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Rough Hesitant Fuzzy Groups

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Abstract — In 2010, Torra introduced the notion of a hesitant fuzzy set, which is a generalization of Zadeh's fuzzy set. In the paper, we define two rough operators on hesitant fuzzy group by means of a normal hesitant fuzzy subgroup, and investigate some of their properties.

Keywords — *Rough set, hesitant fuzzy set, group*

1. Introduction

In 1965, Zadeh proposed the pioneering work of fuzzy subsets of a set [1], and in 1971, Rosenfeld introduced the notion of fuzzy subgroups of a group [2] which led to the fuzzification of algebraic structures. In 1982, Pawlak initiated the rough set theory to study incomplete and insufficient information [3].

Dubois, Prade first investigated fuzzy rough set and rough fuzzy set in [4], which attracting many scholars attentions. From the view of the theory of groups, Davvaz, Kuroki, Biswas, Kuroki, Yaqoob, Chen etc studied the notions of fuzzy groups, fuzzy subgroups, rough groups, rough subgroups, rough fuzzy groups, rough fuzzy subgroups in [5–11].

On the other hand, Torra [12] introduced the notion of a hesitant fuzzy set. After that time, Pei, Thakur et al. investigated some operators on hesitant fuzzy sets [13, 14]. Divakaran, John, et al. studied hesitant fuzzy rough sets, hesitant fuzzy groups [15–18]. Jun and Ahn applied hesitant fuzzy sets to BCK/BCI-algebras [19]. For more references,see [20–27].

In [28], Wang and Chen investigated the theory of rough subgroups by means of a normal subgroup, and obtained some interesting results. In [6], we investigated two rough operators on L-groups. As a generalization of [6, 9, 28], in the paper, we define the notion of rough hesitant fuzzy group, and investigate some of their properties.

The above contents are arranged into three parts, Section 3: Hesitant fuzzy group, and Section 4: Rough hesitant fuzzy group. In Section 2, we give an overview of hesitant fuzzy sets, group, rough sets, which surveys Preliminaries.

2. Preliminaries

In the section, we introduce some main notions for each area, i.e., hesitant fuzzy sets [12–14], groups, rough sets [3, 29, 30].

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2.1. Hesitant Fuzzy Sets

The seminal paper on fuzzy sets is [1]. As a generalization, the notion of a hesitant fuzzy set was introduced in [12].

Definition 2.1. Suppose X is a reference set, and $P[0,1]$ the power set of $[0,1]$, then a mapping $h : X \to P[0,1]$ is called a hesitant fuzzy set on X.

For instance: $h^0 : X \to P[0,1]$, $h^1 : X \to P[0,1]$ are defined as: for all $x \in X$, $h^0(x) = \emptyset, h^1(x) = [0,1]$, respectively.

We use the symbol $HF(X)$ to denote the set of all hesitant fuzzy sets in X. For $h_1, h_2 \in HF(X)$, $h_1 \preceq h_2$ is defined: if $\forall x \in X$, we have $h_1(x) \subseteq h_2(x)$, and $h_1 \approx h_2$, if $h_1 \preceq h_2$, $h_2 \preceq h_1$.

Definition 2.2. Suppose $h_1, h_2 \in HF(X)$, then $h_1 \tilde{\cap} h_2$ and $h_1 \tilde{\cup} h_2$ are defined as follows $(h_1 \tilde{\cap} h_2)(x) = h_1(x) \cap h_2(x)$, $(h_1 \tilde{\cup} h_2)(x) = h_1(x) \cup h_2(x)$ for every $x \in X$.

In special, a hesitant fuzzy point x_λ is defined by

$$x_\lambda(y) = \begin{cases} \lambda \subseteq [0,1] & \text{if } y = x \\ \emptyset & \text{if } y \neq x \end{cases}$$

The collection of all hesitant fuzzy points is denoted by M. For more details, see [17,31].

2.2. Rough Sets

Pawlak proposed the rough set theory in [3]. Let (X, R) be an approximation space, and $R \subseteq X \times X$ be an equivalence relation, then for $A \subseteq X$, two subsets $\underline{R}(A)$ and $\overline{R}(A)$ of X are defined:

$$\underline{R}(A) = \{x \in X \mid [x]_R \subseteq A\}, \quad \overline{R}(A) = \{x \in X \mid [x]_R \cap A \neq \emptyset\}$$

where $[x]_R = \{y \in X \mid xRy\}$.

If $\underline{R}(A) = \overline{R}(A)$, A is called a definable set; if $\underline{R}(A) \neq \overline{R}(A)$, A is called an undefinable set, and $(\underline{R}(A), \overline{R}(A))$ is referred to as a pair of rough set. Therefore, $\underline{R}$ and $\overline{R}$ are called two rough operators.

Furthermore, as generalizations, they also were defined by an arbitrary binary relation in [30, 32], a mapping in [29], and other methods. Dubois, Prade investigated fuzzy rough set and rough fuzzy set in [4].

2.3. Group

We assume familiarity with the notion of a group as used in the intuitive set theory. Suppose G is a multiplicative group with an identity e , and A is a subgroup of G, if $\forall x, y \in A$, we have $xy \in A$.

N is a normal subgroup of G, if $\forall x \in G$, and $y \in N$, we have $xyx^{-1} \in N$.

