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Harmonic Maps and Torse-Forming Vector Fields

Ahmed Mohammed Cherif* and Mustapha Djaa

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ABSTRACT

In this paper, we prove that any harmonic map from a compact orientable Riemannian manifold without boundary (or from complete Riemannian manifold) (M, g) to Riemannian manifold (N, h) is necessarily constant, with (N, h) admitting a torse-forming vector field satisfying some condition.

Keywords: *harmonic maps; bi-harmonic maps; torse-forming vector fields.*
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1. Introduction

Let (M, g) and (N, h) be two Riemannian manifolds, the energy functional of a map $\varphi \in C^\infty(M, N)$ is defined by

$$E(\varphi) = \int_M e(\varphi) v^g, \tag{1.1}$$

where $e(\varphi) = \frac{1}{2}|d\varphi|^2$ is the energy density of φ, $|d\varphi|$ is the Hilbert-Schmidt norm of the differential $d\varphi$ and v^g is the volume element on (M, g). A map $\varphi \in C^\infty(M, N)$ is called harmonic if it is a critical point of the energy functional, that is, if it is a solution of the Euler Lagrange equation associated to (1.1)

$$\tau(\varphi) = \operatorname{trace} \nabla d\varphi = \nabla^\varphi_{e_i} d\varphi(e_i) - d\varphi(\nabla^M_{e_i} e_i) = 0, \tag{1.2}$$

where $\{e_i\}$ is an orthonormal frame on (M, g), ∇^M is the Levi-Civita connection of (M, g), and ∇^φ denote the pull-back connection on $\varphi^{-1}TN$. Harmonic maps are solutions of a second order nonlinear elliptic system and they play a very important rôle in many branches of mathematics and physics where they may serve as a model for liquid crystal. One can refer to [8]-[10] for background on harmonic maps.

Example 1.1. (Hopf map [15]) The restriction of the map

$$\begin{array}{rcl} F : \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3, \\ (x, y) & \longmapsto & (2x\overline{y}, |x|^2 - |y|^2) \end{array}$$

induces a harmonic map $\varphi : S^3 \longrightarrow S^2$ with eigenvalue $\lambda = 8$ (called Hopf construction).

Example 1.2. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ the Torus. We note that the circle $\mathbb{S}^1$ is compact orientable manifold of dimension 1, and without boundary because $\partial\mathbb{S}^1 = \partial(\partial\mathbb{D}^2) = \emptyset$ where $\mathbb{D}^2$ is the unit disk in $\mathbb{R}^2$. So that the product manifold $\mathbb{S}^1 \times \mathbb{S}^1$ is also compact, without boundary, orientable manifold of dimension 2. In [16], the authors proved that the non-constant map $\varphi : (\mathbb{T}^2, dx_1^2 + dx_2^2) \longrightarrow (\mathbb{S}^2, dy_1^2 + \sin^2 y_1 dy_2^2)$, defined by $(x_1, x_2) \longmapsto (\pi/2, mx_1 + nx_2 + l)$ is harmonic, where $m, n, l \in \mathbb{R}$.

We shall consider a torse-forming vector field ξ, that is, a vector field which is always torse-forming along any curve traced in a Riemennian manifold (M, g) (see [14]-[20]). In this case, we have

$$\nabla^M_X \xi = fX + \omega(X)\xi, \quad \forall X \in \Gamma(TM), \tag{1.3}$$

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** Corresponding author*

for some smooth function f and 1-form ω on M, where ∇^M denotes the Levi-Civita connection of (M,g). The 1-form ω is called the generating form and the function f is called the conformal scalar. A torse-forming vector field ξ is called proper torse-forming if the 1-form ω is nowhere zero on a dense open subset of M. A torqued vector field is a torse-forming vector field ξ satisfying (1.3) with $\omega(\xi)=0$ (see [5],[6]). In the case that ω is identically zero, ξ is called a concircular vector field. In particular, if $\omega=0$ and $f=1$, then ξ is called a concurrent vector field. For the existence of torse-forming vector field on Riemannian manifold see for example [7] and [12].

