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Semi-Invariant Submanifolds of Almost α-Cosymplectic f-Manifolds

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Abstract
In this paper, we have and study several properties of semi-invariant submanifolds of an almost α-cosymplectic f-manifold. We give an example and investigate the integrability conditions for the distributions involved in the definition of a semi-invariant submanifold of an almost α-cosymplectic f-manifold.

Keywords: Almost α-cosypmlectic f-manifolds, Semi-invariant submanifolds, Integrability conditions
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1. Introduction

Contact geometry has been seen to underly many physical phenomena and be related to many other mathematical structures. Contact structures first appeared in the work of Sophus Lie [1] on partial differential equations. They reappeared in Gibbs' work on thermodynamics, Huygens' work on geometric optics and in Hamiltonian dynamics. ([2], [3], [4]).

On the other hand, the notion of CR-submanifold of a Kaehler manifold was introduced by Bejancu [5]. Later, semi-invariant (or contact CR-) submanifolds of a Sasakian manifold was studied by Shahid, Sharfuddin and Husain [6], Kobayashi [7], Matsumoto [8] and many others. Submanifolds of cosymplectic manifold have been studied by Ludden [9], A. Cabras, A.Ianus and G.H. Pitis [10].

Later, the subject was considered for Riemannian manifolds with an almost contact structure. In this sense A. Bejancu and N. Papaghiuc study semi-invariant submanifolds of a Sasakian manifold or Sasakian space form ([11],[12], [13], [14]) and C.L. Bejan, A., et.al. study them on cosymplectic manifolds in ([15], [16]). B. B. Sinha and R. N. Yadav studied the integrable conditions of distributions and the geometry of leaves on a semi-invariant submanifolds in a Kenmotsu manifold [17].

In 2014, Öztürk et.al. introduced and studied almost α-cosymplectic f-manifold [18] defined for any real number α which is defined a metric f-manifold with f-structure $(\varphi, \xi_i, \eta^i, g)$ satisfying the condition $d\eta^i = 0$, $d\Omega = 2\alpha\overline{\eta}\wedge\Omega$.

In this paper, we introduce properties of semi-invariant submanifolds of an almost α-cosymplectic f-manifold. In Section 2, we review basic formulas and definitions for almost α-cosymplectic f-manifolds. In Section 3, we define semi-invariant submanifolds of an almost α-cosymplectic f-manifold. We also present a way to build these submanifolds and give an example. In Section 4, we obtain some basic results for semi-invariant submanifolds of an almost α-cosymplectic f-manifold. In Section 5, we investigate the integrability of the distributions involved in the definition of a semi-invariant submanifold. In last section we focus mixed totally geodesic of semi-invariant submanifolds of an almost α-cosymplectic f-manifold.

2. Preliminaries

Let $\widetilde{M}$ be a real (2n+s)-dimensional framed metric manifold [19] with a framed $(\varphi, \xi_i, \eta^i, g)$, $i \in \{1,...,s\}$, that is, φ is a non-vanishing tensor field of type (1,1) on $\widetilde{M}$ which satisfies $\varphi^3 + \varphi = 0$ and has constant rank $r = 2n$; $\xi_1, ...\xi_s$ are s vector fields; $\eta^1, ..., \eta^s$ are 1-forms and g is a Riemannian metric on $\widetilde{M}$ such that

$$\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i \tag{2.1}$$

$$\eta^i(\xi_j) = \delta^i_j,\ \varphi(\xi_i) = 0,\ \eta^i o \varphi = 0, \tag{2.2}$$

$$\eta^i(X) = g(X, \xi_i), \tag{2.3}$$

$$g(X, \varphi Y) + g(\varphi X, Y) = 0, \tag{2.4}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y) \tag{2.5}$$

for all $X, Y \in \Gamma(T\widetilde{M})$ and $i, j \in \{1,...,s\}$. In above case, we say that $\widetilde{M}$ is a metric f-manifold and its associated structure will be denoted by $\widetilde{M}(\varphi, \xi_i, \eta^i, g)$ [19].
A 2-form Ω is defined by $\Omega(X,Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(T\widetilde{M})$, is called the fundamental 2-form. A framed metric structure is called normal [19] if

$$[\varphi, \varphi] + 2d\eta^i \otimes \xi_i = 0$$

where $[\varphi, \varphi]$ is denoting the Nijenhuis tensor field associated to φ. Throughout this paper we denote by $\overline{\eta} = \eta^1 + \eta^2 + ... + \eta^s$, $\overline{\xi} = \xi_1 + \xi_2 + ... + \xi_s$ and $\overline{\delta}^j_i = \delta^1_i + \delta^2_i + ... + \delta^s_i$.