3. Hesitant Fuzzy Group

Suppose G is a group with an identity e, the main notions and propositions of the section are from [17].

Definition 3.1. $h : G \to P[0,1]$ is called a hesitant fuzzy subgroup of G, if for every $x, y \in G$, we have $h(x) \cap h(y) \subseteq h(xy)$, and $h(x) \subseteq h(x^{-1})$.

Example 3.2. Suppose $G = \{e, x, y, z\}$ with the operator as the following table,

$\cdot$	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	e	x
z	z	y	x	e

Then $h_1 = \{e_\lambda, x_\mu, y_\mu, z_\mu\}$ is a hesitant fuzzy subgroup of G, where $\lambda \subseteq [0,1], \mu \subseteq [0,1]$, and $\mu \subseteq \lambda$. For example, we choose $\lambda = [0.3, 0.8], \mu = [0.4, 0.6]$, $h_1 = \{e_{[0.3,0.8]}, x_{[0.4,0.6]}, y_{[0.4,0.6]}, z_{[0.4,0.6]}\}$.

Let $h_2(e) = [0,1]$, $h_2(x) = \{\frac{1}{5}, \frac{1}{4}, \frac{1}{2}\}$, $h_2(y) = \{\frac{1}{7}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\}$, $h_2(z) = \{\frac{1}{10}, \frac{1}{4}, \frac{1}{2}\}$, then h_2 is also a hesitant fuzzy subgroup of G.

In [17], Propositions 3.3, 3.4, 3.5 hold.

Proposition 3.3. h is a hesitant fuzzy subgroup of G if and only if $h(x^{-1}y) \supseteq h(x^{-1}) \cap h(y)$ for all $x, y \in G$.

Proposition 3.4. Suppose h is a hesitant fuzzy subgroup of G, then for all $x \in G$
(1) $h(e) \supseteq h(x)$
(2) $h(x) = h(x^{-1})$
(3) $h(x^n) \supseteq h(x)$

Proposition 3.5. Suppose h_1, h_2 are two hesitant fuzzy subgroups of G, then $h_1 \tilde{\cap} h_2$ is also a hesitant fuzzy subgroup of G.

Definition 3.6. g is called a normal hesitant fuzzy subgroup of G, if for every $x, y \in G$, we have $g(y) \subseteq g(xyx^{-1})$.

Cleraly, $h_3(e) = \{1, \frac{1}{3}, \frac{5}{7}\}$, $h_3(x) = \{\frac{1}{3}, \frac{5}{7}\}, h_3(y) = \emptyset, h_3(z) = \emptyset$ is a normal hesitant fuzzy subgroup of G.

In [17], Propositions 3.7, 3.8 hold.

Proposition 3.7. Suppose g is a hesitant fuzzy subgroup of G, then the following conditions are equivalence.
(1) g is normal.
(2) $g(xy) = g(yx)$, for all $x, y \in G$
(3) $g(xyx^{-1}) = g(y)$, for all $x, y \in G$

Proposition 3.8. Suppose g_1, g_2 are two normal hesitant fuzzy subgroups of G, then $g_1 \tilde{\cap} g_2$ is also a normal hesitant fuzzy subgroup of G.

In the classical case, for two subsets A, B of G, $AB = \{z \mid z = xy, x \in A, y \in B\}$, as a generalization, we give the following definition.

Definition 3.9. For h_1, h_2 two hesitant fuzzy subgroups of G, we define h_1h_2, for every $z \in G$,

$$(h_1h_2)(z) = \bigcup_{z=xy} h_1(x) \cap h_2(y)$$

In special, $(x_\lambda h)(w) = \bigcup\limits_{w=st} \{x_\lambda\}(s) \cap h(t) = \bigcup\limits_{w=xt} \lambda \cap h(t) = \lambda \cap h(x^{-1}w)$.
$x_\lambda y_\mu = z_\nu$, where $z = xy, \nu = \lambda \cap \mu$.

Example 3.10. Following Example 3.2, clearly $h_4 = \{e_{[0.2,0.8]}, y_{[0.5,0.7]}\}$ is also a hesitant fuzzy subgroup of G. Then $h_1h_4 = \{e_{[0.2,0.8]}, x_{[0.4,0.7]}, y_{[0.4,0.7]}, z_{[0.4,0.7]}\}$.

4. Rough Hesitant Fuzzy Group

In the section, we introduce the notion of a rough hesitant fuzzy group defined by a normal hesitant fuzzy subgroup, and investigate some of their properties.

First, we give the notion of a rough hesitant fuzzy group.

Definition 4.1. Suppose N is a hesitant fuzzy normal subgroup of G, for every hesitant fuzzy subset h of G, we define $N^-(h)$ and $N_-(h)$, for every $x \in G$,

$$\begin{aligned} N^-(h)(x) &= \bigcup_{x_\lambda \in M} \{\lambda \mid \bigcup_{z \in G} (x_\lambda N)(z) \cap h(z) \neq \emptyset\} \\ &= \bigcup_{x_\lambda \in M} \{\lambda \mid \bigcup_{z \in G} \lambda \cap N(x^{-1}z) \cap h(z) \neq \emptyset\}, \\ N_-(h)(x) &= \bigcup_{x_\lambda \in M} \{\lambda \mid \bigcap_{z \in G} (x_\lambda N)(z) \subseteq h(z)\} \\ &= \bigcup_{x_\lambda \in M} \{\lambda \mid \bigcap_{z \in G} \lambda \cap N(x^{-1}z) \subseteq h(z)\} \end{aligned}$$

where $M = \{x_\lambda \mid x \in G, \lambda \subseteq [0,1]\}$ of all hesitant fuzzy singletons.

