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Natural decomposition method and coupled systems of nonlinear fractional order partial differential equations

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Abstract

In this article, we present a method known as Natural decomposition transform method (NDTM). This method is the couple of Natural transform and Adomian decomposition method. By means of this new method, we successfully handle some coupled systems of nonlinear fractional order partial differential equations (NFPDEs). We obtain the solutions in the form of series which is rapidly converges to the exact solution. Two test examples are provided for the illustration of our method.

Keywords: Natural transform; Adomian polynomials; Fractional order partial differential equations; Coupled system.
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1. Introduction

In the last few decades fractional calculus has got considerable attention from researchers. This is due to its wide range applications in various branches of mathematics,engineering and all other applied sciences. As most of the phenomenons of physical science ,biological sciences and chemical sciences etc are accurately described by fractional differential equations (FDEs), for detail see [1, 2, 3, 4, 5]. The area devoted to the existence and uniqueness of positive solutions to FDEs has been widely explored by many researchers and plenty of research articles are available in literature, we refer few of them which can be found in [6, 7, 8, 9, 10]. Since most of the physical biological ,chemical phenomena and description of memory hereditary properties can be well studied in the form of mathematical models involving differential equations. However these models are well studied in terms of classical differential equations. But classical differential

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equations can not explain the characteristic behavior of hereditary material and process more accurately. Therefore this area need further exploration and investigations in terms of FDEs as the said equations can the ability to describe the aforementioned characteristic more precisely. Therefore in last few decades, the area devoted to FDEs has given much importance and now plenty of articles are available in literature, we refer few of them as [11, 12, 13, 14, 15]. Sine in most of the problems of dynamics, physics, biological and engineering disciplines are nonlinear in nature and to find their exact solution is very difficult task. To obtain better approximate solutions numerious techniques have been developed in literature. For instance, in [16, 17, 18, 19], authors have solved numerically various PDEs representing mathematical models of various phenomenons. As the coupled systems of linear FPDEs have been studied by many authors in last few years by using Adomian decomposition method [20], Homotopy Analysis method [21] and operational matrices and finite difference method [22, 23, 24, 25]. On the other hand, coupled system of non-linear FPDEs are rarely studied. For example in [34], authors have solved a coupled system of nonlinear PDEs by using Natural transform decomposition method. In same line in [20], the authors have used Adomian decomposition method to find approximate solutions to a coupled system of FDEs. Natural transform was introduced by Khan and his co-author [26] in 2008. The concerned transform has the property of being converging to both Laplace and Sumudu transform. Also the integral transforms like Fourier, Laplace, Mellin, and Sumudu transform can be extracted from Natural transform easily, see [27]. The analytical solution to some ordinary FDEs have been found by using Natural transform, for detail see[28]. In this article, we used Natural transform coupled with Adomian decomposition method to establish an iterative scheme which is helpful in solving coupled system of nonlinear FPDEs. The respective method is applicable to both linear and non linear ordinary as well as PDEs with fractional order. Our procedure is an easy tools and can be applied to a verity of problems of linear and nonlinear FPDEs. In this paper, we solve the following coupled system of FPDEs

$$\begin{aligned}
&\mathcal{D}_t^{\alpha}U(x,t) - \mathcal{D}_x^2U(x,t) - 2U(x,t)\mathcal{D}_x^{\alpha}U(x,t) + \mathcal{D}_x[U(x,t)V(x,t)] = 0, \quad 0 < \alpha \leq 1,\\
&\mathcal{D}_t^{\beta}V(x,t) - \mathcal{D}_x^2V(x,t) - 2V(x,t)\mathcal{D}_x^{\alpha}V(x,t) + \mathcal{D}_x[U(x,t)V(x,t)] = 0, \quad 0 < \beta \leq 1,\\
&\text{subject to the initial conditions } U(x,0) = \sin(x),\ V(x,0) = \sin(x).
\end{aligned} \tag{1.1}$$