Definition 1.1. A special torse-forming vector field or briefly a STF-vector field on a Riemennian manifold (M,g) is a torse-forming vector field ξ satisfying the equation (1.3) with generating form $\omega=\mu\xi^\flat$, for some smooth function μ on M, that is

$$\nabla^M_X\xi = fX+\mu g(X,\xi)\xi, \quad \forall X\in\Gamma(TM). \tag{1.4}$$

Example 1.3. In Euclidian space $\mathbb{R}^n$, we set $\xi=fP$, where $f(x)=\frac{1}{2}|x|^2+\epsilon$, for some constant $\epsilon>0$, and P is the position vector field on $\mathbb{R}^n$, then ξ is a STF-vector field with conformal scalar f and generating form $\omega=(1/f)df$.

Example 1.4. In Kenmotsu manifolds (see [11]) there exists a unit vector field ξ satisfying the condition $\nabla^M_X\xi=X-\eta(X)\xi$, where $\eta(X)=g(X,\xi)$, for all $X\in\Gamma(TM)$. So ξ is a STF-vector field with $f=1$ and $\mu=-1$. As we can assume that the vector field ξ is a unit one, (1.3) is written in the form

$$\nabla^M_X\xi = f\big(X-\eta(X)\xi\big), \quad \forall X\in\Gamma(TM), \tag{1.5}$$

where $\eta(X)=g(X,\xi)$. Thus, ξ is a STF-vector field with $\mu=-f$. When the conformal scalar f takes especially the value -1. Then the manifold in consideration becomes an SP-Sasakian manifold (see [1]).

Remark 1.1.

- A torse-forming vector field $\xi\in\Gamma(TM)$ of conformal scalar $f\neq 0$ (at any point on M) and generating form ω, is parallel if $\nabla^M_X\xi=0$, for all $X\in\Gamma(TM)$, that is $fX+\omega(X)\xi=0$. So that $X=-\frac{\omega(X)}{f}\xi$, thus $\Gamma(TM)=C^\infty(M)\xi$. Conversely, let $\xi\in\Gamma(TM)=C^\infty(M)\xi$, and g be a Riemannian metric on M such that $g(\xi,\xi)=1$, then $\nabla_X\xi=0=X-g(X,\xi)\xi$.
- A special torse-forming vector field ξ with conformal scalar f and generating form $\omega=\mu\xi^\flat$ on Riemennian manifold (M,g) ($\dim M\geq 2$) is conformal vector field (that is $g(\nabla^M_X\xi,Y)+g(\nabla^M_Y\xi,X)=2\tilde{f}g(X,Y)$, $\forall X,Y\in\Gamma(TM)$, for some smooth function $\tilde{f}$ on M) if and only if $\mu=0$. Indeed; by the definition of STF-vector field, we have

$$g(\nabla^M_X\xi,Y)+g(\nabla^M_Y\xi,X)=2fg(X,Y)+2\mu g(X,\xi)g(Y,\xi), \tag{1.6}$$

  for all $X,Y\in\Gamma(TM)$. So, if the vector field ξ is conformal with potential function $\tilde{f}$, by equation (1.6), we obtain

$$2(f-\tilde{f})g(X,Y)+2\mu g(X,\xi)g(Y,\xi)=0, \quad \forall X,Y\in\Gamma(TM). \tag{1.7}$$

  From equation (1.7) with $X=Y\perp\xi$, we get $2(f-\tilde{f})|X|^2=0$, that is $f=\tilde{f}$, and for $X=Y=\xi$ we have $2\mu|\xi|^4=0$, thus $\mu=0$. Conversely, a conformal vector field ξ on Riemennian manifold (M,g) is a special torse-forming vector field if the 1-form $\eta(X)=g(X,\xi)$ is closed on M (see [2]). In this cas, the conformal scalar is the potential function of ξ and $\mu=0$.

