MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

https://doi.org/10.36753/mathenot.710119
9 (3) 124-132 (2020) - Research Article
ISSN: 2147-6268

Geodesics of Twisted-Sasaki Metric

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Abstract
The main purpose of the paper is to investigate geodesics on the tangent bundle with respect to the twisted-Sasaki metric. We establish a necessary and sufficient conditions under which a curve be a geodesic respect. Afterward, we also construct some examples of geodesics.

Keywords: *Tangent bundle, Horizontal lift, Vertical lift, Twisted-Sasaki metric, Geodesics.*

AMS Subject Classification (2020): *Primary: 53C22; 53C25 ; Secondary: 53A45; 53C20; 58E20.*

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1. Introduction

The geometry of the tangent bundle TM equipped with Sasaki metric has been studied by many authors such as Sasaki, S. [18], Yano, K. and Ishihara, S. [20], Dombrowski, p. [6], Salimov, A., Gezer, A., and Cengiz, N. [2, 7, 14–16]. The rigidity of Sasaki metric has incited some geometers to construct and study other metrics on TM. Musso, E. and Tricerri, F. have introduced the notion of Cheeger-Gromoll metric [13], Jian, W. and Yong, W. have introduced the notion of Rescaled Metric [9], Zagane, A. and Djaa, M. have introduced the notion of Mus-Sasaki metric [12, 21, 22].

The main idea in this note consists in the modification of the Sasaki metric. First we introduce a new metric called twisted-Sasaki metric on the tangent bundle TM. This new natural metric will lead us to interesting results. Afterward we establish a necessary and sufficient conditions under which a curve be a geodesic with respect to the twisted-Sasaki metric.

2. Preliminaries

Let (M^m, g) be an m-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=\overline{1,m}}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$

Received : *27-03-2020,* ***Accepted :*** *24-12-2020*

defined by :

$$\mathcal{V}_{(x,u)} = Ker(d\pi_{(x,u)}) = \{a^i \frac{\partial}{\partial y^i}|_{(x,u)}; \quad a^i \in \mathbb{R}\},$$
$$\mathcal{H}_{(x,u)} = \{a^i \frac{\partial}{\partial x^i}|_{(x,u)} - a^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}|_{(x,u)}; \quad a^i \in \mathbb{R}\},$$

where $(x,u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i}, \tag{2.1}$$
$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \{\frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}\}. \tag{2.2}$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=\overline{1,m}}$ is a local adapted frame on TTM.

If $w = w^i \frac{\partial}{\partial x^i} + \overline{w}^j \frac{\partial}{\partial x^j} \in T_{(x,u)}TM$, then its horizontal and vertical parts are defined by

$$w^h = w^i \frac{\partial}{\partial x^i} - w^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \in \mathcal{H}_{(x,u)}, \tag{2.3}$$

$$w^v = (\overline{w}^k + w^i u^j \Gamma_{ij}^k) \frac{\partial}{\partial y^k} \in \mathcal{V}_{(x,u)}. \tag{2.4}$$

Lemma 2.1. *[20] Let (M,g) be a Riemannian manifold, ∇ be the Levi-Civita connection and R its tensor curvature, then for all vector fields $X, Y \in \Gamma(TM)$, we have following relations*

1. $[X^H, Y^H]_p = [X,Y]_p^H - (R_x(X,Y)u)^V$,
2. $[X^H, Y^V]_p = (\nabla_X Y)_p^V$,
3. $[X^V, Y^V]_p = 0$,

where $p = (x,u) \in TM$.

3. Twisted-Sasaki metric

3.1 Twisted-Sasaki metric

Definition 3.1. Let (M,g) be a Riemannian manifold and $f : M \to [0, +\infty[$ be a positive smooth function on M. On the tangent bundle TM, we define a twisted-Sasaki metric noted g^f by

1 $g^f(X^H, Y^H)_{(x,u)} = g_x(X,Y)$,

2 $g^f(X^H, Y^V)_{(x,u)} = 0$,

3 $g^f(X^V, Y^V)_{(x,u)} = g_x(X,Y) + f(x) g_x(X,u) g_x(Y,u)$,

where $X, Y \in \Gamma(TM)$, $(x,u) \in TM$, f is called twisting function.