Further, we extend the proposed method to solve a more general non-linear non-homogenous coupled system of FPDEs given by

$$\begin{aligned}
&\mathcal{D}_t^{\alpha}U(x,y,t) - \mathcal{D}_xV(x,y,t)\mathcal{D}_yW(x,y,t) - \mathcal{D}_xV(x,y,t)\mathcal{D}_xW(x,y,t) = -U(x,y,t), \quad 0 < \alpha \leq 1,\\
&\mathcal{D}_t^{\beta}V(x,y,t) + \mathcal{D}_xW(x,y,t)\mathcal{D}_yU(x,y,t) - \mathcal{D}_yU(x,y,t)\mathcal{D}_xW(x,y,t) = V(x,y,t), \quad 0 < \beta \leq 1,\\
&\mathcal{D}_t^{\gamma}W(x,y,t) - \mathcal{D}_xU(x,y,t)\mathcal{D}_yV(x,t) - \mathcal{D}_xU(x,y,t)\mathcal{D}_xV(x,y,t) = W(x,y,t), \quad 0 < \gamma \leq 1,\\
&\text{subject to the initial conditions}\\
&U(x,y,0) = e^{x+y},\ V(x,y,0) = e^{x-y},\ W(x,y,0) = e^{y-x}.
\end{aligned} \tag{1.2}$$

In the system (1.1) and (4.15), the notation $\mathcal{D}_z$ is used for $\frac{\partial}{\partial z}$. We solve the above coupled systems of FPDEs by means of the proposed method.

2. Basic Definitions and Results

We recall some basic definitions and known results from fractional calculus which can be traced in [5, 6, 7, 23, 24, 25, 26, 30, 31, 32].

Definition 1. *The Rieman-Liouville fractional integral of order $\alpha \in \mathbb{R}_+$ of a function $h(t) \in L([0,1],\mathbb{R})$ is defined by*

$$\mathcal{I}_t^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds$$

provided that the integral on the right side is converging.

Definition 2. *[23], For $\mu \in \mathbb{R}$, a function $f : \mathbb{R} \to \mathbb{R}^+$ is said to be in the space C_μ if it can be written as $f(x) = x^q f_1(x)$ with $q > \mu$, $f_1(x) \in C[0,\infty)$ and it is in the space $f(x) \in C^n_\mu$, if $f^{(n)} \in C_\mu$ for $n \in NU\{0\}$.*

Definition 3. *If $\alpha > 0$, then the Rieman-Liouville fractional order derivative of a function $h \in C^n_{-1}$ with $n \in NU\{0\}$ is defined by*

$$\mathcal{D}^\alpha_t h(t) = \mathcal{D}^n_t[\mathcal{J}^{n-\alpha} h(t)],\ n-1 < \alpha \le n,\ n \in N.$$

.

Definition 4. *If $\alpha > 0$, then the Caputo's fractional order derivative of a function $h \in C^n_{-1}$ with $n \in NU\{0\}$ is defined by*

$$\mathcal{D}^\alpha_t h(t) = \begin{cases} \mathcal{J}^{n-\alpha}[f^{(n)}(t)], & n-1 < \alpha \le n,\ n \in N, \\ \mathcal{D}^n_t h(t),\ \alpha = n,\ n \in N. \end{cases}$$

.

Definition 5. *A two parameter Mittag-Leffler function is defined by*

$$E_{p,q}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(kp+q)}.$$
$$\text{for } p = q = 1,\ E_{1,1}(t) = e^t,\ E_{1,1}(-t) = e^{-t}.$$

Definition 6. *The Natural transform of a function $v(x,t)$ for $t \ge 0$, is defined by*

$$\mathcal{N}[v(x,t)] = R(x,s,u) = \int_0^\infty e^{-st}\, v(x,\, ut)\, dt,$$

where s, u are the transform parameters and are assumed to be real and positive.

