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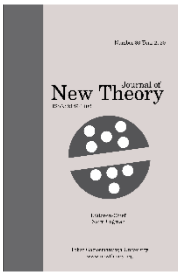

Bipolar Pythagorean Fuzzy Subring of a Ring

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Abstract−In this paper, we study some of the properties of bipolar Pythagorean fuzzy subring of a ring and prove some results on these. We derive some important theorems and intersection and product are applied into the bipolar Pythagorean fuzzy subring of a ring.

Keywords− *Bipolar Pythagorean fuzzy set, bipolar Pythagorean fuzzy subring, bipolar Pythagorean fuzzy product*

1. Introduction

Fuzzy sets were introduced by Zadeh [1], and he discussed only membership function. After the extensions of fuzzy set theory Atanassov [2] generalised this concept and introduced a new concept called intuitionistic fuzzy set (IFS). Yager [3] familiarised the model of Pythagorean fuzzy set. IFSs have its greatest use in practical multiple attribute decision making (MADM) problems, and academic research has achieved significant development [3-5]. However, in some practical problems, the sum of membership degree and non-membership degree to which an alternative satisfying attribute provided by the decision maker (DM) may be bigger than 1, but their square sum is less than or equal to 1. Jun and Song [6] introduced the notion of closed fuzzy ideals in BCI-algebras and discussed their properties.

Bosc and Pivert [7] said that "Bipolarity refers to the propensity of the human mind to reason and make decisions based on positive and negative effects. Positive information states what is possible, satisfactory, permitted, desired, or considered as being acceptable. On the other hand, negative statements express what is impossible, rejected, or forbidden. Negative preferences correspond to constraints since they specify which values or objects have to be rejected (i.e., those that do not satisfy the constraints), while positive preferences correspond to wishes, as they specify which objects are more desirable than others (i.e., satisfy user wishes) without rejecting those that do not meet the wishes". Therefore, Lee [8,9] introduced the concept of bipolar fuzzy sets which is a generalisation of the fuzzy sets. Many authors have studied recently bipolar fuzzy models on algebraic structures such as; Chen et al. [10] studied of m-polar fuzzy set. Then, they examined many results which are related to those concepts can be generalised to the case of m-polar fuzzy sets. They also proposed numerical examples

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to show how to apply m-polar fuzzy sets in real-world problems. In 1982, Liu [11] introduced the concept of the fuzzy ring and fuzzy ideal. After the notion of intuitionistic fuzzy subring by Hur et al. [12], many researchers have tried to generalise the notion of the intuitionistic fuzzy subring. Marashdeh and Salleh [13] introduced the notion of intuitionistic fuzzy rings based on the notion of fuzzy space.

The purpose of this paper is to introduce the concept of bipolar Pythagorean fuzzy subring and established some of their results.

2. Preliminaries

Definition 2.1. [1] Let X be a nonempty set. A fuzzy set A drawn from X is defined as $A = \{(x: \mu_A(x)): x \in X\}$, where $\mu_A: X \to [0,1]$ is the membership function of the set A.

Definition 2.2. [8] Let X be the universe. Then, a bipolar fuzzy set, A on X is defined by a positive membership function μ_A^+, that is $\mu_A^+: X \to [0,1]$, and a negative membership function μ_A^-, that is $\mu_A^-: X \to [-1,0]$. For the sake of simplicity, we shall use the symbol $A = \{(x, \mu_A^+(x), \mu_A^-(x)): x \in X\}$.

Definition 2.3. (Pythagorean Fuzzy Set) [3,4] Let X be a non-empty set and I the unit interval [0,1]. A PF set S is an object having the form $P = \{(x, \mu_P(x), \nu_P(x)): x \in X\}$ where the function $\mu_P: X \to [0,1]$ and $\nu_P: X \to [0,1]$ denote respectively the degree of membership and degree of non-membership of each element x in X to the set P, and $0 \le (\mu_P(x))^2 + (\nu_P(x))^2 \le 1$ for all x in X.

Definition 2.4. (Bipolar Pythagorean Fuzzy Set) [14] Let X be a non-empty set. A bipolar Pythagorean fuzzy set (BPFS) $A = \{(X, T_A^P, F_A^P, T_A^N, F_A^N): x \in X\}$ where $T_A^P: X \to [0,1]$, $F_A^P: X \to [0,1]$, $T_A^N: X \to [-1,0]$, and $F_A^N: X \to [-1,0]$ are the mappings such that $0 \le (T_A^P)^2 + (F_A^P)^2 \le 1$ and $-1 \le -((T_A^N)^2 + (F_A^N)^2) \le 0$ and $T_A^P(x)$ denote the positive membership degree, $F_A^P(x)$ denote the positive non-membership degree, $T_A^N(x)$ denote the negative membership degree, $F_A^N(x)$denote the negative non-membership degree.

Definition 2.5. [14] Let $A = \{(X, T_A^P, F_A^P, T_A^N, F_A^N): x \in X\}$ and $B = \{(X, T_B^P, F_B^P, T_B^N, F_B^N): x \in X\}$ be two BPFNs. Then, their operations are defined as follows:

i. $A \cup B = \{(x, max(T_A^P, T_B^P), min(F_A^P, F_B^P), min(T_A^N, T_B^N), max(F_A^N, F_B^N)): x \in X\}$

ii. $A \cap B = \{(x, min(T_A^P, T_B^P), max(F_A^P, F_B^P), max(T_A^N, T_B^N), min(F_A^N, F_B^N)): x \in X\}$

iii. $A^C = \{(x, F_A^P, T_A^P, F_A^N, T_A^N): x \in X\}$

Definition 2.6 [15] Let R be a ring. An intuitionistic fuzzy subset A= $\{(x, T_A(x), F_A(x))/x \in R\}$ of R is said to be an intuitionistic fuzzy subring of R if the following conditions are satisfied,

i. $T_A(x-y) \ge min\{T_A(x), T_A(y)\}$

ii. $T_A(xy) \ge min\{T_A(x), T_A(y)\}$

iii. $F_A(x-y) \le max\{F_A(x), F_A(y)\}$

iv. $F_A(xy) \le max\{F_A(x), F_A(y)\}$

3. Properties

Definition 3.1. Let R be a ring. A bipolar Pythagorean fuzzy subset A of R is said to be a bipolar Pythagorean fuzzy subring of R if the following conditions are satisfied, for all x and y in R,

i. $T_A^P(x-y) \ge \min\{T_A^P(x), T_A^P(y)\}$

ii. $T_A^P(xy) \geq \min\{T_A^P(x), T_A^P(y)\}$

iii. $F_A^P(x-y) \leq \max\{F_A^P(x), F_A^P(y)\}$

iv. $F_A^P(xy) \leq \max\{F_A^P(x), F_A^P(y)\}$

v. $T_A^N(x-y) \leq \max\{T_A^N(x), T_A^N(y)\}$

vi. $T_A^N(xy) \leq \max\{T_A^N(x), T_A^N(y)\}$

vii. $F_A^N(x-y) \geq \min\{F_A^N(x), F_A^N(y)\}$

viii. $F_A^N(xy) \geq \min\{F_A^N(x), F_A^N(y)\}$

Example 3.2. Let $R = \{0,1\}$ be a set of integers of modulo 2 with two binary operations as follows:

+	0	1
0	0	1
1	1	0

and

·	0	1
0	0	0
1	0	1

Define bipolar Pythagorean fuzzy set $A = \{(X, T_A^P, F_A^P, T_A^N, F_A^N): x \in X\}$ is given by

$$T_A^P(0) = 0.2, T_A^N(0) = -0.3, F_A^P(0) = 0.6, F_A^N(0) = -0.5,$$
$$T_A^P(1) = 0.3, T_A^N(1) = -0.6, F_A^P(1) = 0.9, F_A^N(1) = -0.4.$$

Then, $(R, +, \cdot)$ is a ring.