Remark 3.1. **1** If $f = 0$ g^f is the Sasaki metric [20],

2 $g^f(X^V, U^V) = \alpha g(X,u)$, $\alpha = 1 + fr^2$ and $r^2 = g(u,u)$,
where $X, U \in \Gamma(TM)$, $U_x = u \in T_xM$ and $(x,u) \in TM$.

In the following, we consider $f \neq 0$, $\alpha = 1 + fr^2$ and $r^2 = g(u,u) = \|u\|^2$ where $\|.\|$ denote the norm with respect to (M,g).

Lemma 3.1. *Let (M,g) be a Riemannian manifold and $\rho : \mathbb{R} \to \mathbb{R}$ a smooth function. For all $X, Y \in \Gamma(TM)$, $p = (x,u) \in TM$ and $u \in T_xM$, we have following relations*

1. $X^H(\rho(r^2))_p = 0$,

2. $X^V(\rho(r^2))_p = 2\rho'(r^2)g(X,u)_x$,

3. $X^H(g(Y,u))_p = g(\nabla_X Y, u)_x$,

4. $X^V(g(Y,u)_p = g(X,Y)_x$.

Proof. Locally, if $U : x \in M \to U_x = u = u^i \frac{\partial}{\partial x^i} \in T_x M$ be a local vector field constant on each fiber $T_x M$, then we have

$$\begin{aligned}
1.\quad X^H(\rho(r^2))_p &= \big[X^i \frac{\partial}{\partial x^i}(\rho(r^2)) - \Gamma^k_{ij} X^i y^j \frac{\partial}{\partial y^k}(\rho(r^2))\big]_p \\
&= \big[X^i \rho'(r^2) \frac{\partial}{\partial x^i}(r^2) - \rho'(r^2)\Gamma^k_{ij} X^i y^j \frac{\partial}{\partial y^k}(r^2)\big]_p \\
&= \rho'(r^2)\big[X^i \frac{\partial}{\partial x^i} g_{st} y^s y^t - \Gamma^k_{ij} X^i y^j \frac{\partial}{\partial y^k} g_{st} y^s y^t\big]_p \\
&= \rho'(r^2)\big[X g(U,U)_x - 2(\Gamma^k_{ij} X^i y^j g_{sk} y^s)_p\big] \\
&= \rho'(r^2)[X g(U,U)_x - 2g(U, \nabla_X U)_x] \\
&= 0. \\
2.\quad X^V(\rho(r^2))_p &= [X^i \rho'(r^2) \frac{\partial}{\partial y^i} g_{st} y^s y^t]_p \\
&= 2\rho'(r^2) X^i g_{it} u^t \\
&= 2\rho'(r^2) g(X,u)_x.
\end{aligned}$$

The other formulas are obtained by a similar calculation. □

Lemma 3.2. *Let (M,g) be a Riemannian manifold, we have the following*

$$\begin{aligned}
1)\ X^H g^f(Y^H, Z^H) &= X g(Y,Z), \\
2)\ X^V g^f(Y^H, Z^H) &= 0, \\
3)\ X^H g^f(Y^V, Z^V) &= g^f((\nabla_X Y)^V, Z^V) + g^f(Y^V, (\nabla_X Z)^V) + X(f) g(Y,u) g(Z,u), \\
4)\ X^V g^f(Y^H, Z^H) &= f\big[g(X,Y)g(Z,u) + g(Y,u)g(X,Z)\big],
\end{aligned}$$

where $X, Y, Z \in \Gamma(TM)$.

Proof. Lemma 3.2 follows from Definition 3.1 and Lemma 3.1. □

3.2 The Levi-Civita connection

We shall calculate the Levi-Civita connection ∇^f of TM with twisted-Sasaki metric g^f. This connection is characterized by the Koszul formula

$$\begin{aligned}
2g^f(\nabla^f_{\widetilde{X}} \widetilde{Y}, \widetilde{Z}) &= \widetilde{X} g^f(\widetilde{Y}, \widetilde{Z}) + \widetilde{Y} g^f(\widetilde{Z}, \widetilde{X}) - \widetilde{Z} g^f(\widetilde{X}, \widetilde{Y}) \\
&\quad + g^f(\widetilde{Z}, [\widetilde{X}, \widetilde{Y}]) + g^f(\widetilde{Y}, [\widetilde{Z}, \widetilde{X}]) - g^f(\widetilde{X}, [\widetilde{Y}, \widetilde{Z}]). \qquad (3.1)
\end{aligned}$$

for all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Gamma(TM)$.

