.

Let us prove that R_n satisfy (2.1). Because

$$\left(\frac{(\alpha_n x)^k}{(1+\alpha_n x)^n}\right)' = \frac{(\alpha_n x)^k}{(1+\alpha_n x)^n} \cdot \left(\frac{k}{x} - \frac{n\alpha_n}{1+\alpha_n x}\right),$$

we obtain

$$\begin{aligned}(R_n(f,x))' &= \frac{\beta_n}{x}\sum_{k=0}^{n}\binom{n}{k}\frac{(\alpha_n x)^k}{(1+\alpha_n x)^n} \cdot \left(\frac{k}{\beta_n} - \frac{n\alpha_n x}{\beta_n(1+\alpha_n x)}\right) \cdot f\left(\frac{k}{\beta_n}\right) \\ &= \frac{\beta_n}{x} R_n((t - R_n(t,x))f(t),x),\end{aligned}$$

which proves (2.1) with $b_n = \beta_n$ and $b(x) = x$.

We take $I = (0,\infty)$ and $\theta(x) = x$. The results of Theorem 1, 2 and 3 are valid for the operators R_n in the exponential weighted space $C_{\theta,\alpha}$, because for a fixed $x \geq 0$

$$R_n(e^{\alpha t},x) = \left(\frac{1+\alpha_n x e^{\frac{\alpha}{\beta_n}}}{1+\alpha_n x}\right)^n \to e^{\alpha x}.$$

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