Then $N^-(h), N_-(h)$ are called the upper approximation, the lower approximation of h with respect to the hesitant fuzzy normal subgroup N, respectively.

Example 4.2. $N = h_3$ be a normal hesitant fuzzy subgroup of G, then for h_2, we have

$$N^-(h_2)(e) = \bigcup_{e_\lambda \in M} \{\lambda \mid \bigcup_{w \in G} \lambda \cap N(e^{-1}w) \cap h_2(w) \neq \emptyset\} = [0,1],$$
$$N^-(h_2)(x) = \bigcup_{x_\lambda \in M} \{\lambda \mid \bigcup_{w \in G} \lambda \cap N(x^{-1}w) \cap h_2(w) \neq \emptyset\} = [0,1],$$
$$N^-(h_2)(y) = \bigcup_{y_\lambda \in M} \{\lambda \mid \bigcup_{w \in G} \lambda \cap N(y^{-1}w) \cap h_2(w) \neq \emptyset\} = [0,1],$$
$$N^-(h_2)(z) = \bigcup_{z_\lambda \in M} \{\lambda \mid \bigcup_{w \in G} \lambda \cap N(z^{-1}w) \cap h_2(w) \neq \emptyset\} = [0,1].$$

and

$$N_-(h_2)(e) = \bigcup_{e_\lambda \in M} \{\lambda \mid \bigcap_{w \in G} \lambda \cap N(e^{-1}w) \subseteq h_2(w)\} = [0,1] - \{\tfrac{1}{3}, \tfrac{5}{7}\},$$
$$N_-(h_2)(x) = \bigcup_{x_\lambda \in M} \{\lambda \mid \bigcap_{w \in G} \lambda \cap N(x^{-1}w) \subseteq h_2(w)\} = [0,1] - \{\tfrac{1}{3}, \tfrac{5}{7}\},$$
$$N_-(h_2)(y) = \bigcup_{y_\lambda \in M} \{\lambda \mid \bigcap_{w \in G} \lambda \cap N(y^{-1}w) \subseteq h_2(w)\} = [0,1) - \{\tfrac{1}{3}, \tfrac{5}{7}\},$$
$$N_-(h_2)(z) = \bigcup_{z_\lambda \in M} \{\lambda \mid \bigcap_{w \in G} \lambda \cap N(z^{-1}w) \subseteq h_2(w)\} = [0,1) - \{\tfrac{1}{3}, \tfrac{5}{7}\}.$$

Where $A - B$ denotes the difference set.

Next, we discuss the following properties.

Proposition 4.3. Suppose N is a normal hesitant fuzzy subgroup of G, and $h \in HF(G)$, we have

(1) $N_-(h) \preceq h$

(2) $N^-(h) \succeq Nh$

(3) $N_-(h^1) \approx h^1$

(4) $N^-(h^0) \approx h^0$

Proof. (1) For every $w \in G$, we obtain $h(w) \cap h(w^{-1}w) \subseteq h(w)$; but for $z \in G$, $z \neq w$, $h(w) \cap g(w^{-1}z) \subseteq h(z)$ may be not holds.

$$\begin{aligned} N_-(h)(w) &= \bigcup_{x_\lambda \in M} \{\lambda \mid \bigcap_{z \in G} (x_\lambda N)(z) \subseteq h(z)\} \\ &= \bigcup_{x_\lambda \in M} \{\lambda \mid \bigcap_{z \in G} \lambda \cap N(w^{-1}z) \subseteq h(z)\} \\ &\subseteq \bigcup \{h(w) \mid h(w) \cap N(w^{-1}w) \subseteq h(w)\} \\ &= h(w) \end{aligned}$$

By the above proof, we have $N_-(h) \preceq h$.

(2) For every $w \in G$, if $(Nh)(w) \neq \emptyset$, we have

$$\begin{aligned} N^-(h)(w) &= \bigcup_{w_\lambda \in M} \{\lambda \mid \bigcup_{z \in G} \lambda \cap N(w^{-1}z) \cap h(z) \neq \emptyset\} \\ &= \bigcup_{w_\lambda \in M} \{\lambda \mid \lambda \cap [\bigcup_{z \in G} N(w^{-1}z) \cap h(z)] \neq \emptyset\} \\ &= \bigcup_{w_\lambda \in M} \{\lambda \mid \lambda \cap (Nh)(zw^{-1}z) \neq \emptyset\} \\ &\supseteq \bigcup \{(Nh)(w) \mid (Nh(w) \cap (Nh)(w) \neq \emptyset\} \\ &\qquad (\text{Note: } \lambda = (Nh)(w), z = w) \\ &= (Nh)(w) \end{aligned}$$

(3)and (4) are clearly. □

Proposition 4.4. Suppose $h_1, h_2 \in HF(G)$, and $h_1 \preceq h_2$, N is a normal hesitant fuzzy subgroup, then