Definition 2.1. *Let $\widetilde{M}(\varphi, \xi_i, \eta^i, g)$ be a (2n+s)-dimensional a metric f-manifold for each $\eta^i, (1 \le i \le s)$ 1-forms and each 2-form Ω, if $d\eta^i = 0$ and $d\Omega = 2\alpha\overline{\eta} \wedge \Omega$ satisfy, then $\widetilde{M}$ is called almost α-cosymplectic f-manifold [18].*

Let $\widetilde{M}$ be an almost α-cosypmlectic f-manifold. Since the distribution D is integrable, we have $L_{\xi_i}\eta^j = 0$, $[\xi_i, \xi_j] \in D$ and $[X, \xi_j] \in D$ for any $X \in \Gamma(D)$. Then the Levi-Civita connection is given by [18]:

$$\begin{aligned} 2g((\widetilde{\nabla}_X \varphi)Y, Z) = 2\alpha g\left(\sum_{i=1}^{s} (g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X), Z\right) \\ + g(N(Y,Z), \varphi X) \end{aligned} \tag{2.6}$$

for any $X, Y \in \Gamma(T\widetilde{M})$. Putting $X = \xi_i$ we obtain $\widetilde{\nabla}_{\xi_i}\varphi = 0$ which implies $\widetilde{\nabla}_{\xi_i}\xi_j \in D^\perp$ and then $\widetilde{\nabla}_{\xi_i}\xi_j = \widetilde{\nabla}_{\xi_j}\xi_i$, since $[\xi_i, \xi_j] = 0$. We put $A_i X = -\widetilde{\nabla}_X \xi_i$ and $h_i = \frac{1}{2}(L_{\xi_i}\varphi)$, where L denotes the Lie derivative operator. If $\widetilde{M}$ is almost α-cosymplectic f-manifold with Kaehlerian leaves [20], we have

$$(\widetilde{\nabla}_X \varphi)Y = \sum_{i=1}^{s} \left[-g(\varphi A_i X, Y)\xi_i + \eta^i(Y)\varphi A_i X\right]$$

or

$$(\widetilde{\nabla}_X \varphi)Y = \sum_{i=1}^{s} \left[\alpha\left(g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\right) + g(h_i X, Y)\xi_i - \eta^i(Y)h_i X\right]. \tag{2.7}$$

Proposition 2.2. *([18]) For any $i \in \{1,...,s\}$ the tensor field A_i is a symmetric operator such that* *(i)* $A_i(\xi_j) = 0$, *for any* $j \in \{1, ..., s\}$

(ii) $A_i o\varphi + \varphi o A_i = -2\alpha\varphi$

(iii) $tr(A_i) = -2\alpha n$

(iv) $\widetilde{\nabla}_X \xi_i = -\alpha\varphi^2 X - \varphi h_i X$.

Proposition 2.3. *([21]) For any* $i \in \{1, ..., s\}$ *the tensor field* h_i *is a symmetric operator and satisfies*

(i) $h_i(\xi_j) = 0$, *for any* $j \in \{1, ..., s\}$

(ii) $h_i o\varphi + \varphi o h_i = 0$

(iii) $trh_i = 0$

(iv) $tr(\varphi h_i) = 0$.