Lemma 3.3. *Let (M,g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with the twisted-Sasaki metric.*

If ∇ (resp ∇^f) denotes the Levi-Civita connection of (M,g) (resp (TM,g^f)), then we have following relations

$$\begin{aligned}
&1)\ g^f(\nabla^f_{X^H}Y^H,Z^H) = g^f\big((\nabla_X Y)^H,Z^H\big),\\
&2)\ g^f(\nabla^f_{X^H}Y^H,Z^V) = -\frac{1}{2}g^f\big((R(X,Y)u)^V,Z^V\big),\\
&3)\ g^f(\nabla^f_{X^H}Y^V,Z^H) = \frac{1}{2}g^f\big((R(u,Y)X)^H,Z^H\big),\\
&4)\ g^f(\nabla^f_{X^H}Y^V,Z^V) = g^f\big((\nabla_X Y)^V,Z^V\big)+\frac{1}{2\alpha}X(f)g(Y,u)g^f(U^V,Z^V),\\
&5)\ g^f(\nabla^f_{X^V}Y^H,Z^H) = \frac{1}{2}g^f\big((R(u,X)Y)^H,Z^H\big),\\
&6)\ g^f(\nabla^f_{X^V}Y^H,Z^V) = \frac{1}{2\alpha}Y(f)g(X,u)g^f(U^V,Z^V),\\
&7)\ g^f(\nabla^f_{X^V}Y^V,Z^H) = \frac{-1}{2}g(X,u)g(Y,u)g^f((grad\, f)^H,Z^H),\\
&8)\ g^f(\nabla^f_{X^V}Y^V,Z^V) = \frac{f}{\alpha}g(X,Y)g^f(U^V,Z^V),
\end{aligned}$$

for all vector fields $X,Y,U\in\Gamma(TM)$, $U_x=u\in T_xM$ and $(x,u)\in TM$, where R denotes the curvature tensor of (M,g).

Proof. The proof of Lemma 3.3 follows directly from Kozul formula (3.1), Lemma 2.1, Definition 3.1 and Lemma 3.2.
1) The statement is obtained as follows

$$\begin{aligned}
2g^f(\nabla^f_{X^H}Y^H,Z^H) =& X^Hg^f(Y^H,Z^H)+Y^Hg^f(Z^H,X^H)-Z^Hg^f(X^H,Y^H)\\
&+g^f(Z^H,[X^H,Y^H])+g^f(Y^H,[Z^H,X^H])-g^f(X^H,[Y^H,Z^H])\\
=& Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)+g^f(Z^H,[X,Y]^H)\\
&+g^f(Y^H,[Z,X]^H)-g^f(X^H,[Y,Z]^H)\\
=& Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)+g(Z,[X,Y])\\
&+g(Y,[Z,X])-g(X,[Y,Z])\\
=& 2g(\nabla_X Y,Z)\\
=& 2g^f((\nabla_X Y)^H,Z^H).
\end{aligned}$$

2) Direct calculations give

$$\begin{aligned}
2g^f(\nabla^f_{X^H}Y^H,Z^V) =& X^Hg^f(Y^H,Z^V)+Y^Hg^f(Z^V,X^H)-Z^Vg^f(X^H,Y^H)\\
&+g^f(Z^V,[X^H,Y^H])+g^f(Y^H,[Z^V,X^H])-g^f(X^H,[Y^H,Z^V])\\
=& g^f(Z^V,[X^H,Y^H])\\
=& -g^f((R(X,Y)u)^V,Z^V).
\end{aligned}$$

The other formulas are obtained by a similar calculation. □

As a direct consequence of Lemma 3.3, we get the following theorem.