Definition 3.3. Let $A = \{(X, T_A^P, F_A^P, T_A^N, F_A^N): x \in X\}$ and $B = \{(X, T_B^P, F_B^P, T_B^N, F_B^N): x \in X\}$ be any two bipolar Pythagorean fuzzy subsets of sets G and H respectively. The product of A and B, denoted by $A \times B$, is defined as

$$A \times B = \left\{\left((x,y), T_{A\times B}^P(x,y), F_{A\times B}^P(x,y), T_{A\times B}^N(x,y), F_{A\times B}^N(x,y)\right): \ x \in G, y \in H\right\}$$

where $T_{A\times B}^P(x,y) = \min\{T_A^P(x), T_B^P(y)\}$, $F_{A\times B}^P(x,y) = \max\{F_A^P(x), F_B^P(y)\}$, $T_{A\times B}^N(x,y) = \max\{T_A^N(x), T_B^N(y)\}$ and $F_{A\times B}^N(x,y) = \min\{F_A^N(x), F_B^N(y)\}$, for all x in G and y in H.

Definition 3.4. Let $A = \{(X, T_A^P, F_A^P, T_A^N, F_A^N) : x \in X\}$ be a bipolar Pythagorean fuzzy subset in a set S, the strongest bipolar Pythagorean fuzzy relation on S that is a bipolar Pythagorean fuzzy relation on A is

$$V = \left\{\left((x,y), T_V^P(x,y), F_V^P(x,y), T_V^N(x,y), F_V^N(x,y)\right): \ x, y \in S\right\}$$

given by $T_V^P(x,y) = \min\{T_A^P(x), T_A^P(y)\}$, $F_V^P(x,y) = \max\{F_A^P(x), F_A^P(y)\}$, $T_V^N(x,y) = \max\{T_A^N(x), T_A^N(y)\}$, and $F_V^N(x,y) = \min\{F_A^N(x), F_A^N(y)\}$, for all x, y in S.

Theorem 3.5. Let $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy subring of a ring R. Then, for all x in R and the identity element e in R

i. $T_A^P(-x) = T_A^P(x)$

ii. $F_A^P(-x) = F_A^P(x)$

iii. $T_A^N(-x) = T_A^N(x)$

iv. $F_A^N(-x) = F_A^N(x)$

v. $T_A^P(x) \leq T_A^P(e)$

vi. $F_A^P(x) \geq F_A^P(e)$

vii. $T_A^N(x) \geq T_A^N(e)$

viii. $F_A^N(x) \leq F_A^N(e)$

Proof. For all x in R,

i. $T_A^P(x) = T_A^P(-(-x)) \geq T_A^P(-x) \geq T_A^P(x)$. Therefore, $T_A^P(x) = T_A^P(-x)$.

ii. $F_A^P(x) = F_A^P(-(-x)) \leq F_A^P(-x) \leq F_A^P(x)$. Therefore, $F_A^P(x) = F_A^P(-x)$.

iii. $T_A^N(x) = T_A^N(-(-x)) \leq T_A^N(-x) \leq T_A^N(x)$. Therefore, $T_A^N(x) = T_A^N(-x)$.

iv. $F_A^N(x) = F_A^N(-(-x)) \geq F_A^N(-x) \geq F_A^N(x)$. Therefore, $F_A^N(x) = F_A^N(-x)$.

v. $T_A^P(e) = T_A^P(x-x) \geq \min\{T_A^P(x), T_A^P(x)\} = T_A^P(x)$. Therefore, $T_A^P(e) \geq T_A^P(x)$.

vi. $F_A^P(e) = F_A^P(x-x) \leq \max\{F_A^P(x), F_A^P(x)\} = F_A^P(x)$. Therefore, $F_A^P(e) \leq F_A^P(x)$.

vii. $T_A^N(e) = T_A^N(x-x) \leq \max\{T_A^N(x), T_A^N(x)\} = T_A^N(x)$. Therefore, $T_A^N(e) \leq T_A^N(x)$.

viii. $F_A^N(e) = F_A^N(x-x) \geq \min\{F_A^N(x), F_A^N(x)\} = F_A^N(x)$. Therefore, $F_A^N(e) \geq F_A^N(x)$.

Theorem 3.6. Let $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy subring of a ring R. Then, for all $x, y \in R$

i. $T_A^P(x-y) = T_A^P(e)$ implies that $T_A^P(x) = T_A^P(y)$

ii. $F_A^P(x-y) = F_A^P(e)$ implies that $F_A^P(x) = F_A^P(y)$

iii. $T_A^N(x-y) = T_A^N(e)$ implies that $T_A^N(x) = T_A^N(y)$

iv. $F_A^N(x-y) = F_A^N(e)$ implies that $F_A^N(x) = F_A^N(y)$

Proof. For all x and y in R,

i. $T_A^P(x) = T_A^P(x-y+y) \geq \min\{T_A^P(x-y), T_A^P(y)\} = \min\{T_A^P(e), T_A^P(y)\} = T_A^P(y)$. $T_A^P(y) = T_A^P(y-x+x) \geq \min\{T_A^P(y-x), T_A^P(x)\} = \min\{T_A^P(e), T_A^P(x)\} = T_A^P(x)$. Therefore, $T_A^P(x) = T_A^P(y)$.

ii. $F_A^P(x) = F_A^P(x-y+y) \leq \max\{F_A^P(x-y), F_A^P(y)\} = \max\{F_A^P(e), F_A^P(y)\} = F_A^P(y)$. $F_A^P(y) = F_A^P(y-x+x) \leq \max\{F_A^P(y-x), F_A^P(x)\} = \max\{F_A^P(e), F_A^P(x)\} = F_A^P(x)$. Therefore, $F_A^P(x) = F_A^P(y)$.

iii. $T_A^N(x) = T_A^N(x-y+y) \leq \max\{T_A^N(x-y), T_A^N(y)\} = \max\{T_A^N(e), T_A^N(y)\} = T_A^N(y)$. $T_A^N(y) = T_A^N(y-x+x) \leq \max\{T_A^N(y-x), T_A^N(x)\} = \max\{T_A^N(e), T_A^N(x)\} = T_A^N(x)$. Therefore, $T_A^N(x) = T_A^N(y)$.

iv. $F_A^N(x) = F_A^N(x-y+y) \geq \min\{F_A^N(x-y), F_A^N(y)\} = \min\{F_A^N(e), F_A^N(y)\} = F_A^N(y)$. $F_A^N(y) = F_A^N(y-x+x) \geq \min\{F_A^N(y-x), F_A^N(x)\} = \min\{F_A^N(e), F_A^N(x)\} = F_A^N(x)$. Therefore, $F_A^N(x) = F_A^N(y)$.