Let $\widetilde{M}$ be an almost α-cosymplectic f-manifold with respect to the curvature tensor field $\widetilde{R}$ of $\widetilde{\nabla}$, the following formulas are proved in [18], for all $X, Y \in \Gamma(T\widetilde{M})$, $i, j \in \{1, ..., s\}$.

$$\begin{aligned}\widetilde{R}(X,Y)\xi_i &= \alpha^2 \sum_{k=1}^{s} (\eta^k(Y)\varphi^2 X - \eta^k(X)\varphi^2 Y) \\ &- \alpha \sum_{k=1}^{s} (\eta^k(X)\varphi h_k Y - \eta^k(Y)\varphi h_k X) \\ &+ (\widetilde{\nabla}_Y \varphi h_i)X - (\widetilde{\nabla}_X \varphi h_i)Y,\end{aligned} \tag{2.8}$$

$$\begin{aligned}\widetilde{R}(X,\xi_j)\xi_i &= \sum_{k=1}^{s} \delta_j^k (\alpha^2\varphi^2 X + \alpha\varphi h_k X) \\ &+ \alpha\varphi h_i X - h_i h_j X + \varphi(\widetilde{\nabla}_{\xi_j} h_i)X\end{aligned} \tag{2.9}$$

$$\widetilde{R}(\xi_j, X)\xi_i - \varphi\widetilde{R}(\xi_j, \varphi X)\xi_i = 2(-\alpha^2\varphi^2 X + h_i h_j X). \tag{2.10}$$

Moreover, by using the above formulas, in [18] it is obtained that

$$\widetilde{S}(X, \xi_i) = -2n\alpha^2 \sum_{k=1}^{s} \eta^k(X) - (div\varphi h_i)X \tag{2.11}$$

$$\widetilde{S}(\xi_i, \xi_j) = -2n\alpha^2 - tr(h_j h_i) \tag{2.12}$$

for all $X, Y \in \Gamma(T\widetilde{M})$, $i, j \in \{1, ..., s\}$, where $\widetilde{S}$ denote, the Ricci tensor field of the Riemannian connection.
From [18], we have the following result.

Proposition 2.4. *Let* $\widetilde{M}$ *be an almost* α*-cosymplectic* f*-manifold and* M *be an integral manifold of* D*. Then*

(i) when $\alpha = 0$, M *is totally geodesic if and only if all the operators* h_i *vanish;*

(ii) when $\alpha \neq 0$, M *is totally umbilic if and only if all the operators* h_i *vanish.*

3. Semi-Invariant Submanifolds of Almost α-Cosymplectic f-Manifolds

The submanifold M of the almost α-cosymplectic f-manifold $\widetilde{M}$ is said to be semi-invariant [22] if it is endowed with two pair of ortogonal distribution $D, D^{\perp}$ satisfying the conditions

(i) $TM = D \oplus D^{\perp} \oplus \{\xi_1, \xi_2, ..., \xi_s\}$

(ii) the distribution D is invariant under φ, that is

$$\varphi D_x = D_x, for \quad each \quad x \in M,$$

(iii) the distribution $D^\perp$ is anti-invariant under φ, that is

$$\varphi D_x^\perp \subset T_x M^\perp for \quad each \quad x \in M.$$

The distribution D $(resp.D^\perp)$ is called the horizantal (resp. vertical) distribution. A semi-invariant submanifold M is said to be invariant (resp. anti-invariant) submanifold if we have $(D_x^\perp = 0)$ respectively $(D_x = 0)$ for each $x \in M$. We say that M is proper semi-invariant submanifold if it is a semi-invariant submanifold which is neither an invariant nor anti-invariant submanifold [22].

We denote by same symbol g both metrices on $\widetilde{M}$ and M. The projection morphism of TM to D and $D^\perp$ are denoted by P and Q respectively. For any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$ we have

$$X = PX + QX + \sum_{i=1}^{s} \eta^i(X)\xi_i \tag{3.1}$$

$$\varphi N = CN + DN \tag{3.2}$$

and

$$h_i X = t_i X + f_i X \tag{3.3}$$

where CN and t_iX(resp.DN and f_iX) denotes the tangential (resp. normal) of φN and h_iX, respectively.

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y) \tag{3.4}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla_X^\perp N \tag{3.5}$$

for any $X,Y \in \Gamma(TM)$ and $N \in TM^\perp$, where ∇ is the Levi-civita connection on M, $\nabla^\perp$ is the linear connection induced by $\widetilde{\nabla}$ on the normal bundle $TM^\perp$, B is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section N. Also we have

$$g(B(X,Y),N) = g(A_N X,Y) \tag{3.6}$$

for any $X,Y \in \Gamma(TM), N \in \Gamma(TM^\perp)$ [19].
We now give an example of semi-invariant submanifold of an almost α-cosymplectic f-manifold.