(1) $N^-(h_1) \preceq N^-(h_2)$
(2) $N_-(h_1) \preceq N_-(h_2)$

Proof. By Definition 4.1 □

Proposition 4.5. Suppose N is a normal hesitant fuzzy subgroup of G, and $h_1, h_2 \in HF(G)$, we have

(1) $N^-(h_1 \tilde{\cup} h_2) \approx N^-(h_1) \tilde{\cup} N^-(h_2)$
(2) $N^-(h_1 \tilde{\cap} h_2) \preceq N^-(h_1) \tilde{\cap} N^-(h_2)$
(3) $N_-(h_1 \tilde{\cup} h_2) \succeq N_-(h_1) \tilde{\cup} N_-(h_2)$
(4) $N_-(h_1 \tilde{\cap} h_2) \approx N_-(h_1) \tilde{\cap} N_-(h_2)$

Proof. By Definition 4.1. □

Proposition 4.6. Suppose N is a normal hesitant fuzzy subgroup of G, and h is a (normal) hesitant fuzzy subgroup of G, we have $N^-(h)$ is a (normal) hesitant fuzzy subgroup of G.

Proof. For $s, t \in G$, we obtain

$$\begin{aligned}
N^-(h)(s) \cap N^-(h)(t) &= \bigcup_{s_\lambda \in M} \{\lambda \mid \bigcup_{x \in G} \lambda \cap N(s^{-1}x) \cap h(x) \neq \emptyset\} \\
&\quad \cap \bigcup_{t_\mu \in M} \{\mu \mid \bigcup_{y \in G} \mu \cap N(t^{-1}y) \cap h(y) \neq \emptyset\} \\
&= \bigcup_{s_\lambda \in M} \bigcup_{t_\mu \in M} [\{\lambda \mid \bigcup_{x \in G} \lambda \cap N(s^{-1}x) \cap h(x) \neq \emptyset\} \\
&\quad \cap \{\mu \mid \bigcup_{y \in G} \mu \cap N(t^{-1}y) \cap h(y) \neq \emptyset\}] \\
&= \bigcup_{s_\lambda \in M} \bigcup_{t_\mu \in M} \{\lambda \cap \mu \mid \bigcup_{x \in G} \bigcup_{y \in G} \lambda \cap \mu \cap N(s^{-1}x) \cap N(t^{-1}y) \cap h(x) \cap h(y) \neq \emptyset\} \\
&= \bigcup_{w_\nu \in M} \{\nu \mid \bigcup_{z = xy \in G} \nu \cap N(s^{-1}x) \cap N(t^{-1}y) \cap h(x) \cap h(y) \neq \emptyset\} \\
&\subseteq \bigcup_{w_\nu \in M} \{\nu \mid \bigcup_{z = xy \in G} \nu \cap N(w^{-1}z) \cap h(z) \neq \emptyset\} \\
&= N^-(h)(w) \text{ (Note } w = st, z = xy)
\end{aligned}$$

So, $N^-(h)$ is a hesitant fuzzy subgroup of G.

Furthermore, if h is a normal hesitant fuzzy subgroup of G, then for $s, t \in G$, let $w = s^{-1}ts$, we have

$$\begin{aligned}
N^-(h)(s^{-1}ts) &= N^-(h)(w) \\
&= \bigcup_{w_\nu \in M} \{\nu \mid \bigcup_{z \in G} \nu \cap N(w^{-1}z) \cap h(w) \neq \emptyset\} \\
&= \bigcup_{w_\nu \in M} \{\nu \mid \bigcup_{z \in G} \nu \cap N((s^{-1}ts)^{-1}z) \cap h(s^{-1}ts) \neq \emptyset\} \\
&= \bigcup_{w_\nu \in M} \{\nu \mid \bigcup_{z \in G} \nu \cap N(st^{-1}s^{-1}z) \cap h(t) \neq \emptyset\} \\
&= \bigcup_{w_\nu \in M} \{\nu \mid \bigcup_{z \in G} \nu \cap N(st^{-1}zs^{-1}) \cap h(t) \neq \emptyset\} \\
&= \bigcup_{w_\nu \in M} \{\nu \mid \bigcup_{z \in G} \nu \cap N(t^{-1}z) \cap h(t) \neq \emptyset\} \\
&= \bigcup_{t_\lambda \in M} \{\lambda \mid \bigcup_{z \in G} \lambda \cap N(t^{-1}z) \cap h(t) \neq \emptyset\} \\
&= N^-(h)(t)
\end{aligned}$$

By the above proof, we obtain $N^-(h)$ is a normal hesitant fuzzy subgroup of G.

□

In general, $N_-(h)$ is not a hesitant fuzzy subgroup of G. But if $N_-(h)$ is a hesitant fuzzy subgroup of G, and h is a normal hesitant fuzzy subgroup of G, in the similar method, we can prove $N_-(h)$ is also a normal hesitant fuzzy subgroup of G.

