

Analytic Families of Self-Adjoint Compact Operators Which Commute with Their Derivative

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Abstract

Spectral properties of analytic families of compact operators on a Hilbert space are studied. The results obtained are then used to establish that an analytic family of self-adjoint compact operators on a Hilbert space \mathcal{H} , which commute with their derivative, must be functionally commutative.

Keywords: Compact operator, Spectral decomposition, Analytic projection, Analytic eigenvalue, Functional commutativity, Analytic operator-valued function.

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1. Preliminaries

In [6], Stuart Goff studied analytic hermitian function matrices which commute with their derivative on some real interval I , i.e., $A(t)A'(t) = A'(t)A(t)$ for all $t \in I$. He obtained as a main result that these matrices are functionally commutative on I , i.e.,

$$A(s)A(t) = A(t)A(s)$$

for all $s, t \in I$ [[6], Theorem 3.6].

Subsequently, in [5], Jean-Claude Evard while studying the nonlinear differential equation

$$A(t) \frac{dA(t)}{dt} = \frac{dA(t)}{dt} A(t), t \in \Omega,$$

where Ω is an open interval in \mathbb{R} and A is a differentiable map from Ω into the \mathbb{C} -Banach space M_n of all $n \times n$ matrices $(\alpha_{i,j})$, with $\alpha_{i,j} \in \mathbb{C}$ for $i, j \in \{1, \dots, n\}$, was led to consider the more general problem where Ω is an open connected subset of a Banach space on \mathbb{R} or \mathbb{C} . In his paper Evard generalized Goff's theorem in ([5], Theorem 4.3) and summarizes the history and motivations behind the problem on matrix functions commuting with their derivative from 1950 to 1982. It also suggests further paths of investigations such as the one of interest to us, indeed our main result, Theorem 3, extends the final dimensional result of Goff [6] to the infinite-dimensional situation of compact self-adjoint operators on a Hilbert space.

In this paper, we study analytic families of compact self-adjoint operators, on a complex Hilbert space, which commute with their derivative on some real interval I . Our main result establishes that these operators must be functionally commutative on I , that is,

$$A(s)A(t) = A(t)A(s)$$

for all $s, t \in I$, extending the main result of [6] and [5] from the case of matrices to the infinite dimensional situation of operators on a Hilbert space. Indeed, Stuart Goff studied the case of Hermitian function matrices which commute with their derivative in [6] and Jean-Claude Evard extended Goff's result to matrix functions defined on an open subset of a normed space in [5].

We recall that $\mathcal{B}(\mathcal{H})$ denotes the Banach algebra of all bounded operators on the complex Hilbert space \mathcal{H} and by definition the *spectrum* of T , denoted by $\text{Sp}T$, is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible in $\mathcal{B}(\mathcal{H})$. In this paper $A(t)$ will denote an analytic family of compact self-adjoint operators on a complex Hilbert space and defined on a real interval $I = (a, b)$ with $a < 0 < b$. Let λ be an eigenvalue of $A(0)$. In fact, more generally $A(t)$ can be an analytic family of bounded self-adjoint operators on \mathcal{H} with the property that λ is an isolated point of the spectrum $\text{Sp}A(0)$, and such that the λ -eigenspace of $A(0)$ is finite-dimensional. Let D be a closed disk centered at λ such that $\text{Sp}A(0) \cap D = \{\lambda\}$. It follows that, for t sufficiently small, $\text{Sp}A(t) \cap \gamma = \emptyset$ where $\gamma = \partial D$ is the boundary of D . For such t , we have the orthogonal Riesz projections

$$P = \frac{1}{2\pi i} \int_{\gamma} (\xi I - A)^{-1} d\xi$$

with range $\mathcal{H}(t)$, depending analytically on t , such that $P(0)$ is the orthogonal projection of \mathcal{H} onto the λ -eigenspace of $A(0)$.