Example 3.1. *Let us denote the standart coordinates of R^{2n+s} $(x_1,...,x_n,y_1,...,y_n,z_1,...,z_s)$ and take $(2n+s)$-dimensional manifold $\widetilde{M} \subset R^{2n+s}$ defined by*

$$\widetilde{M} = \{(x_1,...,x_n,y_1,...,y_n,z_1,...,z_s) | z_1,...,z_s \neq 0\}.$$

Consider following vector fields as a global basis of $\widetilde{M}$:

$$X_i = e^{\sum_{i=1}^n z_i}\frac{\partial}{\partial x_i}, \qquad Y_i = \frac{\partial}{\partial y_i}, \qquad \xi_j = \frac{\partial}{\partial z_j}, \;\; i = 1,...,n \;\; j = 1,...,s.$$

The brackets of these vector fields are

$$[\xi_j, X_i] = e^{\sum_{i=1}^n z_i}\frac{\partial}{\partial x_i}, \qquad [\xi_j, Y_i] = [X_k, X_i] = [X_i, Y_k] = [Y_i, Y_k] = 0$$

for any $i,k \in \{1,...,n\}$ and $j \in \{1,...,s\}$. One may easily verify that putting

$$\eta^j = dz_j, \qquad g = \sum_{i=1}^{n}[e^{-2(z_1+...+z_s)}dx_i^2 + dy_i^2] + \sum_{j=1}^{s} dz_j^2,$$

$$\varphi(\xi_j) = 0, \qquad \varphi(\frac{\partial}{\partial x_i}) = e^{-(z_1+...+z_s)}\frac{\partial}{\partial yi}, \qquad \varphi(\frac{\partial}{\partial y_i}) = -e^{(z_1+...+z_s)}\frac{\partial}{\partial x_i},$$

$(\varphi, \xi_i, \eta^i, g)$ is an almost contact metric f- structure on $\widetilde{M}$. We şhall check that $(\widetilde{M}, \varphi, \xi_i, \eta^i, g)$ is an almost α-cosymplectic f-manifold. Obviously, $\eta^j = dz_j \Rightarrow d\eta^j = d^2 z_j = 0$ from poincare metric we get $d\eta^j = 0$. To verify the condition $d\Phi = 2\alpha\bar{\eta}\wedge\Phi$, considering that all Φ_{ij}' s are zero except for $\Phi_{ii} = g(\frac{\partial}{\partial x_i}, \varphi\frac{\partial}{\partial y_i}) = -e^{-(z_1+...+z_s)}$ and hence

$$\Phi = -\frac{1}{e^{(z_1+...+z_s)}}\sum_{i=1}^{n} dx_i \wedge dy_i$$

holds. As a result, the exterior derivative $d\Phi$ is given by

$$d\Phi = -e^{-(z_1+...+z_s)}\sum_{i=1}^{n} dx_i \wedge dy_i \wedge (dz_1 + ... + dz_s)$$
$$d\Phi = e^{-(z_1+...+z_s)}e^{(z_1+...+z_s)}\Phi \wedge (\eta^1 + ... + \eta^s)$$
$$d\Phi = \bar{\eta}\wedge\Phi = 2(\frac{1}{2})\bar{\eta}\wedge\Phi.$$

Since the Nijenhuis torsion of φ is not zero, the manifold is an almost $(\frac{1}{2})$-cosymplectic f-manifold.

Now, we definite the distributions

$$D = sp\{X_1, Y_1, X_2, Y_2, ..., X_m, Y_m\}$$

and

$$D^{\perp} = sp\{X_{m+1}, X_{m+2}, ..., X_{m+p}\}(m < n).$$

It is clear that $TM = D \oplus D^{\perp} \oplus \{\xi_1, ..., \xi_s\}$, $dimM = 2m + p + s$. Let

$$TM^{\perp} = \{Y_{m+1}, Y_{m+2}, ..., Y_{m+p}, Y_{m+p+1}, ..., Y_n, X_{m+p+1}, ..., X_n\}$$

then we have $\varphi D = D$ and $\varphi D^{\perp} \subset TM^{\perp}$. Consequently, M is a semi-invariant submanifold of an almost $\frac{1}{2}$-cosymplectic f-manifold.

