Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (5) (2020), 1777 – 1787
DOI : /10.15672/hujms.535246

Research Article

On fourth Hankel determinant for functions associated with Bernoulli's lemniscate

Muhammad Arif[1], Sadaf Umar[1], Mohsan Raza*[2], Teodor Bulboacă[3], Muhammad Umar Farooq[1], Hasan Khan[1]

[1] *Department of Mathematics, Abdul Wali Khan University, Mardan, Pakistan*
[2] *Department of Mathematics, Government College University, Faisalabad, Faisalabad, Pakistan*
[3] *Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania*

Abstract

The aim of this paper is to find an upper bound of the fourth Hankel determinant $H_4(1)$ for a subclass of analytic functions associated with the right half of the Bernoulli's lemniscate of the form $\left(x^2+y^2\right)^2-2\left(x^2-y^2\right)=0$. The problem is also discussed for 2-fold and 3-fold symmetric functions. The key tools in the proof of our main results are the coefficient inequalities for class $\mathcal{P}$ of functions with positive real part.

Mathematics Subject Classification (2010). 30C45, 30C50

Keywords. starlike functions, differential subordination, Bernoulli's lemniscate, Hankel determinants

1. Introduction

Let $\mathcal{A}$ denote the family of all functions f which are analytic in the open unit disc $\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Therefore, each function $f \in \mathcal{A}$ has a power series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \mathcal{U}. \tag{1.1}$$

Also, let $\mathcal{S}$ denote a subclass of $\mathcal{A}$ which contains the univalent functions.

If f and g are analytic functions in $\mathcal{U}$, then we say that f is subordinate to g, denoted by $f \prec g$, if there exists an analytic function w in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < |z|$ such that $f(z) = g(w(z))$. Moreover if the function g is univalent in $\mathcal{U}$, then we have

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathcal{U}) \subset g(\mathcal{U}).$$

Consider the subclass $\mathcal{SL}$ of $\mathcal{A}$ defined by

$$\mathcal{SL} := \left\{ f \in \mathcal{A} : \left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1,\ z \in \mathcal{U} \right\}.$$

*Corresponding Author.
Email addresses: marifmaths@awkum.edu.pk (M. Arif), sadaf_umar90@yahoo.com (S. Umar), mohsan976@yahoo.com (M. Raza), bulboaca@math.ubbcluj.ro (T. Bulboacă), umar4207590@gmail.com (M.U. Farooq), hassankhan@awkum.edu.pk (H. Khan)
Received: 04.03.2019; Accepted: 13.01.2020

The geometrical interpretation of the fact $f \in \mathcal{SL}$ is that, for any $z \in \mathcal{U}$, the ratio $\dfrac{zf'(z)}{f(z)}$ lies in the region bounded by the right half side of the Bernoulli's lemniscate by the inequality $\left|w^2-1\right| < 1$. We can easily see that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}$, if and only if

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z}, \tag{1.2}$$

where the square root function is considered at principal branch, that is

$$\left.\sqrt{1+z}\right|_{z=0} = 1. \tag{1.3}$$

Remark that the class $\mathcal{SL}$ was introduced by Sokól and Stankiewicz [21], and further studied by different authors in [2, 11, 17–20].

For a function $f \in \mathcal{A}$ of the form (1.1), the q-th Hankel determinant $H_q(n)$, with $q \geq 1$ and $n \geq 1$, was studied by Noonan and Thomas [14] and it is defined by

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \ldots & a_{n+q} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n+q-1} & a_{n+q} & \ldots & a_{n+2q-2} \end{vmatrix}.$$

Remarks 1.1. (i) It is well-known that the Fekete-Szegő functional $\left|a_3-a_2^2\right|$ is $H_2(1)$, and Fekete and Szegö [9] generalized the estimate as $\left|a_3-\lambda a_2^2\right|$ with $\lambda \in \mathbb{R}$ and $f \in \mathcal{S}$.

(ii) Moreover, we also know that the functional $\left|a_2a_4-a_3^2\right|$ is in fact $H_2(2)$.

(iii) The sharp upper bound of the second Hankel determinant for the familiar classes of starlike and convex functions was studied by Janteng, Halim, and Darus [12]. Thus, for $f \in \mathcal{S}^*$ and $f \in \mathcal{C}$ they obtained that $\left|a_2a_4-a_3^2\right| \leq 1$ and $8\left|a_2a_4-a_3^2\right| \leq 1$, respectively. For second Hankel determinant see also [8].

(iv) In 2010, Babalola [5] considered the third Hankel determinant $H_3(1)$ and obtained the upper bound of the well-known classes of bounded-turning, starlike, and convex functions. Later, in 2013 Raza and Malik [16] investigated the upper bound of $H_3(1)$ for the class $\mathcal{SL}$, and they obtained that $|H_3(1)| \leq \dfrac{43}{576}$.