Theorem 3.1. *Let (M,g) be a Riemannian manifold and (TM,g^f) its tangent bundle equipped with the twisted-Sasaki metric. If ∇ (resp ∇^f) denotes the Levi-Civita connection of (M,g) (resp (TM,g^f)), then we have:*

$$\begin{aligned}
1.\ (\nabla^f_{X^H}Y^H)_p &= (\nabla_X Y)^H_p-\frac{1}{2}(R_x(X,Y)u)^V,\\
2.\ (\nabla^f_{X^H}Y^V)_p &= (\nabla_X Y)^V_p+\frac{1}{2\alpha}X_x(f)g_x(Y,u)U^V_p+\frac{1}{2}(R_x(u,Y)X)^H,\\
3.\ (\nabla^f_{X^V}Y^H)_p &= \frac{1}{2\alpha}Y_x(f)g_x(X,u)U^V_p+\frac{1}{2}(R_x(u,X)Y)^H,\\
4.\ (\nabla^f_{X^V}Y^V)_p &= \frac{-1}{2}g_x(X,u)g_x(Y,u)(grad\, f)^H_p+\frac{f}{\alpha}g_x(X,Y)U^V_p,
\end{aligned}$$

for all vector fields $X,Y,U\in\Gamma(TM)$, $U_x=u\in T_xM$ and $p=(x,u)\in TM$, where R denotes the curvature tensor of (M,g).

4. Geodesics of twisted-Sasaki metric.

Lemma 4.1. *Let (M,g) be a Riemannian manifold. If $X,Y\in\Gamma(TM)$ are vector fields on M and $(x,u)\in TM$ such that $Y_x=u$, then we have*

$$d_xY(X_x)=X^H_{(x,u)}+(\nabla_XY)^V_{(x,u)}.$$

Proof. Let (U,x^i) be a local chart on M in $x\in M$ and $\pi^{-1}(U),x^i,y^j)$ be the induced chart on TM, if $X_x=X^i(x)\frac{\partial}{\partial x^i}|_x$ and $Y_x=Y^i(x)\frac{\partial}{\partial x^i}|_x=u$, then

$$d_xY(X_x)=X^i(x)\frac{\partial}{\partial x^i}|_{(x,u)}+X^i(x)\frac{\partial Y^k}{\partial x^i}(x)\frac{\partial}{\partial y^k}|_{(x,u)}.$$

Thus the horizontal part is given by:

$$\begin{aligned}(d_xY(X_x))^h &= X^i(x)\frac{\partial}{\partial x^i}|_{(x,u)}-X^i(x)Y^j(x)\Gamma^k_{ij}(x)\frac{\partial}{\partial y^k}|_{(x,u)}\\ &= X^H_{(x,u)},\end{aligned}$$

and the vertical part is given by:

$$\begin{aligned}(d_xY(X_x))^v &= \{X^i(x)\frac{\partial Y^k}{\partial x^i}(x)+X^i(x)Y^j(x)\Gamma^k_{ij}(x)\}\frac{\partial}{\partial y^k}|_{(x,u)}\\ &= (\nabla_XY)^V_{(x,u)}.\end{aligned}$$

□

Let (M,g) be a Riemannian manifold and $x:I\to M$ be a curve on M. We define a curve $C:I\to TM$ by for all $t\in I, C(t)=(x(t),y(t))$ where $y(t)\in T_{x(t)}M$ i.e. $y(t)$ is a vector field along $x(t)$.

Definition 4.1. ([17, 20]) Let (M,g) be a Riemannian manifold. If $x(t)$ is a curve on M, the curve $C(t)=(x(t),\dot{x}(t))$ is called the natural lift of curve $x(t)$.

Definition 4.2. ([20]) Let (M,g) be a Riemannian manifold and ∇ denotes the Levi-Civita connection of (M,g). A curve $C(t)=(x(t),y(t))$ is said to be a horizontal lift of the cure $x(t)$ if and only if $\nabla_{\dot{x}}y=0$.