4. Basic Lemmas

For any $X, Y \in \Gamma(TM)$, we put

$$u(X,Y) = \nabla_X \varphi PY - A_{\varphi QY}X. \tag{4.1}$$

We start with proving the following lemma.

Lemma 4.1. *Let M be a semi-invariant submanifold of almost α- cosymplectic f- manifold with Kaehlerian leaves $\widetilde{M}$. Then we have*

$$P(u(X,Y)) = \varphi P\nabla_X Y - \sum_{i=1}^{s}[\alpha\eta^i(Y)\varphi PX + \eta^i(Y)Pt_iX] \tag{4.2}$$

$$Q(u(X,Y)) = QCB(X,Y) - \sum_{i=1}^{s}\eta^i(Y)Qt_iX \tag{4.3}$$

$$B(X,\varphi PY) + \nabla^{\perp}_X\varphi QY = \varphi Q\nabla_X Y + DB(X,Y) - \sum_{i=1}^{s}[\alpha\eta^i(Y)\varphi QX - \eta^i(Y)f_iX] \tag{4.4}$$

$$\eta^i(u(X,Y))\xi_i = \sum_{i=1}^{s}[\alpha g(\varphi PX,Y)\xi_i + g(h_iX,Y)\xi_i] - \sum_{i,j=1}^{s}\eta^i(Y)\eta^j(t_iX)\xi_i. \tag{4.5}$$

Proof. For $X,Y \in \Gamma(TM)$, putting (3.1), (3.2) and (3.3) in the equation (2.7) we get

$$\begin{aligned}(\widetilde{\nabla}_X\varphi)Y &= \sum_{i=1}^{s}[\alpha(g(\varphi PX,Y)\xi_i - \eta^i(Y)\varphi PX - \eta^i(Y)\varphi QX)\\ &+ g(h_iX,Y)\xi_i - \eta^i(Y)h_iX]\\ &= \sum_{i=1}^{s}[\alpha(g(\varphi PX,Y)\xi_i - \eta^i(Y)\varphi PX - \eta^i(Y)\varphi QX) + g(h_iX,Y)\xi_i\\ &- \eta^i(Y)Pt_iX - \eta^i(Y)Qt_iX - \eta^i(Y)\sum_{j=1}^{s}\eta^j(t_iX)\xi_j - \eta^i(Y)f_iX.\end{aligned}$$

On the other hand, by using (3.1), (3.2), (3.4) and (3.5) we have

$$\begin{aligned}(\widetilde{\nabla}_X\varphi)Y &= \widetilde{\nabla}_X\varphi Y - \varphi\widetilde{\nabla}_XY\\ &= \widetilde{\nabla}_X\varphi PY + \widetilde{\nabla}_X\varphi QY - \varphi(\nabla_XY + B(X,Y))\\ &= \nabla_X\varphi PY + B(X,\varphi PY) - A_{\varphi QY}X + \nabla^{\perp}_X\varphi QY\\ &- \varphi P\nabla_XY - \varphi Q\nabla_XY - CB(X,Y) - DB(X,Y)\end{aligned}$$

$$\begin{aligned}(\widetilde{\nabla}_X\varphi)Y &= P\nabla_X\varphi PY + Q\nabla_X\varphi PY + \sum_{i=1}^{s}\eta^i(\nabla_X\varphi PY)\xi_i + B(X,\varphi PY)\\ &- PA_{\varphi QY}X - QA_{\varphi QY}X + \nabla^{\perp}_X\varphi QY - \sum_{i=1}^{s}\eta^i(A_{\varphi QY}X)\xi_i\\ &- \varphi P\nabla_XY - \varphi Q\nabla_XY - CB(X,Y) - DB(X,Y).\end{aligned}$$

Taking the components of D, ξ_i, $D^{\perp}$ and $TM^{\perp}$ in above equations, we have our assertion. □

