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Research Article

New properties of the generalized Dini function

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Abstract

In this work we study some properties of the normalized form of generalized Dini function like close-to-convexity of some order and close-to-convex with respect to another convex function. Furthermore, we investigate sufficient conditions which these functions are uniformly k-starlike functions of complex order b in the open unit disk, and some consequences of the main results are also presented.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{S}$ denote the class of all functions of $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Furthermore, let $\mathcal{P}(\gamma)$ denote the class of all analytic functions consisting of functions p with $p(0) = 1$ such that

$$\operatorname{Re} p(z) > \gamma,\ z \in \mathbb{U},\ (0 \le \gamma < 1),$$

and in particular, $\mathcal{P} := \mathcal{P}(0)$ is the well-known *Caratheódory class of functions* with positive real part in $\mathbb{U}$.

We denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ the subclasses of $\mathcal{A}$ consisting of functions which are *starlike of order* α, and *convex of order* α, that is

$$\mathcal{S}^*(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha,\ z \in \mathbb{U} \right\},\ (0 \le \alpha < 1)$$

and

$$\mathcal{C}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{(zf'(z))'}{f'(z)} > \alpha,\ z \in \mathbb{U} \right\},\ (0 \le \alpha < 1)$$

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respectively. In particular, $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{C} := \mathcal{C}(0)$ are the class of *starlike functions* and *convex functions* in the unit disk $\mathbb{U}$, respectively.

Also, we denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are *close-to-convex of order* α, that is

$$\operatorname{Re}\frac{zf'(z)}{g(z)} > \alpha,\ z \in \mathbb{U},\ (0 \le \alpha < 1)$$

for some function $g \in \mathcal{S}^*$. In particular, $\mathcal{K} := \mathcal{K}(0)$ is the class of *close-to-convex functions* in the unit disk $\mathbb{U}$.

There has been a continuous interest shown on the geometric and other related properties as univalency, starlikeness, convexity, and uniformly convexity of various special functions such as Bessel, Struve, Lommel, Wright, and Bessel functions. Several authors obtained many applications in the geometric functions theory for these special functions, see for example [1–5, 7, 12, 13].

Special functions, like Bessel functions of the first kind play an important role in pure and applied mathematics. First, we will define the *generalized Bessel function of first kind of order* ν by

$$J_\nu^c(z) := \sum_{n=0}^{\infty} \frac{(-c)^n\,(z/2)^{2n+\nu}}{n!\Gamma(\nu+n+1)},\ z \in \mathbb{U}.$$

In the present paper we will use the following normalized form of *generalized Dini function*:

$$\begin{aligned} r_\nu^c(z) &:= 2^\nu\Gamma(\nu+1)z^{1-\frac{\nu}{2}}\left[(1-\nu)J_\nu^c(\sqrt{z}) + \sqrt{z}\,(J_\nu^c)'\,(\sqrt{z})\right] \\ &= z + \sum_{n=1}^{\infty} \frac{(-c)^n(2n+1)\Gamma(\nu+1)}{4^n n!\Gamma(\nu+n+1)} z^{n+1},\ z \in \mathbb{U}\quad (c \in \mathbb{C},\ \nu > -1). \end{aligned} \tag{1.2}$$

By taking $c = -k$, where $k > 0$, we get the *modified Dini function* which has the form

$$R_\nu^k(z) = z + \sum_{n=1}^{\infty} \frac{k^n(2n+1)\Gamma(\nu+1)}{4^n n!\Gamma(\nu+n+1)} z^{n+1},\ z \in \mathbb{U}.$$

In 2018, Bansal et al. [1] investigated some certain geometric properties of the modified Dini function R_ν^k like close-to-convexity, starlikeness, and strongly starlikeness in the open unit disk. In this paper our aim is to study some properties of the normalized form of generalized Dini function r_ν^c. For this work, the following lemmas and definition will be used in our investigation.