2. STF-vector fields and harmonic maps

In the seminal work [13], where we proved that, if (M,g) is a compact Riemannian manifold without boundary, (N,h) is a Riemannian manifold, $\varphi:(M,g)\to(N,h)$ a harmonic map, assume that there is a proper homothetic vector field ξ on (N,h), that is $\mathcal{L}_\xi h=2kh$, for some constant $k\in\mathbb{R}^*$, where $\mathcal{L}_\xi h$ is the Lie derivative of the metric h with respect to ξ. Then φ is a constant map. In the case of STF-vector field we obtain the following results.

Theorem 2.1. *Let (M,g) be a compact orientable Riemannian manifold without boundary, and (N,h) be a Riemannian manifold admitting a STF-vector field ξ with conformal scalar f and generating form $\mu\xi^\flat$. If $f>0$ and $\mu\geq 0$ on N, then any harmonic map φ from (M,g) to (N,h) is constant.*

Proof. Define a 1-form on M by

$$\eta(X) = h\big(\xi \circ \varphi, d\varphi(X)\big), \quad \forall X \in \Gamma(TM). \tag{2.1}$$

Let $\{e_i\}$ be a normal orthonormal frame at $x \in M$, we have

$$\operatorname{div}^M \eta = e_i\big[h\big(\xi \circ \varphi, d\varphi(e_i)\big)\big]. \tag{2.2}$$

By equation (2.2), and the harmonicity condition of φ, we get

$$\operatorname{div}^M \eta \quad = \quad h\big(\nabla^\varphi_{e_i}(\xi \circ \varphi), d\varphi(e_i)\big), \tag{2.3}$$

and since ξ is a STF-vector field on (N, h), we find that

$$\operatorname{div}^M \eta = (f \circ \varphi)h\big(d\varphi(e_i), d\varphi(e_i)\big) + (\mu \circ \varphi)h(\xi \circ \varphi, d\varphi(e_i))^2. \tag{2.4}$$

The Theorem 2.1 follows from (2.4), and the Green Theorem, with $f > 0$ and $\mu \geq 0$ on N. □

Remark 2.1.

- If $(M, g) = (N, h)$ and $\varphi = Id_M$, from Theorem 2.1, we get; Every orientable Riemannian manifold without boundary admitting a STF-vector field with strictly positive conformal scalar and $\mu \geq 0$ is necessarily non-compact.
- An harmonic map from a compact orientable Riemannian manifold without boundary to a Riemannian manifold admitting a STF-vector field is not necessarily constant. For example, the identity map on the unit n-dimensional sphere $\mathbb{S}^n$ on $\mathbb{R}^{n+1}$, note that the sphere $\mathbb{S}^n$ admits a STF-vector field given by $\xi = \operatorname{grad} \lambda$, where $\lambda(x) =< x, \alpha >_{\mathbb{R}^{n+1}}$ for all $x \in \mathbb{S}^n$, and $\alpha \in \mathbb{R}^{n+1}$, the conformal scalar of ξ is the function $f = -\lambda$, and the generating form ω is null, that is $\mu = 0$ (see [17]).

From Theorem 2.1 we get the following result:

Corollary 2.1. *Let $(\overline{N}, \overline{h})$ be an n-dimensional Riemannian manifold which admits a STF-vector field $\overline{\xi}$ with conformal scalar f and generating form $\omega = \mu\overline{\xi}^\flat$. Consider (N, h) a Riemannian hypersurface of $(\overline{N}, \overline{h})$ such that h is the induced metric of $\overline{h}$ on N. Suppose that*

- *(N, h) is totally umbilical, that is:*

$$B(X, Y) = \rho h(X, Y)\eta, \quad \forall X, Y \in \Gamma(TN),$$

for some smooth function ρ on N, where B is the second fundamental form of N on $\overline{N}$ given by $B(X, Y) = (\overline{\nabla}_X Y)^\perp$, $\overline{\nabla}$ is the Levi-Civita connection on $\overline{N}$, and η is the unit normal to N;
- *the functions $f + \rho\overline{h}(\overline{\xi}, \eta) > 0$ and $\mu \geq 0$ on N.*

Then, any harmonic map from a compact orientable Riemannian manifold without boundary to (N, h) is constant.