We have the following important result in the general setting of a Banach algebra. Let f be an analytic function from a domain D of \mathbb{C} into the algebra of compact operators $\mathcal{K}(X)$ on a Banach space X and let $\lambda_0 \in D$, $\alpha_0 \in \text{Sp}f(\lambda_0)$ with $\alpha_0 \neq 0$. Suppose α_0 is an eigenvalue of multiplicity one, or equivalently that the Riesz projection associated to the null space $\mathcal{N}(f(\lambda_0)I - \alpha_0 I)$ has rank one. Then there exist $r, \delta > 0$ such that $|\lambda - \lambda_0| < \delta$ implies that $\text{Sp}(f(\lambda)) \cap B(\alpha_0, r)$ contains only one eigenvalue $\alpha(\lambda)$. What can be said about α ? In this particular case it is known that α is holomorphic on $B(\lambda_0, \delta)$. A proof of the next theorem is given in [1], pp 59-60.

Theorem 1.1 (Holomorphic Variation of Isolated Spectral Values). *Let f be an analytic function from a domain D of \mathbb{C} into a Banach algebra \mathcal{A} . Suppose there exists $\lambda_0 \in D$, $\alpha_0 \in \text{Sp}f(\lambda_0)$ and $r, \delta > 0$ such that $|\lambda - \lambda_0| < \delta$ implies that $\lambda \in D$ and that $\text{Sp}(f(\lambda)) \cap B(\alpha_0, r)$ contains only one point $\alpha(\lambda)$. Then α is holomorphic in a neighbourhood of λ_0 .*

A more general discussion about the behaviour of isolated parts of the spectrum is dealt with in [2], Chapter 10. Another source of interest on this topic is [8], Chapter XII, where a proof of the analyticity of discrete eigenvalues in the nondegenerate case for analytic families of operators is given. We recall that a point $\lambda \in \text{Sp}(A)$ is called discrete if λ is isolated and its associated Riesz projection P_λ is finite-dimensional; if P_λ is one-dimensional, we say that λ is a nondegenerate eigenvalue. These results are contained in the Kato-Rellich theorem ([8], Chapter III, p.15). If $A(z)$ is an operator depending analytically on a complex parameter z near $z = 0$ and σ_0 is a component of $\text{Sp}(A(0))$, then the Riesz projection $P(z)$ will still be defined for sufficiently small z , and will represent an idempotent depending analytically on z . Our main result will be based essentially on the possibility of decomposing an operator like $A(z)$ into a sum $\sum \mu_k P_k(z)$, where the $P_k(z)$ are mutually orthogonal analytic projections, i.e. $(P_k(z)P_l(z) = 0$ for $k \neq l$), such that $A(z)P_k(z) = P_k(z)A(z)$. It is a spectral decomposition with the added condition of analyticity. For a more complete and comprehensive modern reference for spectral theory we refer the reader to [3], Chapter VIII.

2. Self-Adjoint Compact Operators on a Hilbert Space

It is well known that self-adjoint $n \times n$ matrices can be diagonalized, i.e. can be written as $\sum_{\alpha=1}^k \lambda_\alpha P_\alpha$ where the P_α are self-adjoint orthogonal projections and the λ_α are real numbers. This result can be extended to self-adjoint compact operators on a Hilbert space. A proof of the next theorem is given in [1], pp 25-26.

Theorem 2.1 (Spectral Theorem for Self-Adjoint Compact Operators on a Hilbert Space). *Let \mathcal{H} be a Hilbert space and let T be a self-adjoint compact operator on \mathcal{H} . Let $\{\lambda_k\}_{k \geq 1}$ be the discrete set of nonzero eigenvalues of T . Also let $E_0 = \mathcal{N}(T)$ and $E_k = \mathcal{N}(T - \lambda_k I)$, for $k \geq 1$. Then we have the following properties:*

(i) *for $k \geq 0$ the closed subspaces E_k are orthogonal and their Hilbertian direct sum is \mathcal{H} . Moreover, if P_k denotes the self-adjoint projection on E_k we have $TP_k = P_k T$ for all k ,*

(ii) *the series $\sum_{k \geq 1} \lambda_k P_k$ converges in norm in $\mathcal{B}(\mathcal{H})$ and we have*

$$T = \sum_{k \geq 1} \lambda_k P_k.$$

3. Analytic self-adjoint compact operators on a Hilbert space

The proof of the following lemma is suggested by the argument used in the proof of Theorem 3.5 of [5]. Here $z \in I$ where I is an interval in \mathbb{R} . As in [5], we say that $A(t)$ commutes with its derivative on a real interval I if it satisfies the commutation equation

$$A(t) \frac{dA(t)}{dt} = \frac{dA(t)}{dt} A(t), t \in I,$$

that we write simply as

$$A(t)A'(t) = A'(t)A(t), t \in I.$$

Lemma 3.1. *Let $A(t)$ be an analytic family of self-adjoint compact operators on a Hilbert space \mathcal{H} which commute with its derivative. Then the projections associated to the eigenvalues of $A(t)$ commute with their derivative.*