Lemma 4.2. *Let (M,g) be a Riemannian manifold and ∇ denotes the Levi-Civita connection of (M,g). If $x(t)$ be a curve on M and $C(t)=(x(t),y(t))$ be a curve on TM, then*

$$\dot{C}=\dot{x}^H+(\nabla_{\dot{x}}y)^V. \tag{4.1}$$

Proof. Locally, if $Y\in\Gamma(TM)$ is a vector field such $Y(x(t))=y(t)$, then we have

$$\dot{C}(t)=dC(t)=dY(x(t)).$$

Using Lemma 4.1, we obtain

$$\dot{C}(t)=dY(x(t))=\dot{x}^H+(\nabla_{\dot{x}}y)^V.$$

□

Theorem 4.1. *Let (M,g) be a Riemannian manifold and (TM,g^f) its tangent bundle equipped with the twisted-Sasaki metric. If ∇ (resp. ∇^f) denotes the Levi-Civita connection of (M,g) (resp. (TM,g^f)) and $C(t)=(x(t),y(t))$ is the cure on TM such $y(t)$ is a vector field along $x(t)$, then*

$$\begin{aligned}\nabla^f_{\dot{C}}\dot{C} &= (\nabla_{\dot{x}}\dot{x})^H+(R(y,\nabla_{\dot{x}}y)\dot{x})^H-\frac{1}{2}g(\nabla_{\dot{x}}y,y)^2(grad\, f)^H\\ &\quad +(\nabla_{\dot{x}}\nabla_{\dot{x}}y)^V+\frac{1}{\alpha}\big[\dot{x}(f)g(\nabla_{\dot{x}}y,y)+f\|\nabla_{\dot{x}}y\|^2\big]y^V.\end{aligned} \tag{4.2}$$

Proof. Using Lemma 4.2, we obtain

$$\begin{aligned}
\nabla^f_{\dot{C}}\dot{C} &= \nabla^f_{[\dot{x}^H+(\nabla_{\dot{x}}y)^V]}[\dot{x}^H+(\nabla_{\dot{x}}y)^V]\\
&= \nabla^f_{\dot{x}^H}\dot{x}^H+\nabla^f_{\dot{x}^H}(\nabla_{\dot{x}}y)^V+\nabla^f_{(\nabla_{\dot{x}}y)^V}\dot{x}^H+\nabla^f_{(\nabla_{\dot{x}}y)^V}(\nabla_{\dot{x}}y)^V\\
&= (\nabla_{\dot{x}}\dot{x})^H-\frac{1}{2}(R(\dot{x},\dot{x})y)^V+(\nabla_{\dot{x}}\nabla_{\dot{x}}y)^V+\frac{1}{2\alpha}\dot{x}(f)g(\nabla_{\dot{x}}y,y)y^V\\
&\quad+\frac{1}{2}(R(y,\nabla_{\dot{x}}y)\dot{x})^H+\frac{1}{2\alpha}\dot{x}(f)g(\nabla_{\dot{x}}y,y)y^V+\frac{1}{2}(R(y,\nabla_{\dot{x}}y)\dot{x})^H\\
&\quad-\frac{1}{2}g(\nabla_{\dot{x}}y,y)g(\nabla_{\dot{x}}y,y)(grad\,f)^H+\frac{f}{\alpha}g(\nabla_{\dot{x}}y,\nabla_{\dot{x}}y)y^V\\
&= (\nabla_{\dot{x}}\dot{x})^H+(R(y,\nabla_{\dot{x}}y)\dot{x})^H-\frac{1}{2}g(\nabla_{\dot{x}}y,y)^2(grad\,f)^H\\
&\quad+(\nabla_{\dot{x}}\nabla_{\dot{x}}y)^V+\frac{1}{\alpha}\big[\dot{x}(f)g(\nabla_{\dot{x}}y,y)+f\|\nabla_{\dot{x}}y\|^2\big]y^V.
\end{aligned}$$