(v) Recently Arif et al. [3, 4] have investigated $H_4(1)$ for some subclasses of univalent functions.

In the present investigation, we determine the upper bound of $H_4(1)$ for the subclass $\mathcal{SL}$ of analytic and normalized functions in $\mathcal{U}$. To prove our main results we need the following definition and lemmas.

We recall the class $\mathcal{P}$ of analytic functions p of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,\ z \in \mathcal{U}, \tag{1.4}$$

with $Re\, p(z) > 0$ in $\mathcal{U}$. The class $\mathcal{P}$ is known as the class of functions with positive real part.

It is well-known (see, for example, [6] or [10, p. 80]) that, if $p \in \mathcal{P}$ and has the form (1.4), then the following sharp coefficient estimates hold:

$$|c_n| \leq 2,\ n \in \mathbb{N}. \tag{1.5}$$

Lemma 1.2. *If $p \in \mathcal{P}$ and has the form* (1.4), *then*

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{\left|c_1^2\right|}{2},$$

where the above inequality is proved in [1].

Lemma 1.3. [7] *If $p \in \mathcal{P}$ and has the form* (1.4), *then*

$$|c_{n+k} - \mu c_n c_k| < 2 \, for \, 0 \leq \mu \leq 1.$$

This result is due to Ravichandran and Verma [15].

Lemma 1.4. *If $p \in \mathcal{P}$ and has the form* (1.4), *then*

$$\left|Jc_1^3 - Kc_1c_2 + Lc_3\right| \leq 2\left(|J| + |K-2J| + |J-K+L|\right).$$

Proof. It is easy to see that

$$\begin{aligned}
&\left|Jc_1^3 - Kc_1c_2 + Lc_3\right| = \left|J\left(c_3 - 2c_1c_2 + c_1^3\right) + (K-2J)\left(c_3 - c_1c_2\right) + (J-K+L)\, c_3\right| \\
&\leq |J|\left|c_3 - 2c_1c_2 + c_1^3\right| + |K-2J|\left|c_3 - c_1c_2\right| + |J-K+L|\left|c_3\right| \\
&\leq 2\left(|J| + |K-2J| + |J-K+L|\right),
\end{aligned}$$

where we have used the Lemma 1.3 for $\mu = 1$, $n = 1$, $k = 2$, and a result due to Libra and Zlotkiewicz [13]. □

Lemma 1.5. [16] *If $f \in \mathcal{SL}$ and has the form* (1.1), *then*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{4}.$$

Lemma 1.6. *If $f \in \mathcal{SL}$ and has the form* (1.1), *then*

$$|a_2| \leq \frac{1}{2}, \quad |a_3| \leq \frac{1}{4}, \quad |a_4| \leq \frac{1}{6}, \quad |a_5| \leq \frac{1}{8}.$$

These estimates are sharp.

The first three bounds were obtained by Sokól [19] and the bound for $|a_5|$ was proved in [15].

Lemma 1.7. *If $f \in \mathcal{SL}$ and has the form* (1.1), *then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{16}.$$

This result was found by Sokól [19].

2. Main results

Theorem 2.1. *If $f \in \mathcal{SL}$ and of the form* (1.1), *then*

$$\left|a_3a_5 - a_4^2\right| \leq 0.080574496.$$

Proof. If $f \in \mathcal{SL}$, by using the subordination relation (1.2), it follows that

$$\frac{zf'(z)}{f(z)} \prec \Phi(z), \tag{2.1}$$

where $\Phi(z) = \sqrt{1+z}$ is considered at principal branch (1.3). From (2.1), there exists a function w, analytic in the unit disk $\mathcal{U}$, with $|w(z)| \leq 1$ in $\mathcal{U}$, such that

$$\frac{zf'(z)}{f(z)} = \Phi(w(z)), \; z \in \mathcal{U}. \tag{2.2}$$

Thus, if we define the function p by

$$p(z) := \frac{1+w(z)}{1-w(z)} = 1 + c_1z + c_2z^2 + \ldots, \; z \in \mathcal{U}, \tag{2.3}$$

it follows that $p \in \mathcal{P}$ and

$$w(z) = \frac{p(z)-1}{p(z)+1}, \; z \in \mathcal{U}.$$

From (2.2) and the above relation we obtain

$$\frac{zf'(z)}{f(z)}=\sqrt{\frac{2p(z)}{p(z)+1}},\ z\in\mathcal{U}. \tag{2.4}$$

Now, according to the power series expansions (1.1) and (1.4), a simple computation shows that