Proposition 4.7. Suppose N, H are two normal hesitant fuzzy subgroups of G, the corresponding rough operators $N^-, N_-; H^-, H_-$ respectively, and $h, k \in HF(G)$, we have
(1) $N^-(h)N^-(k) \preceq N^-(hk)$
(2) $N_-(h)N_-(k) \preceq N_-(hk)$
(3) $(N\tilde{\cap}H)^-(h) \succeq N^-(h)\tilde{\cap}H^-(h)$
(4) $(N\tilde{\cap}H)_-(h) \preceq N_-(h)\tilde{\cap}H_-(h)$
where $(N\tilde{\cap}H)^-$, $(N\tilde{\cap}H)_-$ are two rough operators induced by the normal hesitant fuzzy subgroup $N\tilde{\cap}H$.

Proof. (1) For every $w \in G$,

$$N^-(hk)(w) = \bigcup_{w_\lambda \in M} \{\lambda \mid \bigcup_{z\in G} \lambda \cap N(w^{-1}z) \cap (hk)(z) \neq \emptyset\}$$
$$= \bigcup_{w_\lambda \in M} \{\lambda \mid \bigcup_{z\in G} \lambda \cap N(w^{-1}z) \cap [\bigcup_{z=xy} h(x)\cap k(y)] \neq \emptyset\}$$
$$= \bigcup_{w_\lambda \in M} \{\lambda \mid \bigcup_{z=xy} \lambda \cap N(w^{-1}z) \cap h(x)\cap k(y) \neq \emptyset\}$$

$$(N^-(h)N^-(k))(w) = \bigcup_{w=st} N^-(h)(s)\cap N^-(k)(t)$$
$$= \bigcup_{w=st}[\bigcup_{s_\mu\in M}\{\mu \mid \bigcup_{x\in G}\mu\cap N(s^{-1}x)\cap h(x)\neq\emptyset\}$$
$$\cap \bigcup_{t_\nu\in M}\{\nu \mid \bigcup_{y\in G}\nu\cap N(t^{-1}y)\cap k(y)\neq\emptyset\}]$$
$$= \bigcup_{w=st}\bigcup_{s_\mu\in M}\bigcup_{t_\nu\in M}[\{\mu \mid \bigcup_{x\in G}\mu\cap N(s^{-1}x)\cap h(x)\neq\emptyset\}$$
$$\cap\{\nu \mid \bigcup_{y\in G}\nu\cap N(t^{-1}y)\cap k(y)\neq\emptyset\}]$$
$$= \bigcup_{w=st}\bigcup_{s_\mu\in M}\bigcup_{t_\nu\in M}\{\mu\cap\nu \mid \bigcup_{x\in G}\bigcup_{y\in G}\mu\cap N(s^{-1}x)\cap h(x)\cap\nu\cap N(t^{-1}y)\cap k(y)\neq\emptyset\}$$
$$= \bigcup_{w=st}\bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z=xy}\lambda\cap N(s^{-1}x)\cap N(t^{-1}y)\cap h(x)\cap k(y)\neq\emptyset\}$$
(Note $w_\lambda = s_\mu t_\nu$)
$$\subseteq \bigcup_{w=st}\bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z=xy}\lambda\cap N(s^{-1}xt^{-1}y)\cap h(x)\cap k(y)\neq\emptyset\}$$
$$= \bigcup_{w=st}\bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z=xy}\lambda\cap N(s^{-1}t^{-1}xy)\cap h(x)\cap k(y)\neq\emptyset\}$$
$$= \bigcup_{w=st}\bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z=xy}\lambda\cap N(w^{-1}z)\cap h(x)\cap k(y)\neq\emptyset\}$$
$$= N^-(hk)(w)$$

(2) For every $w \in G$,

$$N_-(hk)(w) = \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcap_{z\in G}(w_\lambda N)(z)\subseteq (hk)(z)\}$$
$$= \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcap_{z\in G}(w_\lambda N)(z)\subseteq \bigcup_{z=xy}h(x)\cap k(y)\}$$
$$= \bigcup_{z=xy}\bigcup_{w=st}[\bigcup_{s_\mu\in M}\{\mu \mid \bigcap_{z\in G}(s_\mu N)(z)\subseteq k(x)\}]$$
$$\cap[\bigcup_{t_\nu\in M}\{\nu \mid \bigcap_{z\in G}(t_\nu N)(z)\subseteq k(y)\}] \quad \text{(Note } w_\lambda = s_\mu t_\nu)$$
$$= \bigcup_{z=xy}\bigcup_{w=st}[\bigcup_{s_\mu\in M}\{\mu \mid \bigcap_{z\in G}\mu\cap N(s^{-1}z)\subseteq h(x)\}]$$

$$\cap[\bigcup_{t_\nu \in M}\{\nu \mid \bigcap_{z\in G}\nu\cap N(t^{-1}z)\subseteq k(y)\}]$$
$$\supseteq \bigcup_{w=st}[\bigcup_{s_\mu\in M}\{\mu \mid \bigcap_{x\in G}\mu\cap N(s^{-1}x)\subseteq h(x)\}]$$
$$\cap[\bigcup_{t_\nu \in M}\{\nu \mid \bigcap_{y\in G}\nu\cap N(t^{-1}y)\subseteq k(y)\}]$$
$$= \bigcup_{w=st} N_-(h)(s)\cap N_-(k)(t)$$
$$= (N_-(h)N_-(k))(w)$$

Which implies that $N_-(h)N_-(k)\succeq N_-(hk)$.