Definition 7. *The Natural transform of Mittag-Leffler function $E_{p,q}(t)$ is defined by*

$$\mathcal{N}[E_{p,q}(t)] = \sum_{k=0}^{\infty} \frac{u^{k+1}\Gamma(k+1)}{s^{k+1}\Gamma(kp+q)}.$$

.

Definition 8. *The Natural transform of $\mathcal{D}^\alpha f(t)$ is defined by:*

$$\begin{aligned} \mathcal{N}(\mathcal{D}^\alpha f(t)) &= \mathcal{N}\left(\mathcal{J}^{n-\alpha} f^{(n)}(t)\right) \\ &= \mathcal{N}\left[\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds\right] \\ &= \frac{u^{n-\alpha}}{s^{n-\alpha}} N\{f^{(n)}(t)\} = \frac{u^{n-\alpha}}{s^{n-\alpha}}\left[\frac{s^n}{u^n} R(s,u) - \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k}} f^{(k)}(0)\right]. \end{aligned} \tag{2.1}$$

Lemma 1. *[30, 31, 32] The Natural transform of $\frac{\partial^\alpha f(x,t)}{\partial t^\alpha}$ w.r.t tcan be calculated as:*

$$\mathcal{N}\left[\frac{\partial^\alpha f(x,t)}{\partial t^\alpha}\right] = \frac{s^\alpha}{u^\alpha} \bar{F}(x,s,u) - \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k}} \left[\lim_{t\to 0} \frac{\partial^\alpha f(x,t)}{\partial t^\alpha}\right]. \tag{2.2}$$

Lemma 2. *[31] The Natural transform of α order partial derivative of $f(x,t)$ w.r.t x is defined by*

$$\mathcal{N}\left[\frac{\partial^\alpha f(x,t)}{\partial x^\alpha}\right] = \mathcal{D}^\alpha_x \bar{F}(x,s,u).$$

Lemma 3. *The dual relation between Natural transform and Laplace transform is given by*

$$\mathcal{N}[f(x,t)] = R(x,s,u) = \frac{1}{u}\int_0^\infty e^{\frac{-st}{u}} f(x,t) dt = \frac{1}{u} F(x, \frac{s}{u}).$$

3. Natural Decomposition Transform Method(NDTM)

In this section, we are carried out the general procedure of our proposed method Natural decomposition transform method (NDTM) by considering the following coupled system of FPDEs

$$\begin{aligned} \mathcal{D}_t^{\alpha} U(x,t) + \mathcal{D}_x V(x,t) + F(U,V)(x,t) = f(x,t),\ 0<\alpha\leq 1, \\ \mathcal{D}_t^{\beta} V(x,t) + \mathcal{D}_x U(x,t) + G(U,V)(x,t) = g(x,t),\ 0<\beta\leq 1, \end{aligned} \tag{3.1}$$

subject to the initial conditions

$$U(x,0) = h_1(x), V(x,0) = h_2(x), \tag{3.2}$$

where $\mathcal{D}_t^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$, $\mathcal{D}_x = \frac{\partial}{\partial x}$ and $\mathcal{D}_t^{\beta} = \frac{\partial^{\beta}}{\partial t^{\beta}}$, partial differential operators and $F(U,V)(x,t)$ and $G(U,V)(x,t)$ are nonlinear operators and $f(x,t)$ $g(x,t)$ are the non-homogenous source terms. Taking the Natural transform of (3.1) and (3.2), we get

$$\begin{aligned} \frac{s^{\alpha}}{u^{\alpha}}\overline{U}(x,s,u) - \sum_{k=0}^{n-1}\frac{s^{\alpha-k-1}}{u^{\alpha-k}}U^{(k)}(x,t)\mid_{t=0} + \mathcal{N}[\mathcal{D}_x V(x,t)] + \mathcal{N}[F(U,V)(x,t)] = \mathcal{N}[f(x,t)], \\ \frac{s^{\beta}}{u^{\beta}}\overline{U}(x,s,u) - \sum_{k=0}^{n-1}\frac{s^{\beta-k-1}}{u^{\beta-k}}V^{(k)}(x,t)\mid_{t=0} + \mathcal{N}[\mathcal{D}_x U(x,t)] + \mathcal{N}[F(U,V)(x,t)] = \mathcal{N}[g(x,t)]. \end{aligned} \tag{3.3}$$