Theorem 3.7. Let $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy subring of a ring R. For all x and y in R,

i. If $T_A^P(x-y) = 1$, then $T_A^P(x) = T_A^P(y)$.

ii. If $F_A^P(x-y) = 0$, then $F_A^P(x) = F_A^P(y)$.

iii. If $T_A^N(x-y) = -1$, then $T_A^N(x) = T_A^N(y)$.

iv. If $F_A^N(x-y) = 0$, then $F_A^N(x) = F_A^N(y)$.

Proof. For all x and y in R,

i. $T_A^P(x) = T_A^P(x-y+y) \geq \min\{T_A^P(x-y), T_A^P(y)\} = \min\{1, T_A^P(y)\} = T_A^P(y) = T_A^P(-y) = T_A^P(-x+x-y) \geq \min\{T_A^P(x), T_A^P(x-y)\} = \min\{T_A^P(x), 1\} = T_A^P(x)$. Therefore, $T_A^P(x) = T_A^P(y)$.
ii. $F_A^P(x) = F_A^P(x-y+y) \leq \max\{F_A^P(x-y), F_A^P(y)\} = \max\{0, F_A^P(y)\} = F_A^P(y) = F_A^P(-y) = F_A^P(-x+x-y) \leq \max\{F_A^P(x), F_A^P(x-y)\} = \max\{F_A^P(x), 0\} = F_A^P(x)$. Therefore, $F_A^P(x) = F_A^P(y)$.
iii. $T_A^N(x) = T_A^N(x-y+y) \leq \max\{T_A^N(x-y), T_A^N(y)\} = \max\{-1, T_A^N(y)\} = T_A^N(y) = T_A^N(-y) = T_A^N(-x+x-y) \leq \max\{T_A^N(x), T_A^N(x-y)\} = \max\{T_A^N(x), -1\} = T_A^N(x)$. Therefore, $T_A^N(x) = T_A^N(y)$.
iv. $F_A^N(x) = F_A^N(x-y+y) \geq \min\{F_A^N(x-y), F_A^N(y)\} = \min\{0, F_A^N(y)\} = F_A^N(y) = F_A^N(-y) = F_A^N(-x+x-y) \geq \min\{F_A^N(x) F_A^N(x-y)\} = \min\{F_A^N(x), 0\} = F_A^N(x)$. Therefore, $F_A^N(x) = F_A^N(y)$.

Theorem 3.8. Let $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy subring of a ring G. for x and y in G,

i. $T_A^P(xy^{-1}) = 0$, then either $T_A^P(x) = 0$ or $T_A^P(y) = 0$.

ii. $F_A^P(xy^{-1}) = 0$, then either $F_A^P(x) = 0$ or $F_A^P(y) = 0$.

iii. $T_A^N(xy^{-1}) = 0$, then either $T_A^N(x) = 0$ or $T_A^N(y) = 0$.

iv. $F_A^N(xy^{-1}) = 0$, then either $F_A^N(x) = 0$ or $F_A^N(y) = 0$.

Proof. Let x and y in G. Then, by the definition

i. $T_A^P(xy^{-1}) \geq \min\{T_A^P(x), T_A^P(y)\}$, which implies that $0 \geq \min\{T_A^P(x), T_A^P(y)\}$. Therefore, either $T_A^P(x) = 0$ or $T_A^P(y) = 0$.
ii. $F_A^P(xy^{-1}) \leq \max\{F_A^P(x), F_A^P(y)\}$, which implies that $0 \leq \max\{F_A^P(x), F_A^P(y)\}$. Therefore, either $F_A^P(x) = 0$ or $F_A^P(y) = 0$.
iii. $T_A^N(xy^{-1}) \leq \max\{T_A^N(x), T_A^N(y)\}$, which implies that $0 \leq \max\{T_A^N(x), T_A^N(y)\}$. Therefore, either $T_A^N(x) = 0$ or $T_A^N(y) = 0$.
iv. $F_A^N(xy^{-1}) \geq \min\{F_A^N(x), F_A^N(y)\}$, which implies that $0 \geq \min\{F_A^N(x), F_A^N(y)\}$. Therefore, either $F_A^N(x) = 0$ or $F_A^N(y) = 0$.

Theorem 3.9. If $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy subring of a ring G, then, for x and y in G

i. $T_A^P(xy) = T_A^P(yx)$ if and only if $T_A^P(x) = T_A^P(y^{-1}xy)$.

ii. $F_A^P(xy) = F_A^P(yx)$ if and only if $F_A^P(x) = F_A^P(y^{-1}xy)$.

iii. $T_A^N(xy) = T_A^N(yx)$ if and only if $T_A^N(x) = T_A^N(y^{-1}xy)$.

iv. $F_A^N(xy) = F_A^N(yx)$ if and only if $F_A^N(x) = F_A^N(y^{-1}xy)$.

Proof. Let x and y in G.

i. Assume that $T_A^P(xy) = T_A^P(yx)$, so $T_A^P(y^{-1}xy) = T_A^P(y^{-1}yx) = T_A^P(x)$. Therefore, $T_A^P(x) = T_A^P(y^{-1}xy)$, for x and y in G. Conversely, assume that $T_A^P(x) = T_A^P(y^{-1}xy)$, we get $T_A^P(xy) = T_A^P(xyxx^{-1}) = T_A^P(yx)$. Therefore, $T_A^P(xy) = T_A^P(yx)$.

ii. Assume that $F_A^P(xy) = F_A^P(yx)$, so $F_A^P(y^{-1}xy) = F_A^P(y^{-1}yx) = F_A^P(x)$. Therefore, $F_A^P(x) = F_A^P(y^{-1}xy)$. Conversely, assume that $F_A^P(x) = F_A^P(y^{-1}xy)$, we get $F_A^P(xy) = F_A^P(xyxx^{-1}) = F_A^P(yx)$. Therefore, $F_A^P(xy) = F_A^P(yx)$.
iii. Assume that $T_A^N(xy) = T_A^N(yx)$, so $T_A^N(y^{-1}xy) = T_A^N(y^{-1}yx) = T_A^N(x)$. Therefore, $T_A^N = T_A^N(y^{-1}xy)$. Conversely, assume that $T_A^N(x) = T_A^N(y^{-1}xy)$, we get $T_A^N(xy) = T_A^N(xyxx^{-1}) = T_A^N(yx)$. Therefore, $T_A^N(xy) = T_A^N(yx)$.
iv. Assume that $F_A^N(xy) = F_A^N(yx)$, so $F_A^N(y^{-1}xy) = F_A^N(y^{-1}yx) = F_A^N(x)$. Therefore, $F_A^N(x) = F_A^N(y^{-1}xy)$. Conversely, assume that $F_A^N(x) = F_A^N(y^{-1}xy)$, we get $F_A^N(xy) = F_A^N(xyxx^{-1}) = F_A^N(yx)$. Therefore, $F_A^N(xy) = F_A^N(yx)$.