$$\begin{aligned}\sqrt{\frac{2p(z)}{p(z)+1}} &= 1+\frac{1}{4}c_1z+\left(\frac{1}{4}c_2-\frac{5}{32}c_1^2\right)z^2+\left(\frac{1}{4}c_3-\frac{5}{16}c_1c_2+\frac{13}{128}c_1^3\right)z^3\\ &\quad+\left(-\frac{141}{2048}c_1^4+\frac{39}{128}c_1^2c_2-\frac{5}{32}c_2^2+\frac{1}{4}c_4-\frac{5}{16}c_1c_3\right)z^4+\ldots,\end{aligned} \tag{2.5}$$

and

$$\frac{zf'(z)}{f(z)}=1+a_2z+\left(2a_3-a_2^2\right)z^2+\left(3a_4-3a_2a_3+a_2^3\right)z^3+\ldots,\ z\in\mathcal{U}. \tag{2.6}$$

By comparing (2.5) and (2.6) , we have

$$a_2 = \frac{1}{4}c_1, \tag{2.7}$$

$$a_3 = \frac{1}{8}\left(c_2-\frac{3}{8}c_1^2\right), \tag{2.8}$$

$$a_4 = \frac{1}{12}\left(c_3-\frac{7}{8}c_1c_2+\frac{13}{64}c_1^3\right), \tag{2.9}$$

$$a_5 = \left(-\frac{49}{6144}c_1^4+\frac{17}{384}c_1^2c_2-\frac{11}{192}c_1c_3-\frac{1}{32}c_2^2+\frac{1}{16}c_4\right), \tag{2.10}$$

$$\begin{aligned}a_6 &= -\frac{223}{7680}c_1^3c_2+\frac{3}{80}c_1^2c_3+\frac{77}{1920}c_1c_2^2-\frac{3}{64}c_1c_4-\frac{5}{96}c_2c_3\\ &\quad+\frac{181}{40960}c_1^5+\frac{1}{20}c_5,\end{aligned} \tag{2.11}$$

$$\begin{aligned}a_7 &= \frac{323}{4608}c_1c_2c_3-\frac{17}{384}c_2c_4-\frac{19}{480}c_1c_5-\frac{13}{576}c_3^2+\frac{19}{1536}c_2^3+\frac{1}{24}c_6\\ &\quad-\frac{32303}{11796480}c_1^6-\frac{4717}{184320}c_1^3c_3+\frac{33}{1024}c_1^2c_4-\frac{7457}{184320}c_1^2c_2^2+\frac{30211}{1474560}c_1^4c_2.\end{aligned} \tag{2.12}$$

From (2.8) , (2.9), and (2.10) , we obtain

$$\begin{aligned}\left|a_3a_5-a_4^2\right| &= \left|-\frac{89}{147456}c_2c_1^4+\frac{31}{18432}c_1^2c_2^2+\frac{23}{4608}c_2c_1c_3-\frac{1}{256}c_2^3+\frac{1}{128}c_2c_4+\frac{103}{1179648}c_1^6\right.\\ &\quad\left.-\frac{5}{36864}c_1^3c_3-\frac{3}{1024}c_1^2c_4-\frac{1}{144}c_3^2\right|.\end{aligned}$$

Now, re-arranging the above equation, we have

$$\begin{aligned}\left|a_3a_5-a_4^2\right| &= \left|-\frac{103}{589824}c_1^4\left(c_2-\frac{c_1^2}{2}\right)+\frac{27}{16384}c_2\left(\frac{253}{486}c_1^2-c_2\right)\left(c_2-\frac{c_1^2}{2}\right)\right.\\ &\quad-\frac{23}{9216}c_1c_3\left(\frac{5}{92}c_1^2-c_2\right)+\frac{1}{144}c_3\left(\frac{23}{64}c_1c_2-c_3\right)\\ &\quad\left.-\frac{1}{256}c_2\left(\frac{37}{64}c_2^2-c_4\right)-\frac{1}{256}c_4\left(\frac{3}{4}c_1^2-c_2\right)\right|.\end{aligned}$$

Applying the triangle inequality, Lemma 1.2, and Lemma 1.3, we have

$$\left|a_3a_5-a_4^2\right|\le\frac{103}{589824}\left|c_1\right|^4\left(2-\frac{\left|c_1\right|^2}{2}\right)+\frac{27}{4096}\left(2-\frac{\left|c_1\right|^2}{2}\right)+\frac{23}{2304}\left|c_1\right|+\frac{17}{288}. \tag{2.13}$$