(3) For every $w\in G$, we have

$$(N\tilde{\cap}H)^-(h)(w) = \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}(w_\lambda(N\tilde{\cap}H))(z)\cap h(z)\neq\emptyset\}$$
$$= \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}\lambda\cap(N\tilde{\cap}H)(w^{-1}z)\cap h(z)\neq\emptyset\}$$
$$= \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}\lambda\cap N(w^{-1}z)\cap H(w^{-1}z)\cap h(z)\neq\emptyset\}$$
$$\supseteq [\bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}\lambda\cap N(w^{-1}z)\cap h(z)\neq\emptyset\}]$$
$$\cap[\bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}\lambda\cap H(w^{-1}z)\cap h(z)\neq\emptyset\}]$$
$$= N^-(h)(w)\cap H^-(h)(w)$$
$$= (N^-(h)\tilde{\cap}H^-(h))(w)$$

(4) For every $w\in G$, we have

$$(N\cap H)_-(h)(w) = \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcap_{z\in G}(w_\lambda(N\tilde{\cap}H))(z)\subseteq h(z)\}$$
$$= \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcap_{z\in G}\lambda\cap(N\tilde{\cap}H)(w^{-1}z)\subseteq h(z)\}$$
$$= \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcap_{z\in G}\lambda\cap N(w^{-1}z)\cap H(w^{-1}z)\subseteq h(z)\}$$
$$\subseteq \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcap_{z\in G}\lambda\cap N(w^{-1}z)\subseteq h(z)\}$$
$$\cap \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcap_{z\in G}\lambda\cap H(w^{-1}z)\subseteq h(z)\}$$
$$= N_-(h)(w)\cap H_-(h)(w)$$
$$= (N_-(h)\tilde{\cap}H_-(h))(w)$$

□

Proposition 4.8. Suppose N, H are two normal hesitant fuzzy subgroups of G, and for every hesitant fuzzy subgroup h of G, we have $N^-(h)H^-(h)\preceq(NH)^-(h)$.

PROOF. For every $w\in G$, we have

$$(NH)^-(h)(w) = \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}(w_\lambda(NH))(z)\cap h(z)\neq\emptyset\}$$
$$= \bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}\lambda\cap(NH)(w^{-1}z)\cap h(z)\neq\emptyset\}$$

$$(N^-(h)H^-(h))(w) = \bigcup_{w=st} N^-(h)(s)\wedge H^-(h)(t)$$
$$= \bigcup_{w=st}[\bigcup_{s_\mu\in M}\{\mu \mid \bigcup_{x\in G}(s_\mu N)(x)\cap h(x)\neq\emptyset\}]$$
$$\cap[\bigcup_{t_\nu\in M}\{\nu \mid \bigcup_{y\in G}(t_\nu H)(y)\cap h(y)\neq\emptyset\}]$$
$$= \bigcup_{w=st}[\bigcup_{s_\mu\in M}\{\mu \mid \bigcup_{x\in G}\mu\cap N(s^{-1}x)\cap h(x)\neq\emptyset\}]$$

$$\cap[\bigcup_{t_\nu \in M}\{\nu \mid \bigcup_{y\in G}\nu \cap H(t^{-1}y)\cap h(y)\neq\emptyset\}]$$
$$=\bigcup_{w=st}\bigcup_{s_\mu\in M}\bigcup_{t_\nu\in M}\{\mu\wedge\nu \mid \bigcup_{x\in G}\bigcup_{y\in G}\mu\cap\nu\cap N(s^{-1}x)\cap H(t^{-1}y)\cap h(x)\cap h(y)\neq\emptyset\}$$
$$=\bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{x\in G}\bigcup_{y\in G}\lambda\cap N(s^{-1}x)\cap H(t^{-1}y)\cap h(x)\cap h(y)\neq\emptyset\}$$
$$=\bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z=xy\in G}\lambda\cap N(s^{-1}x)\cap H(t^{-1}y)\cap h(x)\cap h(y)\neq\emptyset\}$$
$$=\bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z=xy\in G}\lambda\cap (NH)(w^{-1}z)\cap h(x)\cap h(y)\neq\emptyset\}\quad (w=st)$$
$$\subseteq\bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z=xy\in G}\lambda\cap (NH)(w^{-1}z)\cap h(z)\neq\emptyset\}$$
$$=(NH)^-(h)(w)$$
□

Proposition 4.9. Suppose N, H are two normal hesitant fuzzy subgroups of G, and for every hesitant fuzzy subgroup h of G, we have $(NH)^-(h)\succeq(N^-(h))H\tilde{\cap}(H^-(h))N$.