Proof. It is possible to express $\overline{\xi}$ as $\overline{\xi} = \xi + \lambda\eta$, where ξ is tangent to N and λ is a smooth function on N. Thus we have

$$\overline{\nabla}_X \overline{\xi} = \nabla^N_X \xi + \rho h(X, \xi)\eta + X(\lambda)\eta - \rho\lambda X, \tag{2.5}$$

where $X \in \Gamma(TN)$. By equation (2.5) with $\overline{\xi}$ is STF-vector field of conformal scalar f and generating form $\omega = \mu\overline{\xi}^\flat$, we get

$$\nabla^N_X \xi = (f + \rho\lambda)X + \mu h(X, \xi)\xi, \quad \forall X \in \Gamma(TN). \tag{2.6}$$

Thus ξ is STF-vector field of conformal scalar $f + \rho\lambda$ and generating form $\omega = \mu\xi^\flat$ on (N, h). Note that the function $f + \rho\lambda = f + \rho\overline{h}(\overline{\xi}, \eta)$. The Corollary follows by Theorem 2.1. □

In the case of non-compact Riemannian manifold, we obtain the following result:

Theorem 2.2. *Let (M, g) be a complete non-compact Riemannian manifold, and (N, h) be a Riemannian manifold admitting a STF-vector field ξ of conformal scalar $f > 0$ and generating form $\omega = \mu\xi^\flat$ with $\mu \geq 0$. If $\varphi : (M, g) \longrightarrow (N, h)$ is harmonic map, satisfying:*

$$\int_M \frac{|\xi \circ \varphi|^2}{(f \circ \varphi)} v^g < \infty, \tag{2.7}$$

then φ is constant. Furthermore, if (M, g) has an infinite volume we have $\varphi(x) = y_0$ for all $x \in M$ and $\xi(y_0) = 0$.

Proof. Let ρ be a smooth function with compact support on M, we set

$$\eta(X) = h\big(\xi \circ \varphi, \rho^2 d\varphi(X)\big), \quad X \in \Gamma(TM). \tag{2.8}$$

Let $\{e_i\}$ be a normal orthonormal frame at $x \in M$, we have

$$\operatorname{div}^M \eta = e_i\Big[h\big(\xi \circ \varphi, \rho^2 d\varphi(e_i)\big)\Big]. \tag{2.9}$$

By equation (2.9), and the harmonicity condition of φ, we get

$$\begin{aligned}\operatorname{div}^M \eta &= h\big(\nabla^\varphi_{e_i}(\xi \circ \varphi), \rho^2 d\varphi(e_i)\big) + h\big(\xi \circ \varphi, \nabla^\varphi_{e_i}\rho^2 d\varphi(e_i)\big) \\ &= \rho^2 h\big(\nabla^\varphi_{e_i}(\xi \circ \varphi), d\varphi(e_i)\big) + 2\rho e_i(\rho) h\big(\xi \circ \varphi, d\varphi(e_i)\big).\end{aligned} \tag{2.10}$$

Since ξ is a STF-vector field with conformal scalar f and generating form $\omega = \mu\xi^\flat$, we find that

$$\rho^2 h\big(\nabla^\varphi_{e_i}(\xi \circ \varphi), d\varphi(e_i)\big) = \rho^2 (f \circ \varphi) h\big(d\varphi(e_i), d\varphi(e_i)\big) + \rho^2(\mu \circ \varphi) h(\xi \circ \varphi, d\varphi(e_i))^2. \tag{2.11}$$