Lemma 4.2. *Let M be a semi-invariant submanifold of almost α- cosymplectic f- manifold with Kaehlerian leaves $\widetilde{M}$. Then we have*

$$\varphi P(A_NX) + P(\nabla_XCN) = P(A_{DN}X) \tag{4.6}$$
$$Q((C\nabla^{\perp}_XN) + A_{DN}X - \nabla_XCN) = 0 \tag{4.7}$$
$$\eta(A_{DN}X - \nabla_XCN) = \alpha g(X,CN) + g(h_iX,N)\xi_i \tag{4.8}$$
$$B(X,CN) + \varphi Q(A_NX) + \nabla^{\perp}_XDN = D\nabla^{\perp}_XN \tag{4.9}$$

for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$

Proof. By using the decompositions (3.1), (3.2) and the equations of Gauss and Weingarten in (2.7) we have

$$(\widetilde{\nabla}_X\varphi)N = \widetilde{\nabla}_X\varphi N - \varphi\widetilde{\nabla}_XN = \sum_{i=1}^{s}[\alpha g(\varphi X,N)\xi_i + g(h_iX,N)\xi_i]$$

$$\nabla_XCN + B(X,CN) - A_{DN}X + \nabla^{\perp}_XDN + \varphi A_NX - \varphi\nabla^{\perp}_XN = \sum_{i=1}^{s}[\alpha g(\varphi X,N)\xi_i + g(h_iX,N)\xi_i]$$

$$\begin{aligned}&= P\nabla_XCN + Q\nabla_XCN + \sum_{i=1}^{s}\eta^i(\nabla_XCN)\xi_i + B(X,CN) - PA_{DN}X - QA_{DN}X - \sum_{i=1}^{s}(A_{DN}X)\xi_i\\ &+ \nabla^{\perp}_XDN + \varphi PA_NX + \varphi QA_NX - C\nabla^{\perp}_XN - D\nabla^{\perp}_XN\\ &= -\sum_{i=1}^{s}[\alpha g(X,CN)\xi_i + g(h_iX,N)\xi_i]\end{aligned}$$

Then (4.6)- (4.9) follows by taking the components on each of the vector bundle D, $D^{\perp}$, ξ_i and respectively $TM^{\perp}$. □

Lemma 4.3. *Let M be a semi-invariant submanifold of almost* α*- cosymplectic* f*- manifold* $\widetilde{M}$*. Then we have*

$$\nabla_X \xi_i = \alpha X - \varphi t_i X - C f_i X \quad \forall X \in \Gamma(D) \tag{4.10}$$

$$\nabla_X \xi_i = \alpha X - \varphi t_i X - C f_i X \quad \forall X \in \Gamma(D^\perp) \tag{4.11}$$

$$\nabla_{\xi_i} \xi_j = 0, \;\; B(X, \xi_i) = -D f_i X. \tag{4.12}$$

Proof. For $X \in \Gamma(TM)$, using (3.2), (3.3) and (3.4) we obtain

$$\begin{aligned}
\widetilde{\nabla}_X \xi_i = \nabla_X \xi_i + B(X, \xi_i) &= -\alpha \varphi^2 X - \varphi h_i X \\
&= \alpha X - \alpha \sum_{i=1}^{s} \eta^i(X) \xi_i - \varphi h_i X \\
&= \alpha X - \alpha \sum_{i=1}^{s} \eta^i(X) \xi_i - \varphi t_i X - \varphi f_i X \\
&= \alpha X - \alpha \sum_{i=1}^{s} \eta^i(X) \xi_i - \varphi t_i X - C f_i X - D f_i X.
\end{aligned} \tag{4.13}$$