□

Theorem 4.2. *Let (M,g) be a Riemannian manifold and (TM,g^f) its tangent bundle equipped with the twisted-Sasaki metric. If $C(t)=(x(t),y(t))$ is the cure on (TM,g^f) such $y(t)$ is a vector field along $x(t)$, then $C(t)$ is a geodesic on TM if and only if*

$$\begin{cases}
\nabla_{\dot{x}}\dot{x}=\dfrac{1}{2}g(\nabla_{\dot{x}}y,y)^2grad\,f-R(y,\nabla_{\dot{x}}y)\dot{x}\\[2ex]
\nabla_{\dot{x}}\nabla_{\dot{x}}y=-\dfrac{1}{\alpha}\big[\dot{x}(f)g(\nabla_{\dot{x}}y,y)+f\|\nabla_{\dot{x}}y\|^2\big]y.
\end{cases}\tag{4.3}$$

Proof. The statement is a direct consequence of Theorem 4.1 and definition of geodesic. □

Using Theorem 4.2, we deduce following.

Corollary 4.1. *Let (M,g) be a Riemannian manifold and (TM,g^f) its tangent bundle equipped with the twisted-Sasaki metric. The natural lift $C(t)=(x(t),\dot{x}(t))$ of any geodesic $x(t)$ on (M,g) is a geodesic on (TM,g^f).*

Corollary 4.2. *Let (M,g) be a Riemannian manifold, (TM,g^f) its tangent bundle equipped with the twisted-Sasaki metric. The horizontal lift $C(t)=(x(t),y(t))$ of the curve $x(t)$ is a geodesic on (TM,g^f) if and only if $x(t)$ is a geodesic on (M,g).*

Remark 4.1. Let (M^m,g) be an m-dimensional Riemannian manifold. If $C(t)=(x(t),y(t))$ horizontal lift of the curve $x(t)$, locally we have

$$\begin{aligned}
\nabla_{\dot{x}}y=0 \;&\Leftrightarrow\; \frac{dy^k}{dt}+\Gamma^k_{ij}y^i\frac{dx^j}{dt}=0\\
&\Leftrightarrow\; y'(t)=A(t).y(t),
\end{aligned}$$

where, $A(t)=[a_{kj}]$, $a_{kj}=\sum_{i=1}^{m}-\Gamma^k_{ij}\dfrac{dx^j}{dt}$.

Remark 4.2.
Using the Remark 4.1, we can construct an infinity of examples of geodesics on (TM,g^f).

Example 4.1. We consider on $\mathbb{R}$ the metric $g=e^x dx^2$.
The Christoffel symbols of the Levi-cita connection associated with g are

$$\Gamma^1_{11}=\frac{1}{2}g^{11}\Big(\frac{\partial g_{11}}{\partial x^1}+\frac{\partial g_{11}}{\partial x^1}-\frac{\partial g_{11}}{\partial x^1}\Big)=\frac{1}{2}.$$

1)The geodesics $x(t)$ such that $x(0)=a\in\mathbb{R},x'(0)=v\in\mathbb{R}$ of g satisfies the equation

$$\frac{d^2x^s}{dt^2}+\sum_{i,j=1}^{n}\frac{dx^i}{dt}\frac{dx^j}{dt}\Gamma^s_{ij}=0\Leftrightarrow x''+\frac{1}{2}(x')^2=0.$$

Hence, we get $x'(t)=\dfrac{2v}{2+vt}$ and $x(t)=a+2\ln(1+\dfrac{vt}{2})$.
Then, the natural lift

$$C_1(t)=(x(t),x'(t))=\big(a+2\ln(1+\frac{vt}{2}),\frac{2v}{2+vt}\big)$$

is a geodesic on $T\mathbb{R}$.
2) The curve $C_2(t)=(x(t),y(t))$ such $\nabla_{\dot{x}}y=0$ satisfies the equation

$$\frac{dy^s}{dt}+y^i\Gamma^s_{ij}\frac{dx^j}{dt}=0\Leftrightarrow y'+\frac{1}{2}yx'=0,$$

after that $y(t)=k.\exp(-\dfrac{v}{2+tv})$, $k\in\mathbb{R}$.
Then, the horizontal lift

$$C_2(t)=(x(t),y(t))=\big(a+2\ln(1+\frac{vt}{2}),k.\exp(-\frac{v}{2+tv})\big)$$

is a geodesic on $T\mathbb{R}$.