By the Young's inequality we have

$$-2\rho e_i(\rho) h\big(\xi \circ \varphi, d\varphi(e_i)\big) \le \lambda \rho^2 |d\varphi|^2 + \frac{1}{\lambda} e_i(\rho)^2 |\xi \circ \varphi|^2, \tag{2.12}$$

for any smooth function $\lambda > 0$ on M. From (2.10), (2.11) and (2.12) we deduce the inequality

$$\rho^2 (f \circ \varphi)|d\varphi|^2 + \rho^2(\mu \circ \varphi) h(\xi \circ \varphi, d\varphi(e_i))^2 - \operatorname{div}^M \eta \le \lambda \rho^2 |d\varphi|^2 + \frac{1}{\lambda} e_i(\rho)^2 |\xi \circ \varphi|^2. \tag{2.13}$$

Since $\rho^2(\mu \circ \varphi) h(\xi \circ \varphi, d\varphi(e_i))^2 \ge 0$, by (2.13) with $\lambda = \frac{1}{2}(f \circ \varphi)$, we have

$$\frac{1}{2}\rho^2 (f \circ \varphi)|d\varphi|^2 - \operatorname{div}^M \eta \le 2 e_i(\rho)^2 \frac{|\xi \circ \varphi|^2}{(f \circ \varphi)}. \tag{2.14}$$

By the divergence Theorem, and (2.14) we have

$$\frac{1}{2}\int_M \rho^2 (f \circ \varphi)|d\varphi|^2 v^g \le 2\int_M e_i(\rho)^2 \frac{|\xi \circ \varphi|^2}{(f \circ \varphi)} v^g. \tag{2.15}$$

Consider the smooth function $\rho = \rho_R$ such that, $\rho \le 1$ on M, $\rho = 1$ on the ball $B(p, R)$, $\rho = 0$ on $M \backslash B(p, 2R)$ and $|\operatorname{grad}^M \rho| \le \frac{2}{R}$ (see [18]). Then, from (2.15) we get

$$\frac{1}{2}\int_M \rho^2 (f \circ \varphi)|d\varphi|^2 v^g \le \frac{8}{R^2}\int_M \frac{|\xi \circ \varphi|^2}{(f \circ \varphi)} v^g, \tag{2.16}$$

and since $\int_M \frac{|\xi \circ \varphi|^2}{(f \circ \varphi)} v^g < \infty$, when $R \to \infty$, we obtain

$$\int_M (f \circ \varphi)|d\varphi|^2 v^g = 0. \tag{2.17}$$

Consequently, $|d\varphi| = 0$, that is φ is constant. □

Corollary 2.2. *Let (M, g) be a complete non-compact Riemannian manifold has a STF-vector field of conformal scalar $f > 0$ and generating form $\omega = \mu\xi^\flat$ with $\mu \ge 0$. Then,*

$$\int_M \frac{|\xi|^2}{f} v^g = \infty.$$

Using the similar technique we have the following results:

Theorem 2.3. *Let (M, g) be a complete non-compact Riemannian manifold has a STF-vector field ξ of conformal scalar $f > 0$ and generating form $\omega = \mu\xi^\flat$ with $\mu \ge 0$, and (N, h) be a Riemannian manifold. If $\varphi : (M, g) \longrightarrow (N, h)$ is harmonic map, satisfying:*

$$\xi(e(\varphi)) \ge 0, \quad \int_M \frac{|d\varphi(\xi)|^2}{f} v^g < \infty, \tag{2.18}$$

then φ is constant.