Taking $|c_1| = y \in [0,2]$ in (2.13), it gives

$$\left|a_3a_5 - a_4^2\right| \le \frac{103}{589824}y^4\left(2-\frac{y^2}{2}\right)+\frac{27}{4096}\left(2-\frac{y^2}{2}\right)+\frac{23}{2304}y+\frac{17}{288}. \quad (2.14)$$

The above function gets its maximum value at $y = 1.573483035$, in (2.14), we have

$$\left|a_3a_5 - a_4^2\right| \le 0.080574496.$$

□

Theorem 2.2. *If $f \in \mathcal{SL}$ and of the form* (1.1), *then*

$$|a_3a_4 - a_2a_5| \le \frac{173}{532224}\sqrt{39963}+\frac{1}{24} \simeq 0.1066468.$$

Proof. From (2.7), (2.8), (2.9), and (2.10), we have

$$|a_3a_4 - a_2a_5| = \left|-\frac{59}{49152}c_1^5+\frac{17}{3072}c_1^3c_2-\frac{1}{96}c_1^2c_3+\frac{1}{768}c_1c_2^2+\frac{1}{64}c_1c_4-\frac{1}{96}c_2c_3\right|.$$

By re-arrangement of the above equation, we get

$$\begin{aligned}|a_3a_4 - a_2a_5| \quad = \quad & \Big|\frac{77}{12288}c_1\Big(\frac{59}{154}c_1^2-c_2\Big)\Big(c_2-\frac{c_1^2}{2}\Big)+\frac{1}{96}c_2\Big(\frac{93}{128}c_1c_2-c_3\Big)\\ & -\frac{1}{64}c_1\Big(\frac{2}{3}c_1c_3-c_4\Big)\Big|.\end{aligned}$$

Now applying the triangle inequality, Lemma 1.2, and Lemma 1.3, we have

$$|a_3a_4 - a_2a_5| \le \frac{77}{6144}|c_1|\left(2-\frac{|c_1|^2}{2}\right)+\frac{1}{24}+\frac{1}{32}|c_1|. \quad (2.15)$$

Let $|c_1| = y \in [0,2]$, then (2.15), becomes

$$|a_3a_4 - a_2a_5| \le \frac{77}{12288}y\left(2-\frac{y^2}{2}\right)+\frac{1}{24}+\frac{1}{32}y.$$

The above function has its maximum value at $y = \frac{2}{231}\sqrt{39963}$. This implies that

$$|a_3a_4 - a_2a_5| \le \frac{173}{532224}\sqrt{39963}+\frac{1}{24} \simeq 0.1066468.$$

□

Theorem 2.3. *If $f \in \mathcal{SL}$ and of the form* (1.1), *then*

$$|a_5 - a_2a_4| \le \frac{7}{16}.$$

Proof. From (2.7), (2.9), and (2.10), we obtain

$$|a_5 - a_2a_4| = \left|-c_1\left(\frac{25}{2048}c_1^3-\frac{1}{16}c_1c_2+\frac{5}{64}c_3\right)-\frac{1}{16}\left(\frac{c_2^2}{2}-c_4\right)\right|.$$

Now by using the triangle inequality, Lemma 1.3, and Lemma 1.4, we have

$$|a_5 - a_2a_4| \le \frac{7}{16}.$$

□

Theorem 2.4. *If $f \in \mathcal{SL}$ and of the form* (1.1), *then*

$$|a_4 - a_2a_3| \le \frac{1}{6}.$$

This result is sharp for the function $f(z) = z\exp\left(\int_0^z \frac{\sqrt{1+t^3}}{t}dt\right) = z + \frac{1}{6}z^4 - \frac{1}{144}z^7 + \cdots$.

Proof. From (2.7), (2.8), and (2.9), we have

$$|a_4 - a_2a_3| = \left|\frac{11}{384}c_1^3 - \frac{5}{48}c_1c_2 + \frac{1}{12}c_3\right|.$$

Using Lemma 1.4, we obtain

$$|a_4 - a_2a_3| \leq \frac{1}{6}.$$

□

Theorem 2.5. *If $f \in \mathcal{SL}$ and of the form* (1.1), *then*

$$|a_3a_7 - a_4a_6| \leq \frac{125999}{589824}.$$

Proof. From (2.8), (2.9), (2.11), and (2.12), we have

$$\begin{aligned}|a_3a_7 - a_4a_6| \;=\; & \Big|\frac{19}{12288}c_2^4 + \frac{4493}{83886080}c_1^8 - \frac{1}{240}c_3c_5 - \frac{17}{3072}c_2^2c_4 + \frac{7}{4608}c_2c_3^2 \\ & + \frac{1}{192}c_2c_6 - \frac{721}{1474560}c_1^6c_2 - \frac{25}{9216}c_1^2c_2^3 + \frac{9799}{5898240}c_1^4c_2^2 \\ & + \frac{31}{30720}c_1^3c_5 - \frac{127}{61440}c_1^2c_3^2 - \frac{1}{512}c_1^2c_6 + \frac{773}{3932160}c_1^5c_3 - \frac{47}{65536}c_1^4c_4 \\ & + \frac{299}{184320}c_1c_2^2c_3 - \frac{1}{768}c_1c_2c_5 - \frac{331}{737280}c_1^3c_2c_3 + \frac{11}{4096}c_1^2c_2c_4 \\ & + \frac{1}{256}c_1c_3c_4\Big|.\end{aligned}$$