Theorem 3.10. If $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy subring of a ring G, then $H = \{x \in G : T_A^P(x) = T_A^P(e),\ F_A^P(x) = F_A^P(e), T_A^N(x) = T_A^N(e), F_A^N(x) = F_A^N(e)\}$ is a subring of G.
PROOF. Here, $H = \{x \in G : T_A^P(x) = T_A^P(e),\ F_A^P(x) = F_A^P(e), T_A^N(x) = T_A^N(e),\ F_A^N(x) = F_A^N(e)\}$, by Theorem 3.1, $T_A^P(x^{-1}) = T_A^P(x) = T_A^P(e), F_A^P(x^{-1}) = F_A^P(x) = F_A^P(e), T_A^N(x^{-1}) = T_A^N(x) = T_A^N(e)$ and $F_A^N(x^{-1}) = F_A^N(x) = F_A^N(e)$. Therefore, $x^{-1} \in H$. Now, $T_A^P(xy^{-1}) \geq \min\{T_A^P(x), T_A^P(y)\} = \min\{T_A^P(e), T_A^P(e)\} = T_A^P(e)$, and $T_A^P(e) = T_A^P\big((xy^{-1})(xy^{-1})^{-1}\big) \geq \min\{T_A^P(xy^{-1}), T_A^P(xy^{-1})\} = T_A^P(xy^{-1})$.
Hence, $T_A^P(e) = T_A^P(xy^{-1})$. $F_A^P(xy^{-1}) \leq \max\{F_A^P(x), F_A^P(y)\} = \max\{F_A^P(e), F_A^P(e)\} = F_A^P(e)$, and $F_A^P(e) = F_A^P\big((xy^{-1})(xy^{-1})^{-1}\big) \leq \max\{F_A^P(xy^{-1}), F_A^P(xy^{-1})\} = F_A^P(xy^{-1})$.
Hence, $F_A^P(e) = F_A^P(xy^{-1})$. $T_A^N(xy^{-1}) \leq \max\{T_A^N(x), T_A^N(y)\} = \max\{T_A^N(e), T_A^N(e)\} = T_A^N(e)$ and $T_A^N(e) = T_A^N\big((xy^{-1})(xy^{-1})^{-1}\big) \leq \max\{T_A^N(xy^{-1}), T_A^N(xy^{-1})\} = T_A^N(xy^{-1})$.
Hence, $T_A^N(e) = T_A^N(xy^{-1})$. $F_A^N(xy^{-1}) \geq \min\{F_A^N(x), F_A^N(y)\} = \min\{F_A^N(e), F_A^N(e)\} = F_A^N(e)$, and $F_A^N(e) = F_A^N\big((xy^{-1})(xy^{-1})^{-1}\big) \geq \min\{F_A^N(xy^{-1}), F_A^N(xy^{-1})\} = F_A^N(xy^{-1})$.
Hence, $F_A^N(e) = F_A^N(xy^{-1})$. Therefore, $xy^{-1} \in H$. Hence, H is a subring of G.

Theorem 3.11. Let G be a ring. If $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy subring of a ring G, then, for each x and y in G with $T_A^P(x) \neq T_A^P(y)$, $F_A^P(x) \neq F_A^P(y)$, $T_A^N(x) \neq T_A^N(y)$ and $F_A^N(x) \neq F_A^N(y)$,

i. $T_A^P(xy) = \min\{T_A^P(x), T_A^P(y)\}$

ii. $F_A^P(xy) = \max\{F_A^P(x), F_A^P(y)\}$

iii. $T_A^N(xy) = \max\{T_A^N(x), T_A^N(y)\}$

iv. $F_A^N(xy) = \min\{F_A^N(x), F_A^N(y)\}$

PROOF.

i. Assume that $T_A^P(x) > T_A^P(y), F_A^P(x) > F_A^P(y), T_A^N(x) > T_A^N(y)$ and $F_A^N(x) > F_A^N(y)$. Then, $T_A^P(y) = T_A^P(x^{-1}xy) \geq \min\{T_A^P(x^{-1}), T_A^P(xy)\} = \min\{T_A^P(x), T_A^P(xy)\} = T_A^P(xy) \geq \min\{T_A^P(x), T_A^P(y)\} = T_A^P(y)$. Therefore, $T_A^P(xy) = T_A^P(y) = \min\{T_A^P(x), T_A^P(y)\}$.
ii. $F_A^P(y) = F_A^P(x^{-1}xy) \leq \max\{F_A^P(x^{-1}), F_A^P(xy)\} = \max\{F_A^P(x), F_A^P(xy)\} = F_A^P(xy) \leq \max\{F_A^P(x), F_A^P(y)\} = F_A^P(y)$. Therefore, $F_A^P(xy) = F_A^P(y) = \max\{F_A^P(x), F_A^P(y)\}$.
iii. $T_A^N(y) = T_A^N(x^{-1}xy) \leq \max\{T_A^N(x^{-1}), T_A^N(xy)\} = \max\{T_A^N(x), T_A^N(xy)\} = T_A^N(xy) \leq \max\{T_A^N(x), T_A^N(y)\} = T_A^N(y)$. Therefore, $T_A^N(xy) = T_A^N(y) = \max\{T_A^N(x), T_A^N(y)\}$.
iv. $F_A^N(y) = F_A^N(x^{-1}xy) \geq \min\{F_A^N(x^{-1}), F_A^N(xy)\} = \min\{F_A^N(x), F_A^N(xy)\} = F_A^N(xy) \geq \min\{F_A^N(x), F_A^N(y)\} = F_A^N(y)$. Therefore, $F_A^N(xy) = F_A^N(y) = \min\{F_A^N(x), F_A^N(y)\}$.

Theorem 3.12. If $A=(X,T_A^P,F_A^P,T_A^N,F_A^N)$ and $B=(X,T_B^P,F_B^P,T_B^N,F_B^N)$ are two both bipolar Pythagorean fuzzy subrings of a ring G, then their $A\cap B$ is a bipolar Pythagorean fuzzy subring of G.

PROOF. Let $A=\{(X,T_A^P,F_A^P,T_A^N,F_A^N):x\in G\}$ and $B=\{(X,T_B^P,F_B^P,T_B^N,F_B^N):x\in G\}$. Let $C=A\cap B$ and $C=\{(X,T_C^P,F_C^P,T_C^N,F_C^N):x\in G\}$.

$$\begin{aligned}T_C^P(xy^{-1}) &= \min\{T_A^P(xy^{-1}),T_B^P(xy^{-1})\}\\ &\geq \min\left\{\min\{T_A^P(x),T_A^P(y)\},\min\{T_B^P(x),T_B^P(y)\}\right\}\\ &\geq \min\left\{\min\{T_A^P(x),T_B^P(x)\},\min\{T_A^P(y),T_B^P(y)\}\right\}\\ &= \min\{T_C^P(x),T_C^P(y)\}\end{aligned}$$

Also,

$$\begin{aligned}F_C^P(xy^{-1}) &= \max\{F_A^P(xy^{-1}),F_B^P(xy^{-1})\}\\ &\leq \max\left\{\max\{F_A^P(x),F_A^P(y)\},\max\{F_B^P(x),F_B^P(y)\}\right\}\\ &\leq \max\left\{\max\{F_A^P(x),F_B^P(x)\},\max\{F_A^P(y),F_B^P(y)\}\right\}\\ &= \max\{F_C^P(x),F_C^P(y)\},\\ T_C^N(xy^{-1}) &= \max\{T_A^N(xy^{-1}),T_B^N(xy^{-1})\}\\ &\leq \max\left\{max\{T_A^N(x),T_A^N(y)\},max\{T_B^N(x),T_B^N(y)\}\right\}\\ &\leq \max\left\{\max\{T_A^N(x),T_B^N(x)\},\max\{T_A^N(y),T_B^N(y)\}\right\}\\ &= \max\{T_C^N(x),T_C^N(y)\}\\ F_C^N(xy^{-1}) &= \min\{F_A^N(xy^{-1}),F_B^N(xy^{-1})\}\\ &\geq \min\left\{\min\{F_A^N(x),F_A^N(y)\},min\{F_B^N(x),F_B^N(y)\}\right\}\\ &\geq \min\left\{\min\{F_A^N(x),F_B^N(x)\},min\{F_A^N(y),F_B^N(y)\}\right\}\\ &= \min\{F_C^N(x),F_C^N(y)\}\end{aligned}$$

Hence, $A\cap B$ is a bipolar Pythagorean fuzzy subring of G.