In 2018, Bukhari et al. [6] introduced the class $\mathcal{UM}(g,\gamma,b,k)$, which was defined as follows:

Definition 1.1. Let $f \in \mathcal{A}$ be given by (1.1). Then, $f \in \mathcal{UM}(g,\gamma,b,k)$ if for the function $g \in \mathcal{A}$ given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$ with $b_n \ge 0$ for $n \ge 2$, we have

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zF_\gamma'(z)}{F_\gamma(z)} - 1\right)\right\} > k\left|\frac{1}{b}\left(\frac{zF_\gamma'(z)}{F_\gamma(z)} - 1\right)\right|,\ z \in \mathbb{U},$$

where

$$\begin{aligned} F_\gamma(z) &:= (1-\gamma)(f*g)(z)(z) + \gamma z(f*g)'(z) = z + \sum_{n=2}^{\infty} a_n\,d_n(\gamma)\,z^n,\ z \in \mathbb{U}, \\ d_n(\gamma) &:= [1+n(1-\gamma)]b_n,\ (k \ge 0,\ 0 \le \gamma \le 1,\ b \in \mathbb{C}\setminus\{0\}), \end{aligned}$$

and “$*$” represents the *Hadamard (or convolution) product.*

Generally, this class consists of functions F_γ which are uniformly k-starlike functions of complex order b in $\mathbb{U}$. For special choices of parameters of the class $\mathcal{UM}(g,\gamma,b,k)$ like $g(z)=\dfrac{z}{1-z}$ we obtain the subclass $\mathcal{UM}\left(\frac{z}{1-z},\gamma,b,k\right)=:\mathcal{UM}(\gamma,b,k)$.

The following lemmas will be used in the proofs of our main results.

Lemma 1.2. [6, Theorem 3.4] *If $f\in\mathcal{A}$ be given by* (1.1),

$$g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{A},$$

for some $b_k\geq 0$, and

$$\sum_{n=2}^{\infty}\left[(k+1)(n-1)+|b|\,\right]\left[1+\gamma(n-1)\right]b_n|a_n|<|b|,$$

for some $k\geq 0$, $0\leq\gamma\leq 1$, $b\in\mathbb{C}\setminus\{0\}$, then $f\in\mathcal{UM}(g,\gamma,b,k)$.

Lemma 1.3. [16, Corollary 2] *Let $\gamma\in[0,1)$. If $f\in\mathcal{A}$ satisfies the inequality*

$$|zf''(z)|<\frac{1-\gamma}{4},\ z\in\mathbb{U},$$

then

$$\operatorname{Re}f'(z)>\frac{1+\gamma}{2},\ z\in\mathbb{U}.$$

Letting $q\to 1^-$ in Theorem 2.6 from [17] we obtain the next lemma.

Lemma 1.4. *Let $f(z)=z+\sum\limits_{n=1}^{\infty}b_{2n+1}z^{2n+1}$, $z\in\mathbb{U}$, be an odd function. If*

$$1\geq 3b_3\geq\ldots\geq(2n+1)b_{2n+1}\geq\ldots\geq 0,$$

or

$$1\leq 3b_3\leq\ldots\leq(2n+1)b_{2n+1}\leq\ldots\leq 2,$$

then the function f is close-to-convex (univalent) with respect to the convex function $\dfrac{1}{2}\log\dfrac{1+z}{1-z}$.

2. New properties of the generalized Dini function

Theorem 2.1. *If $\gamma\in[0,1)$ and $\nu>\dfrac{|c|(5-2\gamma)}{4(1-\gamma)}-1$, then $\dfrac{r_\nu^c(z)}{z}\in\mathcal{P}(\gamma)$.*

Proof. Let define the function p by

$$p(z)=\frac{\dfrac{r_\nu^c(z)}{z}-\gamma}{1-\gamma},\ z\in\mathbb{U}.$$

Since r_ν^c is given by (1.2), then p is analytic in $\mathbb{U}$, with $p(0)=1$. To prove our result it is sufficient to show that $|p(z)-1|<1$, $z\in\mathbb{U}$. By using the equality

$$\frac{\Gamma(\nu+1)}{\Gamma(\nu+n+1)}=\frac{1}{(\nu+1)(\nu+2)\ldots(\nu+n)}=:\frac{1}{(\nu+1)_n},\quad n\in\mathbb{N},\tag{2.1}$$

the inequalities

$$4^n\geq\frac{4}{3}(2n+1),\quad(\nu+1)_n\geq(\nu+1)^n,\quad n!\geq 2^{n-1},\quad n\in\mathbb{N},$$