By re-arranging the above equation, we obtain

$$\begin{aligned}|a_3a_7 - a_4a_6| \;=\; & \Big|\frac{241}{92160}c_2c_3\left(\frac{299}{964}c_2c_1 - c_3\right) + \frac{149}{24576}c_2^2\left(\frac{299}{2235}c_3c_1 - c_4\right) \\ & - \frac{149}{49152}c_2^2\left(\frac{4537}{47680}c_2^2 - c_4\right) - \frac{1}{768}c_2\left(c_1c_5 - c_6\right) \\ & + \frac{331}{1474560}c_1^3c_3\left(\frac{2319}{2648}c_1^2 - c_2\right) - \frac{31}{30720}c_1^3\left(\frac{331}{1488}c_2c_3 - c_5\right) \\ & - \frac{144139}{188743680}c_1^4\left(\frac{40437}{288278}c_1^2 - c_2\right)\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{240}c_3\left(\frac{15}{16}c_1c_4 - c_5\right) \\ & - \frac{9619}{5242880}c_2^2\left(\frac{169429}{173142}c_1^2 - c_2\right)\left(c_2 - \frac{c_1^2}{2}\right) + \frac{127}{30720}c_3^2\left(c_2 - \frac{c_1^2}{2}\right) \\ & + \frac{41}{16384}c_4\left(\frac{47}{82}c_1^2 - c_2\right)\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{256}c_6\left(c_2 - \frac{c_1^2}{2}\right)\Big|.\end{aligned}$$

Applying the triangle inequality, Lemma 1.2, and Lemma 1.3, the above equation becomes

$$\begin{aligned}|a_3a_7 - a_4a_6| \;\leq\; & \frac{144139}{94371840}|c_1|^4\left(2 - \frac{|c_1|^2}{2}\right) + \frac{96409}{1966080}\left(2 - \frac{|c_1|^2}{2}\right) \\ & + \frac{215}{73728}|c_1|^3 + \frac{10649}{92160}. \qquad (2.16)\end{aligned}$$

Let $|c_1| = y \in [0, 2]$, then (2.16) becomes

$$|a_3a_7 - a_4a_6| \leq \frac{144139}{94371840}y^4\left(2 - \frac{y^2}{2}\right) + \frac{96409}{1966080}\left(2 - \frac{y^2}{2}\right) + \frac{215}{73728}y^3 + \frac{10649}{92160}.$$

Clearly, the above function is decreasing so by putting $y = 2$, we have

$$|a_3a_7 - a_4a_6| \leq \frac{125999}{589824}.$$

□

Theorem 2.6. *If $f \in \mathcal{SL}$ and of the form (1.1), then*

$$|a_4a_7 - a_5a_6| \leq 0.2210481986.$$

Proof. From (2.9), (2.10), (2.11), and (2.12), it follows that

$$\begin{aligned}|a_4a_7 - a_5a_6| &= \Big| -\frac{1}{2304}c_1c_3c_5 + \frac{83}{18432}c_1c_2c_3^2 - \frac{1}{2304}c_2c_3c_4 - \frac{7}{2304}c_1c_2c_6 \\ &+ \frac{583}{737280}c_1^3c_2c_4 + \frac{669}{655360}c_3c_1^4c_2 + \frac{1}{640}c_2^2c_5 - \frac{11}{18432}c_3c_2^3 \\ &- \frac{499}{184320}c_1^2c_3c_2^2 - \frac{137}{184320}c_1c_2^2c_4 - \frac{3}{1280}c_1^2c_3c_4 + \frac{31}{46080}c_1^2c_2c_5 \\ &+ \frac{3}{1024}c_1c_4^2 - \frac{20131}{1811939328}c_1^9 - \frac{1}{320}c_4c_5 - \frac{137}{1310720}c_1^5c_4 + \frac{259}{737280}c_1c_2^4 \\ &- \frac{13}{6912}c_3^3 - \frac{5}{18432}c_1^4c_5 + \frac{13}{18432}c_1^3c_6 + \frac{527}{1105920}c_1^3c_2^3 - \frac{10271}{23592960}c_1^5c_2^2 \\ &+ \frac{439633}{4529848320}c_1^7c_2 - \frac{3}{8192}c_1^3c_3^2 - \frac{515}{4718592}c_1^6c_3 + \frac{1}{288}c_3c_6 \Big|.\end{aligned}$$