PROOF. For every $w\in G$, we have

$$((N^-(h))H\tilde{\cap}(H^-(h))N)(w)=((N^-(h))H)(w)\cap((H^-(h))N)(w)$$
$$=[\bigcup_{w=st}(N^-(h)(s)\cap H(t)]\cap[\bigcup_{w=st}H^-(h)(t)\cap N(s)]$$
$$=[\bigcup_{w=st}\bigcup_{s_\mu\in M}\{\mu \mid \bigcup_{x\in G}\mu\cap N(s^{-1}x)\cap h(x)\neq\emptyset\}\cap H(t)]$$
$$\cap[\bigcup_{w=st}\bigcup_{t_\nu\in M}\{\nu \mid \bigcup_{y\in G}\nu\cap H(t^{-1}y)\cap\nu(y)\neq\emptyset\}\cap N(s)]$$
$$=[\bigcup_{w=st}\bigcup_{s_\mu\in M}\{\mu\cap H(t) \mid \bigcup_{x\in G}\mu\cap N(s^{-1}x)\cap h(x)\neq\emptyset\}]$$
$$\cap[\bigcup_{w=st}\bigcup_{t_\nu\in M}\{\nu\cap N(s) \mid \bigcup_{y\in G}\nu\cap H(t^{-1}y)\cap h(y)\neq\emptyset\}]$$
$$=\bigcup_{w=st}\bigcup_{s_\mu\in M}\bigcup_{t_\nu\in M}[\{\mu\cap H(t) \mid \bigcup_{x\in G}\mu\cap N(s^{-1}x)\cap h(x)\neq\emptyset\}]$$
$$\cap[\{\nu\cap N(s) \mid \bigcup_{y\in G}\nu\cap H(t^{-1}y)\cap h(y)\neq\emptyset\}]$$
$$=\bigcup_{w=st}\bigcup_{s_\mu\in M}\bigvee_{t_\nu\in M}\{\mu\cap H(t)\cap\nu\cap N(s) \mid$$
$$\bigcup_{x\in G}\bigcup_{y\in G}\mu\cap N(s^{-1}x)\cap h(x)\cap\nu\cap H(t^{-1}y)\cap h(y)\neq\emptyset\}$$
$$=\bigcup_{w=st}\bigcup_{w_\lambda\in M}\{\lambda\cap H(t)\cap N(s) \mid \bigcup_{z=xy} \mid$$
$$\lambda\cap N(s^{-1}x)\cap h(x)\cap H(t^{-1}y)\cap h(y)\neq\emptyset\}$$
$$\subseteq\bigcup_{w=st}\bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z=xy}\lambda\cap N(s^{-1}x)\cap H(t^{-1}y)\cap h(z)\neq\emptyset\}$$
$$(h(x)\wedge h(y)\subseteq h(z))$$
$$=\bigcup_{w_\lambda\in M}\{\lambda \mid \bigcup_{z\in G}\lambda\cap(NH)(w^{-1}z)\cap h(z)\neq\emptyset\}$$
$$=(NH)^-(h)(w)$$
□

Proposition 4.10. Suppose N, H are two normal hesitant fuzzy subgroups of G, and for every hesitant fuzzy subgroup h of G, we have $N_-(h)H_-(h)\preceq(NH)_-(h)$.

PROOF. For every $w\in G$,

$$(N_-(h)H_-(h))(w)=\bigcup_{w=st}N_-(h)(s)\cap H_-(h)(t)$$

$$
\begin{aligned}
&= \bigcup_{w=st} [\bigcup_{s_\mu \in M} \{\mu \mid \bigcup_{x\in G} \mu \cap N(s^{-1}x) \subseteq h(x)\}] \\
&\quad \cap [\bigcup_{t_\nu \in M} \{\nu \mid \bigcup_{y\in G} \nu \cap H(t^{-1}y) \subseteq h(y)\}] \\
&= \bigcup_{w=st} \bigcup_{s_\mu \in M} \bigcup_{t_\nu \in M} [\{\mu \mid \bigcup_{x\in G} \mu \cap N(s^{-1}x) \subseteq h(x)\}] \\
&\quad \cap [\{\nu \mid \bigcup_{y\in G} \nu \cap H(t^{-1}y) \subseteq h(y)\}] \\
&= \bigcup_{w=st} \bigcup_{s_\mu \in M} \bigcup_{t_\nu \in M} [\{\mu \cap \nu \mid \bigcup_{x\in G} \bigcup_{y\in G} \mu \cap \nu \cap N(s^{-1}x) \cap H(t^{-1}y) \subseteq h(x) \cap h(y)\}] \\
&= \bigcup_{w=st} \bigcup_{w_\lambda \in M} [\{\lambda \mid \bigcup_{z=xy\in G} \lambda \cap N(s^{-1}x) \cap H(t^{-1}y) \subseteq h(x) \cap h(y)\}] \\
&= \bigcup_{w=st} \bigcup_{w_\lambda \in M} [\{\lambda \mid \bigcup_{z=xy\in G} \lambda \cap (NH)(w^{-1}z) \subseteq h(x) \cap h(y)\}] \\
&\subseteq \bigcup_{w_\lambda \in M} [\{\lambda \mid \bigcup_{z=xy\in G} \lambda \cap (NH)(w^{-1}z) \subseteq h(z)\}] \\
&= (NH)_-(h)(w) \qquad \square
\end{aligned}
$$

5. Conclusion

In [31], the set of all hesitant fuzzy sets forms a Boolean algebra. As a generalization, we defined two rough operators on a hesitant fuzzy group, and discussed some of their properties.

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References

[1] L. A. Zadeh, *Fuzzy Sets*, Information and Control 8(3) (1965) 338–353.

[2] A. Rosenfeld, *Fuzzy Groups*, Journal of Mathematical Analysis and Applications 35 (1971) 512–517.

[3] Z. Pawlak, *Rough Sets*, International Journal of Computer and Information Sciences 11 (1982) 341–356.

[4] D. Dubois, H. Prade, *Rough Fuzzy Sets and Fuzzy Rough Sets*, International Journal of General Systems 17 (1990) 191–209.