Corollary 4.3. *Let (M,g) be a Riemannian manifold and (TM,g^f) its tangent bundle equipped with the twisted-Sasaki metric. If f be a constant function, then the curve $C(t)=(x(t),y(t))$ is a geodesic on (TM,g^f) if and only if*

$$\begin{cases}\nabla_{\dot{x}}\dot{x}=-R(y,\nabla_{\dot{x}}y)\dot{x}\\[2ex] \nabla_{\dot{x}}\nabla_{\dot{x}}y=-\dfrac{f}{\alpha}\|\nabla_{\dot{x}}y\|^2y.\end{cases}\tag{4.4}$$

Proof. The statement is a direct consequence of Theorem 4.2. □

Theorem 4.3.
Let (M,g) be a Riemannian manifold, (TM,g^f) its tangent bundle equipped with the twisted-Sasaki metric and $x(t)$ be a geodesic on M. If $C(t)=(x(t),y(t))$ is a geodesic on TM such that $\|y(t)\|$ is not a constant, then f is a constant along the curve $x(t)$.

Proof. Let $x(t)$ be a geodesic on M , then $\nabla_{\dot{x}}\dot{x}=0$. Using the first equation of formula (4.3), we obtain

$$\begin{aligned}g(\nabla_{\dot{x}}\dot{x},\dot{x})=0\quad&\Rightarrow\quad\frac{1}{2}g(\nabla_{\dot{x}}y,y)^2g(grad\,f,\dot{x})-g(R(y,\nabla_{\dot{x}}y)\dot{x},\dot{x})=0\\&\Rightarrow\quad\frac{1}{2}g(\nabla_{\dot{x}}y,y)^2\dot{x}(f)=0\\&\Rightarrow\quad\dot{x}(f)=0,\end{aligned}$$

as $\|y(t)\|$ is a constant $\Leftrightarrow\dot{x}g(y,y)=0\Leftrightarrow g(\nabla_{\dot{x}}y,y)=0$. □

Corollary 4.4. *Let (M,g) be a Riemannian manifold and (TM,g^f) its tangent bundle equipped with the twisted-Sasaki metric. If $C(t)=(x(t),y(t))$ is the cure on (TM,g^f) such $\|y(t)\|$ is a constant, then the curve $C(t)=(x(t),y(t))$ is a geodesic on (TM,g^f) if and only if*

$$\begin{cases}\nabla_{\dot{x}}\dot{x}=-R(y,\nabla_{\dot{x}}y)\dot{x}\\[2ex] \nabla_{\dot{x}}\nabla_{\dot{x}}y=-\dfrac{f}{\alpha}\|\nabla_{\dot{x}}y\|^2y.\end{cases}\tag{4.5}$$

Proof. The statement is a direct consequence of Theorem 4.2, and we have
$\|y(t)\|$ is a constant $\Leftrightarrow\dot{x}g(y,y)=0\Leftrightarrow g(\nabla_{\dot{x}}y,y)=0$. □

Theorem 4.4. *Let (M,g) be a flat Riemannian manifold and (TM,g^f) its tangent bundle equipped with the twisted-Sasaki metric. Then, the cure $C(t)=(x(t),y(t))$ is a geodesic on TM if and only if*

$$\begin{cases}\nabla_{\dot{x}}\dot{x}=\dfrac{1}{2}g(\nabla_{\dot{x}}y,y)^2grad\,f\\[2ex] \nabla_{\dot{x}}\nabla_{\dot{x}}y=-\dfrac{1}{\alpha}\big[\dot{x}(f)g(\nabla_{\dot{x}}y,y)+f\|\nabla_{\dot{x}}y\|^2\big]y.\end{cases}\tag{4.6}$$

Proof. The statement is a direct consequence of Theorem 4.1. □

Corollary 4.5. *Let (M, g) be a flat Riemannian manifold and (TM, g^f) its tangent bundle equipped with the twisted-Sasaki metric. If f is a constant function, then the curve $C(t) = (x(t), y(t))$ is a geodesic on TM implies that $x(t)$ is a geodesic on M.*

Proof. The statement is a direct consequence of Theorem 4.4. □

Acknowledgment.

We thank the reviewer for their insightful comments and suggestions that helped us improve the paper.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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