This implies that

$$\begin{aligned}|a_4a_7 - a_5a_6| &= \Big| \frac{18934}{11796480}c_1^2c_3\left(\frac{2575}{18934}c_1^2 - c_2\right)\left(c_2 - \frac{c_1^2}{2}\right) \\ &+ \frac{1}{3840}c_2\left(\frac{2167}{256}c_3c_2 - c_5\right)\left(c_2 - \frac{c_1^2}{2}\right) + \frac{3}{1024}c_1c_4^2 \\ &+ \frac{3}{4096}c_1c_3^2\left(c_2 - \frac{c_1^2}{2}\right) + \frac{13}{9216}c_1\left(\frac{5}{13}c_1c_5 - c_6\right)\left(c_2 - \frac{c_1^2}{2}\right) \\ &+ \frac{323519}{566231040}c_1^3c_2^2\left(c_2 - \frac{c_1^2}{2}\right) + \frac{10739}{56623104}c_1c_2^3\left(c_2 - \frac{c_1^2}{2}\right) \\ &+ \frac{3431}{2949120}c_1c_4\left(\frac{1233}{6862}c_1^2 - c_2\right)\left(c_2 - \frac{c_1^2}{2}\right) \\ &+ \frac{169489}{1132462080}c_1^5\left(\frac{100655}{677956}c_1^2 - c_2\right)\left(c_2 - \frac{c_1^2}{2}\right) + \frac{3}{640}c_3c_4\left(c_2 - \frac{c_1^2}{2}\right) \\ &- \frac{1}{288}c_3\left(\frac{13}{24}c_3^2 - c_6\right) + \frac{793056}{283115520}c_2^3\left(\frac{45761}{793056}c_1c_2 - c_3\right) \\ &+ \frac{59}{11520}c_2c_3\left(\frac{695}{944}c_1c_3 - c_4\right) + \frac{1}{320}c_5\left(\frac{7}{12}c_2^2 - c_4\right) \\ &+ \frac{5}{3072}c_1c_2\left(\frac{413}{1600}c_2c_4 - c_6\right) - \frac{1}{2304}c_3c_1c_5 \Big|.\end{aligned}$$

Using the triangle inequality, Lemma 1.2, and 1.3, we have

$$\begin{aligned}|a_4a_7 - a_5a_6| &\leq \frac{9467}{1474560}|c_1|^2\left(2 - \frac{|c_1|^2}{2}\right) + \frac{15109}{368640}\left(2 - \frac{|c_1|^2}{2}\right) \\ &+ \frac{322063}{35389440}|c_1|\left(2 - \frac{|c_1|^2}{2}\right) + \frac{323519}{141557760}|c_1|^3\left(2 - \frac{|c_1|^2}{2}\right) \\ &+ \frac{169489}{566231040}|c_1|^5\left(2 - \frac{|c_1|^2}{2}\right) + \frac{23}{1152}|c_1| + \frac{20677}{184320}. \qquad (2.17)\end{aligned}$$

Let $|c_1| = y \in [0,2]$, then (2.17) becomes

$$\begin{aligned} |a_4a_7 - a_5a_6| \leq & \frac{9467}{1474560}y^2\left(2-\frac{y^2}{2}\right) + \frac{15109}{368640}\left(2-\frac{y^2}{2}\right) + \frac{322063}{35389440}y\left(2-\frac{y^2}{2}\right) \\ & + \frac{323519}{141557760}y^3\left(2-\frac{y^2}{2}\right) + \frac{169489}{566231040}y^5\left(2-\frac{y^2}{2}\right) + \frac{23}{1152}y + \frac{20677}{184320}. \end{aligned}$$

As the above function attains its maximum value at $y = 1.082047787$, so the above equation becomes

$$|a_4a_7 - a_5a_6| \leq 0.2210481986.$$

□

Theorem 2.7. *If $f \in \mathcal{SL}$ and of the form* (1.1)*, then*

$$|H_3(1)| \leq \frac{43}{576}.$$

Proof. Since

$$\begin{aligned} |H_3(1)| &= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \\ &= |a_3|\left|a_2a_4 - a_3^2\right| + |a_4|\,|a_4 - a_2a_3| + |a_5|\left|a_3 - a_2^2\right|. \end{aligned}$$