Using the initial conditions (3.2) and rearranging the terms, we have

$$\begin{aligned} \overline{U}(x,s,u) = \frac{h_1(x)}{s} - \frac{u^{\alpha}}{s^{\alpha}}\mathcal{N}[\mathcal{D}_x V(x,t) + F(U,V)(x,t) - f(x,t)], \\ \overline{V}(x,s,u) = \frac{h_2(x)}{s} - \frac{u^{\beta}}{s^{\beta}}\mathcal{N}[\mathcal{D}_x U(x,t) + G(U,V)(x,t) - g(x,t)]. \end{aligned} \tag{3.4}$$

Taking the inverse Natural transform of (3.3), we have

$$\begin{aligned} U(x,t) = \phi(x,t) - \mathcal{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathcal{N}[\mathcal{D}_x V(x,t) + F(U,V)(x,t) - f(x,t)]\right], \\ V(x,t) = \phi(x,t) - \mathcal{N}^{-1}\left[\frac{u^{\beta}}{s^{\beta}}\mathcal{N}[\mathcal{D}_x U(x,t) + F(U,V)(x,t) - f(x,t)]\right], \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \phi(x,t) = \frac{h_1(x)}{s} + \frac{u^{\alpha}}{s^{\alpha}}\mathcal{N}[f(x,t)], \\ \psi(x,t) = \frac{h_2(x)}{s} + \frac{u^{\beta}}{s^{\beta}}\mathcal{N}[f(x,t)]. \end{aligned} \tag{3.6}$$

Now assume the solution $U(x,t)$ and $V(x,t)$ in terms of infinite series given by

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t),\ V(x,t) = \sum_{n=0}^{\infty} V_n(x,t). \tag{3.7}$$

Also the nonlinear term $F(U,V)$ and $G(U,V)$ are decomposed in term of Adomian polynomials as

$$F(U,V) = \sum_{n=0}^{\infty} P_n(x,t),\ G(U,V) = \sum_{n=0}^{\infty} Q_n(x,t), \tag{3.8}$$

where P_n and Q_n are Adomian polynomials [35], and can be easily computed by the formula

$$
\begin{aligned}
P_n &= \frac{1}{\Gamma(n+1)}\frac{d^n}{d\lambda^n}\left[F\bigg(\sum_{i=0}^{n}\lambda^i U_i(x,t)\bigg)\right]\bigg|_{\lambda=0},\\
Q_n &= \frac{1}{\Gamma(n+1)}\frac{d^n}{d\lambda^n}\left[G\bigg(\sum_{i=0}^{n}\lambda^i V_i(x,t)\bigg)\right]\bigg|_{\lambda=0},
\end{aligned}
$$

where $n = 0, 1, 2, \ldots$ by putting (3.7) and (3.8) in (3.5), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} U_n(x,t) &= \phi(x,t) - \mathcal{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathcal{N}\left[\mathcal{D}_x[\sum_{n=0}^{\infty} U_n(x,t)] + \sum_{n=0}^{\infty} P_n(x,t)\right]\right],\\
\sum_{n=0}^{\infty} V_n(x,t) &= \psi(x,t) - \mathcal{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathcal{N}\left[\mathcal{D}_x[\sum_{n=0}^{\infty} V_n(x,t)] + \sum_{n=0}^{\infty} Q_n(x,t)\right]\right].
\end{aligned}
\tag{3.9}
$$