Theorem 3.13. The intersection of a family of bipolar Pythagorean fuzzy subrings of a ring G is a bipolar Pythagorean fuzzy subring of G.

PROOF. Let $\{V_i: i\in I\}$ be a family of bipolar Pythagorean fuzzy subrings of a ring G and let $A=\cap_{i\in I}V_i$. Let x and y in G. Now,

$$\begin{aligned}T_A^P(xy^{-1}) = inf_{i\in I}T_{V_i}^P(xy^{-1}) &\geq \inf_{i\in I}\min\{T_{V_i}^P(x),T_{V_i}^P(y)\}\\ &= \min\{\inf_{i\in I}T_{V_i}^P(x),\inf_{i\in I}T_{V_i}^P(y)\}\\ &= \min\{T_A^P(x),T_A^P(y)\}\end{aligned}$$

Therefore, $T_A^P(xy^{-1})\geq \min\{T_A^P(x),T_A^P(y)\}$, for all x and y in G, and

$$\begin{aligned}F_A^P(xy^{-1}) = sup_{i\in I}F_{V_i}^P(xy^{-1}) &\leq sup_{i\in I}\max\{F_{V_i}^P(x),F_{V_i}^P(y)\}\\ &= \max\{sup_{i\in I}F_{V_i}^P(x),\sup_{i\in I}F_{V_i}^P(y)\}\\ &= \max\{F_A^P(x),F_A^P(y)\}\end{aligned}$$

Therefore, $F_A^P(xy^{-1})\leq \max\{F_A^P(x),F_A^P(y)\}$, for all x and y in G.

$$\begin{aligned}T_A^N(xy^{-1}) = sup_{i\in I}T_{V_i}^N(xy^{-1}) &\leq \sup_{i\in I}\max\{T_{V_i}^N(x),T_{V_i}^N(y)\}\\ &= \max\{sup_{i\in I}T_{V_i}^N(x),\sup_{i\in I}T_{V_i}^N(y)\}\\ &= \max\{T_A^N(x),T_A^N(y)\}\end{aligned}$$

Therefore, $T_A^N(xy^{-1}) \leq \max\{T_A^N(x), T_A^N(y)\}$, for all x and y in G.

$$F_A^N(xy^{-1}) = \inf_{i\in I} F_{V_i}^N(xy^{-1}) \geq \inf_{i\in I}\min\{F_{V_i}^N(x), F_{V_i}^N(y)\}$$
$$= \min\{\inf_{i\in I} F_{V_i}^N(x), \inf_{i\in I} F_{V_i}^N(y)\}$$
$$= \min\{F_A^N(x), F_A^N(y)\}$$

Therefore, $F_A^N(xy^{-1}) \geq \min\{F_A^N(x), F_A^N(y)\}$, for all x and y in G. Hence, the intersection of a family of bipolar Pythagorean fuzzy subrings of a ring G is a bipolar Pythagorean fuzzy subring of G.

Theorem 3.14. If $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ and $B = (X, T_B^P, F_B^P, T_B^N, F_B^N)$ are any two bipolar Pythagorean fuzzy subrings of rings G_1 and G_2, respectively, then $A \times B = (T_{A\times B}^P, F_{A\times B}^P, T_{A\times B}^N, F_{A\times B}^N)$ is a bipolar Pythagorean fuzzy subring of $G_1 \times G_2$.

PROOF. Let A and B be two bipolar Pythagorean fuzzy subrings of the ring G_1 and G_2, respectively. Let x_1 and x_2 be in G_1, y_1 and y_2 be in G_2. Then, (x_1, y_1) and (x_2, y_2) are in $G_1 \times G_2$. Now,

$$T_{A\times B}^P[(x_1,y_1)(x_2,y_2)^{-1}] = T_{A\times B}^P(x_1{x_2}^{-1}, y_1{y_2}^{-1})$$
$$= min\{T_A^P(x_1{x_2}^{-1}), T_B^P(y_1{y_2}^{-1})\}$$
$$\geq \min\left\{\min\{T_A^P(x_1), T_A^P(x_2)\}, \min\{T_B^P(y_1), T_B^P(y_2)\}\right\}$$
$$= \min\left\{\min\{T_A^P(x_1), T_B^P(y_1)\}, \min\{T_A^P(x_2), T_B^P(y_2)\}\right\}$$
$$= \min\{T_{A\times B}^P(x_1,y_1), T_{A\times B}^P(x_2,y_2)\}$$

Therefore, $T_{A\times B}^P[(x_1,y_1)(x_2,y_2)^{-1}] \geq \min\{T_{A\times B}^P(x_1,y_1), T_{A\times B}^P(x_2,y_2)\}$.

Also,

$$F_{A\times B}^P[(x_1,y_1)(x_2,y_2)^{-1}] = F_{A\times B}^P(x_1{x_2}^{-1}, y_1{y_2}^{-1})$$
$$= \max\{F_A^P(x_1{x_2}^{-1}), F_B^P(y_1{y_2}^{-1})\}$$
$$\leq \max\left\{\max\{F_A^P(x_1), F_A^P(x_2)\}, \max\{F_B^P(y_1), F_B^P(y_2)\}\right\}$$
$$= \max\left\{max\{F_A^P(x_1), F_B^P(y_1)\}, \max\{F_A^P(x_2), F_B^P(y_2)\}\right\}$$
$$= \max\{F_{A\times B}^P(x_1,y_1), F_{A\times B}^P(x_2,y_2)\}$$

Therefore, $F_{A\times B}^P[(x_1,y_1)(x_2,y_2)^{-1}] \leq \max\{F_{A\times B}^P(x_1,y_1), F_{A\times B}^P(x_2,y_2)\}$.

$$T_{A\times B}^N[(x_1,y_1)(x_2,y_2)^{-1}] = T_{A\times B}^N(x_1{x_2}^{-1}, y_1{y_2}^{-1})$$
$$= \max\{T_A^N(x_1{x_2}^{-1}), T_B^N(y_1{y_2}^{-1})\}$$
$$\leq \max\left\{\max\{T_A^N(x_1), T_A^N(x_2)\}, \max\{T_B^N(y_1), T_B^N(y_2)\}\right\}$$
$$= \max\left\{max\{T_A^N(x_1), T_B^N(y_1)\}, \max\{T_A^N(x_2), T_B^N(y_2)\}\right\}$$
$$= \max\{T_{A\times B}^N(x_1,y_1), T_{A\times B}^N(x_2,y_2)\}$$

Therefore, $T_{A\times B}^N[(x_1,y_1)(x_2,y_2)^{-1}] \leq \max\{T_{A\times B}^N(x_1,y_1), T_{A\times B}^N(x_2,y_2)\}$.

$$F_{A\times B}^N[(x_1,y_1)(x_2,y_2)^{-1}] = F_{A\times B}^N(x_1{x_2}^{-1}, y_1{y_2}^{-1})$$
$$= \min\{F_A^N(x_1{x_2}^{-1}), F_B^N(y_1{y_2}^{-1})\}$$
$$\geq \min\left\{\min\{F_A^N(x_1), F_A^N(x_2)\}, \min\{F_B^N(y_1), F_B^N(y_2)\}\right\}$$
$$= \min\left\{\min\{F_A^N(x_1), F_B^N(y_1)\}, \min\{F_A^N(x_2), F_B^N(y_2)\}\right\}$$
$$= \min\{F_{A\times B}^N(x_1,y_1), F_{A\times B}^N(x_2,y_2)\}$$

Therefore, $F_{A\times B}^N[(x_1,y_1)(x_2,y_2)^{-1}] \geq \min\{F_{A\times B}^N(x_1,y_1), F_{A\times B}^N(x_2,y_2)\}$. Hence, $A \times B$ is a bipolar Pythagorean fuzzy subring of $G_1 \times G_2$.