and the well-known triangle inequality, we deduce that

$$\begin{aligned}
|p(z)-1| &= \left|\frac{1}{1-\gamma}\sum_{n=1}^{\infty}\frac{(-c)^n(2n+1)\Gamma(\nu+1)}{4^n n!\Gamma(\nu+n+1)}z^n\right| \\
&\le \frac{1}{1-\gamma}\sum_{n=1}^{\infty}\frac{|c|^n(2n+1)\Gamma(\nu+1)}{4^n n!\Gamma(\nu+n+1)}|z|^n \le \frac{3}{4(1-\gamma)}\sum_{n=1}^{\infty}\frac{|c|^n\Gamma(\nu+1)}{2^{n-1}\Gamma(\nu+n+1)} \\
&= \frac{3}{2(1-\gamma)}\sum_{n=1}^{\infty}\frac{|c|^n}{2^n(\nu+1)_n} \le \frac{3}{2(1-\gamma)}\sum_{n=1}^{\infty}\left(\frac{|c|}{2(\nu+1)}\right)^n, \ z\in\mathbb{U}.
\end{aligned}$$

Using the fact that the assumption $\nu > \dfrac{|c|(5-2\gamma)}{4(1-\gamma)}-1$ implies $\dfrac{|c|}{2(\nu+1)}<1$, from the above inequality it follows that

$$|p(z)-1| \le \frac{3}{2(1-\gamma)}\sum_{n=1}^{\infty}\left(\frac{|c|}{2(\nu+1)}\right)^n = \frac{1}{1-\gamma}\frac{3|c|}{4(\nu+1)-2|c|} := \mu, \ z\in\mathbb{U}.$$

Under our hypothesis, it is easy to check that $\mu<1$, and therefore $\dfrac{r_\nu^c(z)}{z}\in\mathcal{P}(\gamma)$. □

For $\gamma=0$ the Theorem 2.1 reduces to the following result which gives sufficient condition for the function $\dfrac{r_\nu^c(z)}{z}$ to be in the class $\mathcal{P}$.

Corollary 2.2. *If* $\nu > \dfrac{5|c|}{4}-1$, *then* $\dfrac{r_\nu^c(z)}{z}\in\mathcal{P}$.

Example 2.3. Since

$$J_{\frac{1}{2}}^{1}(z)=\sqrt{\frac{2}{\pi z}}\sin z,$$

from Corollary 2.2 it follows that the function

$$\frac{r_{\frac{1}{2}}^{1}(z)}{z}=\cos\sqrt{z}$$

is in the class $\mathcal{P}$, and thus

$$\operatorname{Re}\left(\cos\sqrt{z}\right)>0, \ z\in\mathbb{U}.$$

Theorem 2.4. *If* $\gamma\in[0,1)$ *and* $\nu > \dfrac{|c|\left(7-\gamma+2\sqrt{12-3\gamma}\right)}{2(1-\gamma)}-1$, *then* $r_\nu^c\in\mathcal{K}\left(\frac{1+\gamma}{2}\right)$.

Proof. Using the relation (2.1), the inequalities

$$4^n\ge\frac{2}{3}(n+1)(2n+1), \quad (\nu+1)_n\ge(\nu+1)^n, \quad n!\ge 2^{n-1}, \quad n\in\mathbb{N},$$

and the triangle inequality, we have

$$\begin{aligned}
\left|z(r_\nu^c)''(z)\right| &= \left|\sum_{n=1}^{\infty}\frac{(-c)^n n(n+1)(2n+1)\Gamma(\nu+1)}{4^n n!\Gamma(\nu+n+1)}z^n\right| \\
&\le \sum_{n=1}^{\infty}\frac{|c|^n n(n+1)(2n+1)\Gamma(\nu+1)}{4^n n!\Gamma(\nu+n+1)}|z|^n \le \frac{3}{2}\sum_{n=1}^{\infty}\frac{|c|^n n\Gamma(\nu+1)}{2^{n-1}\Gamma(\nu+n+1)} \\
&= \frac{3}{2}\sum_{n=1}^{\infty}\frac{n|c|^n}{2^{n-1}(\nu+1)_n} \le \frac{3}{2}\sum_{n=1}^{\infty}\frac{n|c|^n}{2^{n-1}(\nu+1)^n} = \frac{3}{2}\frac{|c|}{\nu+1}\sum_{n=1}^{\infty}n\left(\frac{|c|}{2(\nu+1)}\right)^{n-1}, \ z\in\mathbb{U}.
\end{aligned}$$