Proof. Define a 1-form on M by

$$\eta(X) = h\big(d\varphi(\xi), \rho^2 d\varphi(X)\big), \quad X \in \Gamma(TM), \tag{2.19}$$

where ρ is a smooth function with compact support on M. Let $\{e_i\}$ be a normal orthonormal frame at $x \in M$, we have

$$\operatorname{div}^M \eta = e_i\Big[h\big(d\varphi(\xi), \rho^2 d\varphi(e_i)\big)\Big]. \tag{2.20}$$

By equation (2.20), and the harmonicity condition of φ, we get

$$\begin{aligned} \operatorname{div}^M \eta &= h\big(\nabla^\varphi_{e_i} d\varphi(\xi), \rho^2 d\varphi(e_i)\big) + h\big(d\varphi(\xi), \nabla^\varphi_{e_i} \rho^2 d\varphi(e_i)\big) \\ &= \rho^2 h\big(\nabla^\varphi_{e_i} d\varphi(\xi), d\varphi(e_i)\big) + 2\rho e_i(\rho) h\big(d\varphi(\xi), d\varphi(e_i)\big), \end{aligned} \tag{2.21}$$

and by the property $\nabla^\varphi_X d\varphi(Y) = \nabla^\varphi_Y d\varphi(X) + d\varphi([X,Y])$ (see [4]), with $X = e_i$ and $Y = \xi$, we find that

$$\begin{aligned} \operatorname{div}^M \eta &= \rho^2 h\big(\nabla^\varphi_\xi d\varphi(e_i), d\varphi(e_i)\big) + \rho^2 h\big(d\varphi(\nabla^M_{e_i}\xi), d\varphi(e_i)\big) \\ &\quad + 2\rho e_i(\rho) h\big(d\varphi(\xi), d\varphi(e_i)\big). \end{aligned} \tag{2.22}$$

Since ξ is a STF-vector field with conformal scalar f and generating form $\mu\xi^\flat$, we have the following

$$\rho^2 h\big(d\varphi(\nabla^M_{e_i}\xi), d\varphi(e_i)\big) = \rho^2 f|d\varphi|^2 + \rho^2\mu|d\varphi(\xi)|^2 \geq \rho^2 f|d\varphi|^2. \tag{2.23}$$

By the Young's inequality we have

$$-2\rho e_i(\rho) h\big(d\varphi(\xi), d\varphi(e_i)\big) \leq \frac{f}{2}\rho^2|d\varphi|^2 + \frac{2}{f} e_i(\rho)^2 |d\varphi(\xi)|^2. \tag{2.24}$$

From (2.21), (2.23) and (2.24), with $h\big(\nabla^\varphi_\xi d\varphi(e_i), d\varphi(e_i)\big) = \xi(e(\varphi)) \geq 0$, we deduce the inequality

$$\frac{1}{2}\rho^2 f|d\varphi|^2 - \operatorname{div}^M \eta \leq 2 e_i(\rho)^2 \frac{|d\varphi(\xi)|^2}{f}. \tag{2.25}$$

By the divergence Theorem, and inequality (2.25) we have

$$\frac{1}{2}\int_M \rho^2 f|d\varphi|^2 v^g \leq 2\int_M e_i(\rho)^2 \frac{|d\varphi(\xi)|^2}{f} v^g. \tag{2.26}$$

Consider the smooth function $\rho = \rho_R$ such that, $\rho \leq 1$ on M, $\rho = 1$ on the ball $B(p,R)$, $\rho = 0$ on $M \backslash B(p, 2R)$ and $|\operatorname{grad}^M \rho| \leq \frac{2}{R}$. Then, from (2.26) we get

$$\frac{1}{2}\int_M \rho^2 f|d\varphi|^2 v^g \leq \frac{8}{R^2}\int_M \frac{|d\varphi(\xi)|^2}{f} v^g, \tag{2.27}$$

and using the condition $\int_M \frac{|d\varphi(\xi)|^2}{f} v^g < \infty$, when $R \to \infty$, we obtain