Theorem 3.15. Let $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ and $B = (X, T_B^P, F_B^P, T_B^N, F_B^N)$ be any two bipolar Pythagorean fuzzy subsets of a ring G and H respectively. Suppose that e and e' are the identity elements of G and H, respectively. If $A \times B$ is a bipolar Pythagorean fuzzy subring of $G \times H$, then at least one of the following two elements must hold.

i. $T_B^P(e') \geq T_A^P(x), F_B^P(e') \leq F_A^P(x)$for all x in G and $T_B^N(e') \leq T_A^N(x), F_B^N(e') \geq F_A^N(x)$ for all x in G.

ii. $T_B^P(e) \geq T_A^P(y), F_B^P(e) \leq F_A^P(y)$for all x in G and $T_B^N(e) \leq T_A^N(y), F_B^N(e) \geq F_A^N(y)$ for all y in H.

Proof. Let $A \times B$ be a bipolar Pythagorean fuzzy subring of $G \times H$. By contraposition, suppose that none of the statements (i) and (ii) holds. Then, we can find a in G and b in H such that $T_A^P(a) > T_B^P(e')$, $F_A^P(a) < F_B^P(e')$, $T_A^N(a) < T_B^N(e')$, $F_A^N(a) > F_B^N(e')$, and $T_B^P(b) > T_A^P(e)$, $F_B^P(b) < F_A^P(e)$, $T_B^N(b) < T_A^N(e)$, $F_B^N(b) > F_A^N(e)$. We have $T_{A\times B}^P(a,b) = \min\{T_A^P(a), T_B^P(b)\} > \min\{T_A^P(e), T_B^P(e')\} = T_{A\times B}^P(e,e')$. Also,

$$
\begin{aligned}
F_{A\times B}^P(a,b) = \max\{F_A^P(a), F_B^P(b)\} &< \max\{F_A^P(e), F_B^P(e')\} \\
&= F_{A\times B}^P(e,e'). T_{A\times B}^N(a,b) \\
&= \max\{T_A^N(a), T_B^N(b)\} \\
&< \max\{T_A^N(e), T_B^N(e')\} \\
&= T_{A\times B}^N(e,e'). F_{A\times B}^N(a,b) \\
&= \min\{F_A^N(a), F_B^N(b)\} \\
&> \min\{F_A^N(e), F_B^N(e')\} \\
&= F_{A\times B}^N(e,e')
\end{aligned}
$$

Thus, $A \times B$ is not a bipolar Pythagorean fuzzy subring of $G \times H$. Hence, either, for all x in G,

$$T_B^P(e') \geq T_A^P(x),\ F_B^P(e') \leq F_A^P(x), T_B^N(e') \leq T_A^N(x), \text{ and } F_B^N(e') \geq F_A^N(x)$$

or, for all y in H,

$$T_B^P(e) \geq T_A^P(y),\ F_B^P(e) \leq F_A^P(y), T_B^N(e) \leq T_A^N(y), \text{ and } F_B^N(e) \geq F_A^N(y)$$

Theorem 3.16. Let $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ and $B = (X, T_B^P, F_B^P, T_B^N, F_B^N)$ be any two bipolar Pythagorean fuzzy subsets of a ring G and H, respectively and $A \times B$ be a bipolar Pythagorean fuzzy subring of $G \times H$. Then, the following are true

i. If $T_A^P(x) \leq T_B^P(e'), F_A^P(x) \geq F_B^P(e'), T_A^N(x) \geq T_B^N(e')$, and $F_A^N(x) \leq F_B^N(e')$ for all x in G, then A is a bipolar Pythagorean fuzzy subring of G, where e' is the identity element of H.

ii. If $T_B^P(x) \leq T_A^P(e), F_B^P(x) \geq F_A^P(e), T_B^N(x) \geq T_A^N(e)$, and $F_B^N(x) \leq F_A^N(e)$ for all x in H, then B is a bipolar Pythagorean fuzzy subring of H, where e is the identity element of G.

iii. Either A is a bipolar Pythagorean fuzzy subring of G or B is a bipolar Pythagorean fuzzy subring of H, where e and e' is the identity element of G and H, respectively.

Proof. Let $A \times B$ be a bipolar Pythagorean fuzzy subring of $G \times H$ and x and y in G. Then, (x, e') and (y, e') are in $G \times H$. Now, using the property if $T_A^P(x) \leq T_B^P(e'), F_A^P(x) \geq F_B^P(e'), T_A^N(x) \geq T_B^N(e')$, and $F_A^N(x) \leq F_B^N(e')$, for all x in G, where e' is the identity element of H, we get

$$
\begin{aligned}
T_A^P(xy^{-1}) &= \min\{T_A^P(xy^{-1}), T_B^P(e'e')\} \\
&= T_{A\times B}^P\big((xy^{-1}),(e'e')\big) \\
&= T_{A\times B}^P[(x,e')(y^{-1},e')] \\
&\geq \min\{T_{A\times B}^P(x,e'), T_{A\times B}^P(y^{-1},e')\} \\
&= \min\Big\{\min\{T_A^P(x), T_B^P(e')\}, \min\{T_A^P(y^{-1}), T_B^P(e')\}\Big\} \\
&= \min\{T_A^P(x), T_A^P(y^{-1})\} \\
&= \min\{T_A^P(x), T_A^P(y)\}
\end{aligned}
$$

Therefore, $T_A^P(xy^{-1}) \geq \min\{T_A^P(x), T_A^P(y)\}$ for all $x$and y in G. Also,

$$
\begin{aligned}
F_A^P(xy^{-1}) &= \max\{F_A^P(xy^{-1}), F_B^P(e'e')\} \\
&= F_{A\times B}^P\big((xy^{-1}),(e'e')\big)
\end{aligned}
$$

$$= F^P_{A\times B}[(x,e')(y^{-1},e')]$$
$$\leq \max\{F^P_{A\times B}(x,e'), F^P_{A\times B}(y^{-1},e')\}$$
$$= \max\left\{\max\{F^P_A(x), F^P_B(e')\}, \max\{F^P_A(y^{-1}), F^P_B(e')\}\right\}$$
$$= \max\{F^P_A(x), F^P_A(y^{-1})\}$$
$$= \max\{F^P_A(x), F^P_A(y)\}$$

Therefore, $F^P_A(xy^{-1}) \leq \max\{F^P_A(x), F^P_A(y)\}$ for all x and y in G.