Thus (4.10)-(4.12) follows from (4.13). □

Lemma 4.4. *Let M be a semi-invariant submanifold of almost* α*- cosymplectic* f*- manifold with Kaehlerian leaves* $\widetilde{M}$*. Then we have*

$$A_{\varphi X} Y = A_{\varphi Y} X \tag{4.14}$$

for all $X, Y \in \Gamma(D^\perp)$*.*

Proof. For all $X, Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(TM)$, by using (3.4) and (3.6), we get

$$\begin{aligned}
g(A_{\varphi X} Y, Z) &= g(B(Y, Z), \varphi X) = g(\widetilde{\nabla}_Z Y, \varphi X) \\
&= -g(\varphi \widetilde{\nabla}_Z Y, X) = -g(\widetilde{\nabla}_Z \varphi Y - (\widetilde{\nabla}_Z \varphi) Y, X) \\
&= -g(\widetilde{\nabla}_Z \varphi Y, X) = g(\varphi Y, \widetilde{\nabla}_Z X) \\
&= g(\varphi Y, B(Z, X)) = g(A_{\varphi Y} X, Z),
\end{aligned}$$

which proves (4.14). □

Lemma 4.5. *Let M be a semi-invariant submanifold of almost* α*- cosymplectic* f*- manifold* $\widetilde{M}$*. Then we have,*

$$\nabla_{\xi_i} U \in \Gamma(D), \tag{4.15}$$

$$\nabla_{\xi_i} V \in \Gamma(D^\perp), \tag{4.16}$$

$$[U, \xi_i] \in \Gamma(D), \tag{4.17}$$

$$[V, \xi_i] \in \Gamma(D^\perp) \tag{4.18}$$

for any $i \in \{1, 2, ..., s\}$*,* $U \in \Gamma(D)$ *and* $V \in \Gamma(D^\perp)$*.*

Proof. For $U \in \Gamma(D)$ and $V \in \Gamma(D^\perp)$,

$$g(\nabla_{\xi_i} U, \xi_j) = \xi_i g(U, \xi_j) - g(U, \nabla_{\xi_i} \xi_j) = 0$$

and

$$g(\nabla_{\xi_i} U, V) = \xi_i g(U, V) - g(U, \nabla_{\xi_i} V) = g(\varphi^2 U, \nabla_{\xi_i} V) = -g(\varphi U, \varphi \nabla_{\xi_i} V) = -g(\varphi U, \nabla_{\xi_i} \varphi V) = g(\nabla_{\xi_i} \varphi U, \varphi V) = 0,$$

so $\nabla_{\xi_i} U \in \Gamma(D)$. In a similary way is deduced (4.16). On the other hand, using (4.10) and (4.11), we have

$$g([U, \xi_i], \xi_j) = g(\nabla_U \xi_i, -\nabla_{\xi_i} U, \xi_j) = 0$$

and

$$g([U, \xi_i], V) = g(\nabla_U \xi_i, V) - g(\nabla_{\xi_i} U, V) = 0.$$

Thus completes the proof. □

Lemma 4.6. *Let M be a semi-invariant submanifold of almost α- cosymplectic f- manifold $\widetilde{M}$. Then we have*

$$g(X,t_iY) = g(t_iX,Y), \tag{4.19}$$

$$\varphi t_iX + t_i\varphi X + Cf_iX = 0, \tag{4.20}$$

$$Df_iX + f_i\varphi X = 0 \tag{4.21}$$

for any $X,Y \in \Gamma(M)$.

Proof. Since h_i is symmetric, we get

$$g(X,h_iY) = g(h_iX,Y)$$
$$g(X,t_iY + f_iY) = g(t_iX,Y) + g(f_iX,Y)$$
$$g(X,t_iY) + g(X,f_iY) = g(t_iX,Y) + g(f_iX,Y).$$

From above equation we get (4.19). By making use of proposotion 2.3 and using (3.2), (3.3), we get

$$\varphi t_iX + t_i\varphi X + Cf_iX + Df_iX + f_i\varphi X = 0. \tag{4.22}$$

Comparing the tangential and normal part of (4.22), we get (4.20) and (4.21), respectively. □

5. Integrability of distribution on a semi-invariant submanifold in an almost α-cosymplectic f- manifold

Theorem 5.1. *Let M be a semi-invariant submanifold of almost α- cosymplectic f- manifold $\widetilde{M}$. Then the distribution D is never integrable.*