$$\int_M f|d\varphi|^2 v^g = 0. \tag{2.28}$$

So that $|d\varphi| = 0$, that is φ is constant. □

Proposition 2.1. *Let ξ be a STF-vector field of conformal scalar $f \neq 0$ (at any point on M) on n-dimensional Riemannian manifold (M,g). Then ξ is harmonic if and only if*

$$\begin{cases} \operatorname{grad} f = -\big[g(\operatorname{grad}\mu, \xi) + (n+1)\mu f + 2\mu^2|\xi|^2\big]\xi; \\ \operatorname{Ricci}(\xi) = 0, \end{cases}$$

where grad *is the gradient operator on (M,g), and* Ricci *is the Ricci tensor of (M,g).*

Proof. The tension field of ξ is given by (see [3])

$$\tau(\xi) = (\text{trace}\,\nabla^2\xi)^V + (\text{trace}\,R(\xi, \nabla^M\xi))^H, \tag{2.29}$$

where for all $X \in \Gamma(TM)$, X^V (resp. X^H) is the vertical (resp. horizontal) lift of X, and R is the curvature tensor of (M, g). Since ξ is a STF-vector field, we have

$$\nabla_X^M \xi = fX + \mu g(X, \xi)\xi = fX + \omega(X)\xi, \quad \forall X \in \Gamma(TM). \tag{2.30}$$

Let $x \in M$, then for any orthonormal basis $\{e_i\}$ such $\nabla_{e_i}^M e_j = 0$ at x

$$\begin{aligned}
\text{trace}\,\nabla^2\xi &= \nabla_{e_i}^M \nabla_{e_i}^M \xi && (2.31)\\
&= e_i(f)e_i + e_i(\omega(e_i))\xi + \omega(e_i)\nabla_{e_i}^M \xi\\
&= \text{grad}\, f + e_i(\omega(e_i))\xi + \omega(e_i)[fe_i + \omega(e_i)\nabla_{e_i}^M \xi]\\
&= \text{grad}\, f + e_i(\omega(e_i))\xi + f\mu g(e_i, \xi)e_i + \mu^2 g(e_i, \xi)^2\xi\\
&= \text{grad}\, f + e_i(\mu g(e_i, \xi))\xi + f\mu\xi + \mu^2|\xi|^2\xi\\
&= \text{grad}\, f + g(\text{grad}\,\mu, \xi)\xi + \mu g(e_i, \nabla_{e_i}^M \xi)\xi + f\mu\xi + \mu^2|\xi|^2\xi\\
&= \text{grad}\, f + g(\text{grad}\,\mu, \xi)\xi + n\mu f\xi + \mu^2|\xi|^2\xi + f\mu\xi + \mu^2|\xi|^2\xi\\
&= \text{grad}\, f + \big[g(\text{grad}\,\mu, \xi) + (n+1)\mu f + 2\mu^2|\xi|^2\big]\xi. && (2.32)
\end{aligned}$$

For the second term of (2.30), we have

$$\begin{aligned}
\text{trace}\ R(\xi, \nabla^M\xi) &= R(\xi, \nabla_{e_i}^M \xi)e_i\\
&= f\,R(\xi, e_i)e_i + \omega(e_i)R(\xi, \xi)e_i\\
&= f\,R(\xi, e_i)e_i\\
&= f\,\text{Ricci}(\xi). && (2.33)
\end{aligned}$$

The Proposition 2.1 follows from (2.29), (2.31) and (2.33). □

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Affiliations

AHMED MOHAMMED CHERIF
ADDRESS: Mascara University, Dept. of Mathematics, 29000, Algeria.
E-MAIL: a.mohammedcherif@univ-mascara.dz
ORCID ID: 0000-0002-6155-0976

MUSTAPHA DJAA
ADDRESS: Relizane University, Dept. of Mathematics, 48000, Algeria.
E-MAIL: djaamustapha@live.com
ORCID ID: 0000-0002-7330-2144