Using Lemma 1.6, Lemma 1.5, and Lemma 1.7, we get

$$|H_3(1)| \leq \frac{43}{576}.$$

□

Theorem 2.8. *If $f \in \mathcal{SL}$ and of the form* (1.1)*, then*

$$|H_4(1)| \leq 0.06786551485.$$

Proof. Since

$$\begin{aligned} |H_4(1)| \leq & \left|a_2a_4 - a_3^2\right|\,|a_3a_7 - a_4a_6| + |a_2a_3 - a_4|\,|a_4a_7 - a_5a_6| \\ & + |a_5|\left\{|a_3|\left|a_3a_5 - a_4^2\right| + |a_5|\,|a_5 - a_2a_4| + |a_6|\,|a_4 - a_2a_3|\right\} \\ & + |a_4|\left\{|a_4|\left|a_3a_5 - a_4^2\right| + |a_5|\,|a_2a_5 - a_3a_4| + |a_6|\left|a_2a_4 - a_3^2\right|\right\}. \end{aligned}$$

Using Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4, Theorem 2.7, Theorem 2.5, Theorem 2.6, and Lemma 1.6, we have

$$|H_4(1)| \leq 0.06786551485.$$

□

3. Bounds of $|H_{4,1}(f)|$ for the sets $\mathcal{SL}^{(2)}$ and $\mathcal{SL}^{(3)}$

Let $m \in \mathbb{N} = \{1, 2, \ldots\}$. A domain Λ is said to be m-fold symmetric if a rotation of Λ about the origin through an angle $2\pi/m$ carries Λ on itself. It is easy to see that, an analytic function f is m-fold symmetric in $\mathcal{U}$, if

$$f\left(e^{2\pi i/m}z\right) = e^{2\pi i/m}f(z), \ (z \in \mathcal{U}).$$

By $\mathcal{S}^{(m)}$, we mean the set of m-fold univalent functions having the following Taylor series form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1}, \ (z \in \mathcal{U}). \tag{3.1}$$

The sub-family $\mathcal{SL}^{(m)}$ of $\mathcal{S}^{(m)}$ is the set of m-fold symmetric starlike functions associated with lemniscate of Bernouli. More intuitively, an analytic function f of the form (3.1) belongs to the family $\mathcal{SL}^{(m)}$, if and only if

$$\frac{zf'(z)}{f(z)} = \sqrt{\frac{2p(z)}{p(z)+1}} \text{ with } p \in \mathcal{P}^{(m)},$$

where the set $\mathcal{P}^{(m)}$ is defined by

$$\mathcal{P}^{(m)} = \left\{ p \in \mathcal{P} : p\,(z) = 1 + \sum_{k=1}^{\infty} c_{mk} z^{mk}, \;\; (z \in \mathcal{U}) \right\}. \tag{3.2}$$

Now we can prove the following theorem.

Theorem 3.1. *Let* $f \in \mathcal{SL}^{(2)}$ *be of the form* (3.1). *Then*

$$|H_{4,1}\,(f)| \leq \frac{13}{3072}.$$

Proof. Since $f \in \mathcal{SL}^{(2)}$, therefore there exists a function $p \in \mathcal{P}^{(2)}$ such that

$$\frac{zf'(z)}{f(z)} = \sqrt{\frac{2p(z)}{p(z)+1}}.$$

For $f \in \mathcal{SL}^{(2)}$, using the series form (3.1) and (3.2) when $m = 2$, we can write

$$a_3 = \frac{1}{8}c_2, \;\; a_5 = -\frac{1}{32}c_2^2 + \frac{1}{16}c_4, \;\; a_7 = \frac{19}{1536}c_2^3 - \frac{17}{384}c_4c_2 + \frac{1}{24}c_6.$$

It is clear that for $f \in \mathcal{SL}^{(2)}$,

$$H_{4,1}\,(f) := a_3a_5a_7 - a_3^3a_7 + a_3^2a_5^2 - a_5^3.$$

Therefore

$$H_{4,1}\,(f) = -\frac{4}{786432}\Big(\frac{1}{4}c_2^2 - c_4\Big)\Big(20\left(\frac{7}{20}c_2^2 - c_4\right)c_2^2 + \Big(16c_2c_6 + 48\left(c_2c_6 - c_4^2\right)\Big)\Big).$$

Using Lemma 1.3 and the triangle inequality, we get

$$|H_{4,1}\,(f)| \leq \frac{8}{786432}\,(160 + 64 + 192) = \frac{13}{3072}.$$

Hence the proof is complete. □

Theorem 3.2. *If* $f \in \mathcal{SL}^{(3)}$ *be of the form* (3.1) , *then*

$$|H_{4,1}\,(f)| \leq \frac{8}{3456}.$$

Proof. Since $f \in \mathcal{SL}^{(3)}$, therefore there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$\frac{zf'(z)}{f(z)} = \sqrt{\frac{2p(z)}{p(z)+1}}.$$