Proof. For all $X,Y \in \Gamma(D)$, we have

$$\begin{aligned}
g([X,Y],\xi_i) &= g(\nabla_X Y,\xi_i) - g(\nabla_Y X,\xi_i)\\
&= -g(Y,\nabla_X \xi_i) + g(X,\nabla_Y \xi_i)\\
&= -g(Y,\alpha X - \varphi t_iX - Cf_iX) + g(X,\alpha Y - \varphi t_iY - Cf_iY)\\
&= g(Y,\varphi t_iX) + g(Y,Cf_iX) - g(X,\varphi t_iY) - g(X,Cf_iY)\\
&= g(Y,\varphi t_iX + Cf_iX) - g(X,\varphi t_iY + Cf_iY)\\
&= -g(Y,t_i\varphi X) + g(X,t_i\varphi Y)\\
&= -g(t_iY,\varphi X) + g(t_iX,\varphi Y)\\
&= -g(Y,t_i\varphi X) - g(\varphi t_iX,Y)\\
&= -g(Y,t_i\varphi X + \varphi t_iX)\\
&= g(Y,Cf_iX) \neq 0.
\end{aligned}$$

This follows the non-integrability of D. □

Corollary 5.2. *The distribution $D \oplus D^{\perp}$ never involutive.*

Theorem 5.3. *Let M be a semi-invariant submanifold of almost α- cosymplectic f- manifold with Kaehlerian leaves $\widetilde{M}$. The distribution $D \oplus \{\xi_1,...,\xi_s\}$ is integrable if and only if*

$$B(X,\varphi Y) = B(\varphi X,Y) \tag{5.1}$$

is satisfied.

Proof. From (4.4), the distribution $D \oplus \{\xi_1,...,\xi_s\}$ is integrable if and only if

$$B(X,\varphi Y) - B(Y,\varphi X) = \varphi Q[X,Y] = 0$$

is satisfied so, $B(X,\varphi Y) = B(Y,\varphi X)$. □

Theorem 5.4. *Let M be a semi-invariant submanifold of almost α- cosymplectic f- manifold with Kaehlerian leaves $\widetilde{M}$. Then the distribution $D^{\perp}$ is integrable.*

Proof. From (4.1), we have for $X,Y \in \Gamma(D^{\perp})$

$$U(X,Y) = -A_{\varphi QY}X$$

operating φ in (4.2) we get

$$P\nabla_X Y = \varphi P(A_{\varphi Y}X) \tag{5.2}$$

for any $X,Y \in \Gamma(D^{\perp})$. By virtue of Lemma 4.4, (5.2) reduce to

$$P([X,Y]) = 0$$

which is prove that $[X,Y] \in \Gamma(D^{\perp})$. □

6. Mixed totally geodesic semi-invariant submanifolds

Definition 6.1. *A semi-invariant submanifold M of an almost α- cosymplectic f- manifold $\widetilde{M}$ is called mixed totally geodesic if the second fundamental form satisfies $B(X,Y) = 0$ for any $X \in D$ and $Y \in D^{\perp}$[5].*

Theorem 6.2. *Let M be a semi-invariant submanifold of almost α- cosymplectic f- manifold $\widetilde{M}$. Then M is mixed totally geodesic submanifold of almost α- cosymplectic f- manifold $\widetilde{M}$ if and only if*

$$A_V X \in \Gamma(D)\ (\forall X \in \Gamma(D),\ V \in \Gamma(TM)^{\perp}) \tag{6.1}$$

and

$$A_V X \in \Gamma(D)^{\perp}\ (\forall X \in \Gamma(D)^{\perp},\ V \in \Gamma(TM)^{\perp}). \tag{6.2}$$

Proof. Consider $A_V X$, let $X \in \Gamma(D)$ and $V \in \Gamma(TM)^{\perp}$ and $Y \in \Gamma(D^{\perp})$, then we have

$$\begin{aligned} g(B(X,Y),V) &= g(A_V X,Y) \\ &= 0 \Leftrightarrow A_V X \in \Gamma(D). \end{aligned}$$

On the other hand, if $A_V X \in \Gamma(D)$, we get

$$\begin{aligned} g(A_V X,V) &= g(B(X,Y),V) \\ &= 0 \Leftrightarrow B(X,Y) = 0. \end{aligned}$$

In a similar way is deduced (6.2). □

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