Proof. Let $t \in I$ such that $A'(t)A(t) = A(t)A'(t)$ and

$$\text{Sp}A(t) \subset \bigcup_{n=1}^{\infty} (\mathbb{C} - \Gamma_i),$$

where Γ_i is a simple contour which does not meet $\text{Sp}A(t)$. Then as in the proof of Theorem 3.5 of [5],

$$P_k(t) = \frac{1}{2\pi i} \int_{\Gamma_k} (z - A(t))^{-1} dz,$$

commutes with its derivative

$$P'_k(t) = \frac{1}{2\pi i} \int_{\Gamma_k} (z - A(t))^{-1} (-A'(t)) (z - A(t))^{-1} dz,$$

because

$$A(t)A'(t) = A'(t)A(t)$$

and Γ_k is compact (so differentiation inside the integral sign is justified). □

Evard proved in [4] that if $P(t)$ commute with its derivative, its range $\text{ran}(P(t))$ is not only invariant under their derivative, but also constant. Indeed he proved in Theorem 6 of the same paper that the family $P(t)$ itself is constant.

Lemma 3.2. *If a family of projections $P(t)$ commutes with its derivative on an interval $I \in \mathbb{R}$, then $P(t)$ is constant.*

Proof. Since $P^2(t) = P(t)$, it follows by differentiation of the two sides that

$$P'(t)P(t) + P(t)P'(t) = P'(t).$$

Now by hypothesis

$$P'(t)P(t) = P(t)P'(t)$$

so we get

$$2P'(t)P(t) = P'(t),$$

which by multiplication by $P(t)$ yields

$$2P'(t)P(t) = P'(t)P(t).$$

Hence, $P'(t)P(t) = 0$. Going back to the relation

$$2P'(t)P(t) = P'(t),$$

we conclude that $P'(t) = 0$, which means $P(t)$ is constant. □

Using the previous results, we establish our main result in the next theorem.

Theorem 3.3. *Let $A(t)$ be an analytic family of compact self-adjoint operators on a Hilbert space H . Suppose that $A(t)$ commutes with its derivative for all $t \in I \subset \mathbb{R}$. Then $A(t)$ is functionally commutative, i.e. $A(s)A(t) = A(t)A(s)$ for all $s, t \in I$.*

Proof. By Theorem 2, any compact self-adjoint operator on a Hilbert space admits a spectral decomposition, so we can write,

$$A(t) = \sum_{k=1}^{\infty} \lambda_k(t) P_k(t)$$

where $\{\lambda_k\} \subset \text{Sp}A(t)$ and $P_k^2(t) = P_k(t)$. Moreover by Lemma 2, the projections $P_k(t)$ commute with their derivative, and by Lemma 3 they are constant. Hence,

$$A(t) = \sum_{i=1}^{\infty} \lambda_i(t) P_i$$

where $P_i^2 = P_i$ are constant projections. Consequently we get,

$$A(s) = \sum_{i=1}^{\infty} \mu_i(s) P_i$$

and

$$A(s)A(t) = A(t)A(s) \text{ for all } s, t \in I.$$

□

Note. If T is a compact operator, then its point spectrum is nonempty and countable, which may not hold for noncompact (normal) operators. But this is not the main role played by compact operators in the Spectral Theorem - we can deal with an uncountable weighted sum of projections. What is actually special with a compact operator is that a compact normal operator not only has a nonempty point spectrum but it has enough eigenspaces to span H . That makes the difference, since normal (noncompact) operators may have an empty point spectrum or it may have eigenspaces but not enough to span the whole space H . However, the Spectral Theorem survives the lack of compactness if the point spectrum is replaced with the whole spectrum (which is never empty). Such an approach for the general case of the Spectral Theorem (i.e. for normal, not necessarily compact operators) requires measure theory. This is a work that we intend to undertake in a future investigation.

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