For $f \in \mathcal{SL}^{(3)}$, using the series form (3.1) and (3.2) when $m = 3$, we can write

$$a_4 = \frac{1}{12}c_3, \;\; a_7 = -\frac{13}{576}c_3^2 + \frac{1}{24}c_6.$$

It is clear that for $f \in \mathcal{SL}^{(3)}$,

$$H_{4,1}\,(f) := -a_4^2a_7 + a_4^4.$$

Therefore

$$\begin{aligned} H_{4,1}(f) &= \frac{17}{82944}c_3^4 - \frac{1}{3456}c_3^2 c_6 \\ &= -\frac{c_3^2}{3456}\left(c_6 - \frac{58752}{82944}c_3^2\right). \end{aligned}$$

Using Lemma 1.3 and triangle inequality, we get

$$|H_{4,1}(f)| \leq \frac{8}{3456}.$$

Hence the proof is complete. □

References

[1] R.M. Ali, *Coefficients of the inverse of strongly starlike functions*, Bull. Malays. Math. Sci. Soc. **26** (1), 63–71, 2003.

[2] R.M. Ali, N.E. Cho, V. Ravichandran and S.S. Kumar, *First order differential subordination for functions associated with the lemniscate of Bernoulli*, Taiwanese J. Math. **16** (3), 1017–1026, 2012.

[3] M. Arif, L. Rani, M. Raza and P. Zaprawa, *Fourth Hankel determinant for a family of functions with bounded turning*, Bull. Korean Math. Soc. **55** (6), 1703-1711, 2018.

[4] M. Arif, L. Rani, M. Raza and P. Zaprawa, *Fourth Hankel determinant for a set of starlike function*, submitted.

[5] K.O. Babalola, *On $H_3(1)$ Hankel determinant for some classes of univalent functions*, Inequal. Theory Appl. **6**, 1–7, 2007.

[6] C. Carathéodory, *Über den variabilitätsbereich der koeffizienten von potenzreihen die gegebene werte nicht annehmen*, Math. Ann. **64**, 95–115, 1907.

[7] C. Carathéodory, *Über den variabilitätsbereich der fourier'schen konstanten von positiven harmonischen funktionen*, Rend. Circ. Mat. Palermo, **32**, 193-127, 1911.

[8] E. Deniz and L. Budak, *Second Hankel determinat for certain analytic functions satisfying subordinate condition*, Math. Slovaca, **68** (2), 463–471, 2018.

[9] M. Fekete and G. Szegö, *Eine bemerkung uber ungerade schlichte funktionen*, J. London Math. Soc. **8**, 85–89, 1933.

[10] A.W. Goodman, Univalent Functions, Mariner Publications, Tampa, FLorida, 1983.

[11] S.A. Halim and R. Omar, *Applications of certain functions associated with lemniscate Bernoulli*, J. Indones. Math. Soc. **18** (2), 93–99, 2012.

[12] A. Janteng, S.A. Halim and M. Darus, *Hankel determinant for starlike and convex functions*, Int. J. Math. Anal. **1** (13), 619–625, 2007.

[13] R.J. Libera and E.J. Zlotkiewicz, *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc. **85**, 225-230, 1982.

[14] J.W. Noonan and D.K. Thomas, *On second Hankel determinant of a really mean p-valent functions*, Trans. Amer. Math. Soc. **223** (2), 337–346, 1976.

[15] V. Ravichandran and S. Verma, *Bound for the fifth coefficient of certain starlike functions*, C. R. Math. Acad. Sci. Paris, **353** (6), 505–510, 2015.

[16] M. Raza and S.N. Malik, *Upper bound of the third Hankel determinant for a class of analytic functions related with with the lemniscate of Bernoulli*, J. Inequal. Appl. **(2013)**, art. 412, 2013.

[17] J. Sokól, *On application of certain sufficient condition for starlikeness*, J. Math. Appl. **30**, 40–53, 2008.

[18] J. Sokól, *Radius problem in the class $\mathcal{SL}^*$*, Appl. Math. Comput. **214**, 569–573, 2009.

[19] J. Sokól, *Coefficient estimates in a class of strongly starlike functions*, Kyungpook Math. J. **49**, 349–353, 2009.

[20] J. Sokól and D.K. Thomas, *Further results on a class of starlike functions related to the Bernoulli lemniscate*, Houston J. Math. **44**, 83–95, 2018.

[21] J. Sokól and J. Stankiewicz, *Radius of convexity of some subclasses of strongly starlike functions*, Folia Scient. Univ. Tech. Resoviensis, **147**, 101–105, 1996.