$$\begin{aligned} T^N_A(xy^{-1}) &= \max\{T^N_A(xy^{-1}), T^N_B(e'e')\} \\ &= T^N_{A\times B}\big((xy^{-1}),(e'e')\big) \\ &= T^N_{A\times B}[(x,e')(y^{-1},e')] \\ &\leq \max\{T^N_{A\times B}(x,e'), T^N_{A\times B}(y^{-1},e')\} \\ &= \max\left\{\max\{T^N_A(x), T^N_B(e')\}, \max\{T^N_A(y^{-1}), T^N_B(e')\}\right\} \\ &= \max\{T^N_A(x), T^N_A(y^{-1})\} \\ &= \max\{T^N_A(x), T^N_A(y)\} \end{aligned}$$

Therefore, $T^N_A(xy^{-1}) \leq \max\{T^N_A(x), T^N_A(y)\}$ for all x and y in G.

$$\begin{aligned} F^N_A(xy^{-1}) &= \min\{F^N_A(xy^{-1}), F^N_B(e'e')\} \\ &= F^N_{A\times B}\big((xy^{-1}),(e'e')\big) \\ &= F^N_{A\times B}[(x,e')(y^{-1},e')] \\ &\geq \min\{F^N_{A\times B}(x,e'), F^N_{A\times B}(y^{-1},e')\} \\ &= \min\left\{\min\{F^N_A(x), F^N_B(e')\}, \min\{F^N_A(y^{-1}), F^N_B(e')\}\right\} \\ &= \min\{F^N_A(x), F^N_A(y^{-1})\} \\ &= \min\{F^N_A(x), F^N_A(y)\} \end{aligned}$$

Therefore, $F^N_A(xy^{-1}) \geq \min\{F^N_A(x), F^N_A(y)\}$, for all x and y in G. Hence, A is a bipolar Pythagorean fuzzy subring of G. Thus, (i) is proved.

Now, using the property $T^P_B(x) \leq T^P_A(e), F^P_B(x) \geq F^P_A(e), T^N_B(x) \geq T^N_A(e)$, and $F^N_B(x) \leq F^N_A(e)$, for all x in H, we get

$$\begin{aligned} T^P_B(xy^{-1}) &= \min\{T^P_B(xy^{-1}), T^P_A(e.e)\} \\ &= T^P_{A\times B}\big((e.e),(xy^{-1})\big) \\ &= T^P_{A\times B}[(e,x)(e,y^{-1})] \\ &\geq \min\{T^P_{A\times B}(e,x), T^P_{A\times B}(e,y^{-1})\} \\ &= \min\left\{\min\{T^P_A(e), T^P_B(x)\}, \min\{T^P_A(e), T^P_B(y^{-1})\}\right\} \\ &= \min\{T^P_B(x), T^P_B(y^{-1})\} \\ &= \min\{T^P_B(x), T^P_B(y)\} \end{aligned}$$

Therefore, $T^P_B(xy^{-1}) \geq \min\{T^P_B(x), T^P_B(y)\}$ for all x and $y$in H. Also,

$$\begin{aligned} F^P_B(xy^{-1}) &= \max\{F^P_B(xy^{-1}), F^P_A(e.e)\} \\ &= F^P_{A\times B}\big((e.e),(xy^{-1})\big) \\ &= F^P_{A\times B}[(e,x)(e,y^{-1})] \\ &\leq \max\{F^P_{A\times B}(e,x), F^P_{A\times B}(e,y^{-1})\} \\ &= \max\left\{\max\{F^P_A(e), F^P_B(x)\}, \max\{F^P_A(e), F^P_B(y^{-1})\}\right\} \\ &= \max\{F^P_B(x), F^P_B(y^{-1})\} \\ &= \max\{F^P_B(x), F^P_B(y)\} \end{aligned}$$

Therefore, $F^P_B(xy^{-1}) \leq \max\{F^P_B(x), F^P_B(y)\}$ for all x and y in H.

$$\begin{aligned} T_B^N(xy^{-1}) &= \max\{T_B^N(xy^{-1}), T_A^N(e.e)\} \\ &= T_{A\times B}^N\big((e.e),(xy^{-1})\big) \\ &= T_{A\times B}^N[(e,x)(e,y^{-1})] \\ &\leq \max\{T_{A\times B}^N(e,x), T_{A\times B}^N(e,y^{-1})\} \\ &= \max\Big\{\max\{T_A^N(e), T_B^N(x)\}, \max\{T_A^N(e), T_B^N(y^{-1})\}\Big\} \\ &= \max\{T_B^N(x), T_B^N(y^{-1})\} \\ &= \max\{T_B^N(x), T_B^N(y)\} \end{aligned}$$

Therefore, $T_B^N(xy^{-1}) \leq \max\{T_B^N(x), T_B^N(y)\}$, for all x and y in H.

$$\begin{aligned} F_B^N(xy^{-1}) &= \min\{F_B^N(xy^{-1}), F_A^N(e.e)\} \\ &= F_{A\times B}^N\big((e.e),(xy^{-1})\big) \\ &= F_{A\times B}^N[(e,x)(e,y^{-1})] \\ &\geq \min\{F_{A\times B}^N(e,x), F_{A\times B}^N(e,y^{-1})\} \\ &= \min\Big\{\max\{F_A^N(e), F_B^N(x)\}, \min\{F_A^N(e), F_B^N(y^{-1})\}\Big\} \\ &= \min\{F_B^N(x), F_B^N(y^{-1})\} \\ &= \min\{F_B^N(x), F_B^N(y)\} \end{aligned}$$

Therefore, $F_B^N(xy^{-1}) \geq \min\{F_B^N(x), F_B^N(y)\}$, for all x and y in H. Hence, B is a bipolar Pythagorean fuzzy subring of H. Thus (ii) is proved. Hence (iii) is clear.

Theorem 3.17. Let $A = (X, T_A^P, F_A^P, T_A^N, F_A^N)$ be a bipolar Pythagorean fuzzy subset of a ring $(G,.)$ and $V = (X, T_V^P, F_V^P, T_V^N, F_V^N)$ be the strongest bipolar Pythagorean fuzzy relation of G. Then A is a bipolar Pythagorean fuzzy subring of G if and only if V is a bipolar Pythagorean fuzzy subring of $G \times G$.