[5] R. Biswas, S. Nanda, *Rough Groups and Rough Subgroups*, Bulletin of the Polish Academy of Sciences Mathematics 42 (1994) 251–254.

[6] X. Chen, *Rough Approximations on L-Groups*, Annals of Fuzzy Mathematics and Informatics 18(1) (2019) 45–55.

[7] B. Davvaz, *Roughness Based on Fuzzy Ideals*, Information Science 164 (2006) 2417–2437.

[8] N. Kuroki, P. P. Wang, *The Lower and Upper Approximations in a Fuzzy Group*, Information Science 90 (1996) 203–220.

[9] J. Mordeson, K. R. Bhutani, A. Rosenfeld, *Fuzzy Subsets and Fuzzy Subgroups*, Studies in Fuzziness and Soft Computing 182 (2005) 1–39.

[10] N. Yaqoob, M. Aslam, R. Chinram, *Rough Prime bi-Ideals and Rough Fuzzy Prime bi-Ideals in Semigroups*, Annals of Fuzzy Mathematics and Informatics 3(2) (2012) 203–211.

[11] N. Yaqoob, *Approximations in Left Almost Polygroups*, Journal of Intelligent and Fuzzy Systems 36 (2019) 517–526.

[12] V. Torra, *Hesitant Fuzzy Sets*, International Journal of Intelligent Systems 25 (2010) 529–539.

[13] Z. Pei, L. Yi, *A Note on Operators of Hesitant Fuzzy Sets*, International Journal of Computational Intelligence Systems 8(2) (2015) 226–239.

[14] G. S. Thakur, R. Thakur, R. Singh, *New Hesitant Fuzzy Operators*, Fuzzy Information and Engineering 6 (2014) 379–392.

[15] D. Divakaran, S. J. John, *Hestiant Fuzzy Rough Sets through Hesitant Fuzzy Relations*, Annals of Fuzzy Mathematics and Informatics 8(1) (2014) 33–46.

[16] D. Divakaran, S. J. John, *Homomorphism of Hesitant Fuzzy Subgroups*, International Journal of Scientific and Engineering Research 5(9) (2014) 9–14.

[17] D. Divakaran, S. J. John, *Hesitant Fuzzy Subgroups*, Journal of New Theory 11 (2016) 54–68.

[18] V. Deepa , S. J. John, *Dual Hesitant Fuzzy Rough Subrings and Ideals,* Annals of Fuzzy Mathematics and Informatics 13(3) (2017) 437–448.

[19] Y. B. Jun, S. S. Ahn, *Hesitant Fuzzy Sets Theory Applied to* BCK/BCI-*Algebras*, Journal of Computational Analysis and Applications 20(4) (2016) 635–646.

[20] H. Alshehri, N. Alshehri, *Hesitant Anti-Fuzzy Soft Set in BCK-Algebras*, Mathematical Problems in Engineering (2017), Article ID 3634258, 13 pages.

[21] B. Davvaz, *Rough Algebraic Structures Corresponding to Ring Theory*, DOI: 10.1007/978-3-030-01162-8-9, In book: Algebraic Methods in General Rough Sets (2018) 657–695.

[22] D. Deepak, S. J. John, *Information Systems on Hesitant Fuzzy Sets*, International Journal of Rough Sets and Data Analysis 3(1) (2016) 71–97.

[23] D. Deepak, S. J. John, *Rough Approximations on Hesitant Fuzzy sets*, Book Chapter: Generalized and Hybrid Set Structures and Applications for Soft Computing, IGI Global USA, (2016) 269–279.

[24] A. H. Dehmiry, M. Mashinchi, *Hesitant L-Fuzzy Relations*, 6th Iranian Joint Congress on Fuzzy and Intelligent Systems (2018) 102–104.

[25] Y. B. Jun, K. J. Lee, S. Z. Song, *Hesitant Fuzzy bi-Ideals in Semigroups*, Communications of the Korean Mathematical Society 30(3) (2015) 143–154.

[26] A. Solariaju, S.Mahalakshmi, *Hestitant Intuitionistic Fuzzy soft Groups*,International Journal of Pure and Applied Mathematics 118(10) (2018) 223–232.

[27] M. S. Sindhu, T. Rashid, M. Khan, *Group Decision Making Based on Hesitant Fuzzy Ranking of Hesitant Preference Relations*, Journal of Intelligent and Fuzzy Systems 37(2) (2019) 2563–2573.

[28] C. Wang, D.Chen, *A Short Note on Some Properties of Rough Groups*, Computers and Mathematics with Applications 59(2010) 431–436.

[29] X. Chen, Q. Li, *Construction of Rough Approximations in Fuzzy Setting*, Fuzzy Sets and Systems 158 (2007) 2641–2653.

[30] Y. Y. Yao, *Constructive and Algebraic Methods of The Theory of Rough Sets*, Information Science 109 (1998) 21–47.

[31] J. G. Lee, K. Hur, *Hesitant Fuzzy Topological Spaces*, Mathematics 8(2) 188 (2020) pp 1–21.

[32] X. Chen, *Rough Approximations Induced by a Soft Relation*, Annals of Fuzzy Mathematics and Informatics, 11(1) (2016) 135–144.