From the assumption $\nu > \dfrac{|c|\,(7-\gamma+2\sqrt{12-3\gamma})}{2(1-\gamma)} - 1$ it follows that $\dfrac{|c|}{2(\nu+1)} < 1$, and from the above inequality we deduce

$$\begin{aligned} |z(r_\nu^c)''(z)| &\le \frac{3}{2}\frac{|c|}{\nu+1}\sum_{n=1}^{\infty} n\left(\frac{|c|}{2(\nu+1)}\right)^{n-1} \\ &= \frac{3}{2}\frac{|c|}{\nu+1}\frac{4(\nu+1)^2}{[2(\nu+1)-|c|]^2} = \frac{6|c|(\nu+1)}{[2(\nu+1)-|c|]^2} := \lambda,\ z\in\mathbb{U}. \end{aligned}$$

It is easy to check that our assumption implies that $\lambda < \dfrac{1-\gamma}{4}$ thus, from the previous inequality we obtain that

$$|z(r_\nu^c)''(z)| < \frac{1-\gamma}{4},\ z\in\mathbb{U}.$$

Now, by using Lemma 1.3 we conclude that

$$\mathrm{Re}(r_\nu^c)'(z) > \frac{1+\gamma}{2},\ z\in\mathbb{U},$$

and therefore $r_\nu^c \in \mathcal{K}\left(\frac{1+\gamma}{2}\right)$. □

For the special case $\gamma = 0$ the Theorem 2.4 leads to the following result which gives sufficient condition for the function r_ν^c to be of close-to-convex of order $\frac{1}{2}$.

Corollary 2.5. *If $\nu > \dfrac{|c|(7+4\sqrt{3})}{2} - 1$, then $r_\nu^c \in \mathcal{K}\left(\frac{1}{2}\right)$.*

The following result gives us a sufficient conditions for the function r_ν^c to be in the class $\mathcal{UM}(\gamma,b,k) := \mathcal{UM}\left(\frac{z}{1-z},\gamma,b,k\right)$.

Theorem 2.6. *If*

$$\gamma(k+1)(r_\nu^c)''(1)+[k+1+\gamma(|b|-(k+1))]\,(r_\nu^c)'(1)+(|b|-(k+1))(1-\gamma)r_\nu^c(1) < 2|b|, \quad (2.2)$$

then the function $r_\nu^c \in \mathcal{UM}(\gamma,b,k)$.

Proof. To prove our result, since $r_\nu^c(z) = z+\sum_{n=2}^{\infty} a_n z^n$ with $a_n = \dfrac{(-c)^{n-1}(2n-1)\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(\nu+n)}$, according to Lemma 1.2 it is sufficient to show that

$$\sum_{n=2}^{\infty}\left[(k+1)(n-1)+|b|\,\right][1+\gamma(n-1)]\,|a_n| < |b|. \quad (2.3)$$

Using the assumption (2.2) it is easy to prove that

$$\begin{aligned} &\sum_{n=2}^{\infty}\left[(k+1)(n-1)+|b|\right][1+\gamma(n-1)]\,|a_n| = \sum_{n=2}^{\infty} n\,[k+1+\gamma(|b|-(k+1))]\,|a_n| \\ &+\sum_{n=2}^{\infty} n(n-1)\gamma(k+1)|a_n| + \sum_{n=2}^{\infty}(|b|-(k+1))(1-\gamma)|a_n| \\ &= [k+1+\gamma(|b|-(k+1))]\left[(r_\nu^c)'(1)-1\right] + \gamma(k+1)(r_\nu^c)''(1) \\ &+ (|b|-(k+1))(1-\gamma)\left(r_\nu^c(1)-1\right) = \gamma(k+1)(r_\nu^c)''(1) \\ &+ [k+1+\gamma(|b|-(k+1))]\,(r_\nu^c)'(1) + (|b|-(k+1))(1-\gamma)(r_\nu^c)(1) - |b| < |b|, \end{aligned}$$

hence (2.3) holds, and consequently $r_\nu^c \in \mathcal{UM}(\gamma,b,k)$. □

Remark 2.7. We will emphasize a few special cases of the above theorem obtained for different choices of the parameters b, k, and γ.

(i) Letting $b=k=1$ and $\gamma=0$ in Theorem 2.6 we get: if

$$2(r_\nu^c)'(1)-r_\nu^c(1)<2,$$

then the function r_ν^c belongs to the class $\mathcal{UM}(0,1,1)$ of uniformly starlike functions defined and investigated by Goodman [8,9].

(ii) For $b=k=1$ and $\gamma=1$, Theorem 2.6 yields to the next result: if

$$2(r_\nu^c)''(1)+(r_\nu^c)'(1)<2,$$

then the function r_ν^c belongs to the class $\mathcal{UM}(1,1,1)$ of uniformly convex functions defined and studied by Goodman [8,9].