Proof. Suppose that A is a bipolar Pythagorean fuzzy subring of G. Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $G \times G$. We have,

$$\begin{aligned} T_V^P(xy^{-1}) = T_V^P[(x_1,x_2)(y_1,y_2)^{-1}] &= T_V^P({x_1y_1}^{-1}, {x_2y_2}^{-1}) \\ &= \min\{T_A^P({x_1y_1}^{-1}), T_A^P({x_2y_2}^{-1})\} \\ &\geq \min\Big\{\min\{T_A^P(x_1), T_A^P(y_1)\}, \min\{T_A^P(x_2), T_A^P(y_2)\}\Big\} \\ &= \min\Big\{\min\{T_A^P(x_1), T_A^P(x_2)\}, \min\{T_A^P(y_1), T_A^P(y_2)\}\Big\} \\ &= \min\{T_V^P(x_1,x_2), T_V^P(y_1,y_2)\} \\ &= \min\{T_V^P(x), T_V^P(y)\} \end{aligned}$$

Therefore, $T_V^P(xy^{-1}) \geq min\{T_V^P(x), T_V^P(y)\}$, for all x and y in $G \times G$. Also, we have,

$$\begin{aligned} F_V^P(xy^{-1}) = F_V^P[(x_1,x_2)(y_1,y_2)^{-1}] &= F_V^P({x_1y_1}^{-1}, {x_2y_2}^{-1}) \\ &= \max\{F_A^P({x_1y_1}^{-1}), F_A^P({x_2y_2}^{-1})\} \\ &\leq \max\Big\{\max\{F_A^P(x_1), F_A^P(y_1)\}, \max\{F_A^P(x_2), F_A^P(y_2)\}\Big\} \\ &= \max\Big\{\max\{F_A^P(x_1), F_A^P(x_2)\}, \max\{F_A^P(y_1), F_A^P(y_2)\}\Big\} \\ &= \max\{F_V^P(x_1,x_2), F_V^P(y_1,y_2)\} \\ &= \max\{F_V^P(x), F_V^P(y)\} \end{aligned}$$

Therefore, $F_V^P(xy^{-1}) \leq \max\{F_V^P(x), F_V^P(y)\}$, for all x and y in $G \times G$.

$$\begin{aligned} T_V^N(xy^{-1}) = T_V^N[(x_1,x_2)(y_1,y_2)^{-1}] &= T_V^N({x_1y_1}^{-1}, {x_2y_2}^{-1}) \\ &= \max\{T_A^N({x_1y_1}^{-1}), T_A^N({x_2y_2}^{-1})\} \\ &\leq \max\Big\{\max\{T_A^N(x_1), T_A^N(y_1)\}, \max\{T_A^N(x_2), T_A^N(y_2)\}\Big\} \\ &= \max\Big\{\max\{T_A^N(x_1), T_A^N(x_2)\}, \max\{T_A^N(y_1), T_A^N(y_2)\}\Big\} \end{aligned}$$

$$= \max\{T_V^N(x_1,x_2), T_V^N(y_1,y_2)\}$$
$$= \max\{T_V^N(x), T_V^N(y)\}$$

Therefore, $T_V^N(xy^{-1}) \le \max\{T_V^N(x), T_V^N(y)\}$, for all x and y in $G \times G$.

$$F_V^N(xy^{-1}) = F_V^N[(x_1,x_2)(y_1,y_2)^{-1}] = F_V^N(x_1{y_1}^{-1}, x_2{y_2}^{-1})$$
$$= \min\{F_A^N(x_1{y_1}^{-1}), F_A^N(x_2{y_2}^{-1})\}$$
$$\ge \min\left\{\min\{F_A^N(x_1), F_A^N(y_1)\}, \min\{F_A^N(x_2), F_A^N(y_2)\}\right\}$$
$$= \min\left\{\min\{F_A^N(x_1), F_A^N(x_2)\}, \min\{F_A^N(y_1), F_A^N(y_2)\}\right\}$$
$$= \min\{F_V^N(x_1,x_2), F_V^N(y_1,y_2)\}$$
$$= \min\{F_V^N(x), F_V^N(y)\}$$

Therefore, $F_V^N(xy^{-1}) \ge \min\{F_V^N(x), F_V^N(y)\}$, for all x and y in $G \times G$. This proves that V is a bipolar Pythagorean fuzzy subring of $G \times G$. Conversely, assume that V is a bipolar Pythagorean fuzzy subring of $G \times G$, then for any$x = (x_1,x_2)$ and $y = (y_1,y_2)$ are in $G \times G$, we have

$$\min\{T_A^P(x_1y_1^{-1}), T_A^P(x_2y_2^{-1})\} = T_V^P(x_1y_1^{-1}, x_2y_2^{-1})$$
$$= T_V^P[(x_1,x_2)(y_1,y_2)^{-1}]$$
$$= T_V^P(xy^{-1})$$
$$\ge \min\{T_V^P(x), T_V^P(y)\}$$
$$= \min\{T_V^P(x_1,x_2), T_V^P(y_1,y_2)\}$$
$$= \min\left\{\min\{T_A^P(x_1), T_A^P(x_2)\}, \min\{T_A^P(y_1), T_A^P(y_2)\}\right\}$$

If we put $x_2 = y_2 = e$, we get, $T_A^P(x_1y_1^{-1}) \ge \min\{T_A^P(x_1), T_A^P(y_1)\}$, for all x_1 and y_1 in G. Also, we have

$$\max\{F_A^P(x_1y_1^{-1}), F_A^P(x_2y_2^{-1})\} = F_V^P(x_1y_1^{-1}, x_2y_2^{-1})$$
$$= F_V^P[(x_1,x_2)(y_1,y_2)^{-1}]$$
$$= F_V^P(xy^{-1})$$
$$\le \max\{F_V^P(x), F_V^P(y)\}$$
$$= max\{F_V^P(x_1,x_2), F_V^P(y_1,y_2)\}$$
$$= \max\left\{\max\{F_A^P(x_1), F_A^P(x_2)\}, \max\{F_A^P(y_1), F_A^P(y_2)\}\right\}$$

If we put $x_2 = y_2 = e$, we get, $F_A^P(x_1y_1^{-1}) \le \max\{F_A^P(x_1), F_A^P(y_1)\}$, for all x_1 and y_1 in G.

$$\max\{T_A^N(x_1y_1^{-1}), T_A^N(x_2y_2^{-1})\} = T_V^N(x_1y_1^{-1}, x_2y_2^{-1})$$
$$= T_V^N[(x_1,x_2)(y_1,y_2)^{-1}]$$
$$= T_V^N(xy^{-1})$$
$$\le \max\{T_V^N(x), T_V^N(y)\}$$
$$= \max\{T_V^N(x_1,x_2), T_V^N(y_1,y_2)\}$$
$$= \max\left\{\max\{T_A^N(x_1), T_A^N(x_2)\}, \max\{T_A^N(y_1), T_A^N(y_2)\}\right\}$$

If we put $x_2 = y_2 = e$, we get, $T_A^N(x_1y_1^{-1}) \le \max\{T_A^N(x_1), T_A^N(y_1)\}$, for all x_1 and y_1 in G.

$$\min\{F_A^N(x_1y_1^{-1}), F_A^N(x_2y_2^{-1})\} = F_V^N(x_1y_1^{-1}, x_2y_2^{-1})$$
$$= F_V^N[(x_1,x_2)(y_1,y_2)^{-1}]$$
$$= F_V^N(xy^{-1})$$
$$\ge \min\{F_V^N(x), F_V^N(y)\}$$
$$= \min\{F_V^N(x_1,x_2), F_V^N(y_1,y_2)\}$$
$$= \min\left\{\min\{F_A^N(x_1), F_A^N(x_2)\}, \min\{F_A^N(y_1), F_A^N(y_2)\}\right\}$$

If we put $x_2 = y_2 = e$, we get, $F_A^N(x_1y_1^{-1}) \ge \min\{F_A^N(x_1), F_A^N(y_1)\}$, for all x_1 and y_1 in G. Hence, A is a bipolar Pythagorean fuzzy subring of G.

4. Conclusion

In this paper, we define the bipolar Pythagorean fuzzy subring of a ring and investigate the relationship among these bipolar Pythagorean fuzzy subring of a ring. Some characterisation theorems of bipolar Pythagorean fuzzy subring of a ring are obtained.

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