(iii) By taking $b=1-\alpha$ and $\gamma=0$ in Theorem 2.6 we obtain: if

$$(k+1)(r_\nu^c)'(1)-(\alpha+k)r_\nu^c(1)<2(1-\alpha),$$

then the function r_ν^c belongs to the class $\mathcal{UM}(0,1-\alpha,k)$ of k-uniformly starlike functions of order α introduced and investigated by Rønning [14,15].

(iv) Letting $b=1-\alpha$ and $\gamma=1$ in Theorem 2.6 we have: if

$$(k+1)(r_\nu^c)''(1)+(1-\alpha)(r_\nu^c)'(1)<2(1-\alpha),$$

then the function r_ν^c belongs to the class $\mathcal{UM}(1,1-\alpha,k)$ of k-uniformly convex functions of order α introduced and studied by Rönning [14,15].

(v) Taking $b=1$ and $\gamma=0$ in Theorem 2.6 we obtain: if

$$(k+1)(r_\nu^c)'(1)-kr_\nu^c(1)<2,$$

then the function r_ν^c belongs to the class $\mathcal{UM}(0,1,k)$ of k-uniformly starlike functions investigated by Kanas and Wiśniowska [11].

(vi) Letting $b=1$ and $\gamma=1$ in Theorem 2.6 we obtain: if

$$(k+1)(r_\nu^c)''(1)+(r_\nu^c)'(1)<2,$$

then the function r_ν^c is in the class $\mathcal{UM}(1,1,k)$ of k-uniformly convex functions studied by Kanas and Wiśniowska [10].

Theorem 2.8. *If* $\nu\geq\dfrac{9k}{4}-1$, *then the odd function* $\dfrac{R_\nu^k(z^2)}{z}$ *is close-to-convex with respect to the convex function* $\dfrac{1}{2}\log\dfrac{1+z}{1-z}$.

Proof. Since $\dfrac{R_\nu^k(z^2)}{z}=\sum\limits_{n=1}^{\infty}d_{2n-1}z^{2n-1}$, where

$$d_{2n-1}=\frac{k^{n-1}(2n-1)\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(\nu+n)},$$

then $d_{2n-1}>0$ for all $n\geq1$. In view of Lemma 1.4, to prove our result it is sufficient to show that $\{(2n-1)d_{2n-1}\}_{n\geq1}$ is a non-increasing sequence.

First, for $n=2$, the assumption $\nu\geq\dfrac{9k}{4}-1$ implies that

$$3d_3=\frac{9k}{4(\nu+1)}\leq1.$$

If $n \geq 2$, a simple computation shows that

$$(2n-1)d_{2n-1} - (2n+1)d_{2n+1} = \frac{k^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(\nu+n)} \left[(2n-1)^2 - \frac{k(2n+1)^2}{4n(\nu+n)}\right]$$
$$= \frac{k^{n-1}\Gamma(\nu+1)}{4^n n!\Gamma(\nu+n)} \left[4n(2n-1)^2(\nu+n) - k(2n+1)^2\right] = \frac{k^{n-1}\Gamma(\nu+1)}{4^n n!\Gamma(\nu+n)}\varphi(n),$$

where

$$\varphi(n) := 4n(2n-1)^2(\nu+n) - k(2n+1)^2.$$

Thus, it is sufficient to prove $\varphi(n) \geq 0$ for all $n \geq 2$. Using the inequality

$$4n(2n-1)^2 \geq (2n+1)^2,\ n \geq 2,$$

we have

$$\varphi(n) = 4n(2n-1)^2(\nu+n) - k(2n+1)^2 \geq 4n(2n-1)^2(\nu+2) - k(2n+1)^2$$
$$\geq (2n+1)^2(\nu+2) - k(2n+1)^2 = (2n+1)^2(\nu+2-k) \geq (2n+1)^2\left(\frac{5k}{4}+1\right) > 0,$$

whenever $\nu \geq \dfrac{9k}{4} - 1$.

Therefore, $\{(2n-1)c_{2n-1}\}_{n\geq 1}$ is a non-increasing sequence and from Lemma 1.4 our result follows. □

Remark 2.9. Similar results to Theorems 2.1 and 2.8 can be found for the generalized Dini functions $d_{\nu,a,b,c}$ in Theorem 6 (iv) and Theorem 18 in [7].

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