From (3.9), we equate terms on both sides to produce a recurrence relations as

$$
\begin{aligned}
&U_0(x,t) = \phi(x,t),\\
&U_1(x,t) = -\mathcal{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathcal{N}[\mathcal{D}_x U_0(x,t) + P_0(x,t)]\right],\\
&U_2(x,t) = -\mathcal{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathcal{N}[\mathcal{D}_x U_1(x,t) + P_1(x,t)]\right],\\
&\vdots\\
&U_{n+1}(x,t) = -\mathcal{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathcal{N}[\mathcal{D}_x U_n(x,t) + P_n(x,t)]\right],\ n \geq 1.
\end{aligned}
\tag{3.10}
$$

Similarly

$$
\begin{aligned}
&V_0(x,t) = \psi(x,t),\\
&V_1(x,t) = -\mathcal{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathcal{N}[\mathcal{D}_x V_0(x,t) + Q_0(x,t)]\right]\\
&V_2(x,t) = -\mathcal{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathcal{N}[\mathcal{D}_x V_1(x,t) + Q_1(x,t]\right]\\
&\vdots\\
&V_{n+1}(x,t) = -\mathcal{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathcal{N}[\mathcal{D}_x V_n(x,t) + Q_n(x,t)]\right],\ n \geq 1.
\end{aligned}
\tag{3.11}
$$

Hence after evaluation of (3.10) and (3.11), the approximate solutions for the nonlinear system(3.1) are given by

$$
U(x,t) = \sum_{n=0}^{\infty} U_n(x,t),\ V(x,t) = \sum_{n=0}^{\infty} V_n(x,t).
$$

4. Application of Natural Decomposition Transform method

In this section, we test the newly established algorithm of previous section by solving the following systems of FPDEs.

Example 4.

$$\begin{aligned}&\mathcal{D}_t^{\alpha}U(x,t)-\mathcal{D}_x^2U(x,t)-2U(x,t)\mathcal{D}_xU(x,t)+\mathcal{D}_x(UV)=0,\ 0<\alpha\leq 1,\\&\mathcal{D}_t^{\beta}V(x,t)-\mathcal{D}_x^2V(x,t)-2V(x,t)\mathcal{D}_xV(x,t)+\mathcal{D}_x(UV)=0,\ 0<\beta\leq 1,\end{aligned}\tag{4.1}$$

subject to the initial conditions

$$U(x,0)=\sin(x),\ V(x,0)=\sin(x).\tag{4.2}$$

Using the afore mentioned method, we can generate the following recursive relation as

$$\begin{aligned}&U_0(x,t)=\sin(x)\\&U_1(x,t)=\mathcal{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathcal{N}[(U_0)_{xx}+2P_0(U)-Q_0(U,V))]\right]\\&U_2(x,t)=\mathcal{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathcal{N}[U_{1xx}+2P_1(U,U_x)-Q_1(U,V)]\right]\\&\quad\vdots\qquad\vdots\qquad\vdots\\&U_{n+1}(x,t)=\mathcal{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathcal{N}[(U_{nxx}+2P_n(U)-Q_n(U,V)]\right]\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}&V_0(x,t)=\sin(x)\\&V_1(x,t)=N^{-1}[\frac{u^{\alpha}}{s^{\alpha}}N[(V_0)_{xx}+2R_0(V)-(Q_0(U,V))_x]]\\&V_2(x,t)=N^{-1}[\frac{u^{\alpha}}{s^{\alpha}}N[(V_1)_{xx}+2P_1(V)-(Q_1(U,V))_x]]\\&\quad\vdots\qquad\vdots\qquad\vdots\\&V_{n+1}(x,t)=N^{-1}[\frac{u^{\alpha}}{s^{\alpha}}N[(V_n)_{xx}+2R_n(U)-(Q_n(U,V))_x]].\end{aligned}\tag{4.4}$$

Now from the recurrence relation (4.3) and (4.4), we calculate the remaining terms as

$$U_1(x,t)=-\sin(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)},\tag{4.5}$$

$$U_2(x,t)=\sin(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\tag{4.6}$$

$$U_3(x,t)=-\sin(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\tag{4.7}$$

$$\vdots\tag{4.8}$$

and

$$V_1(x,t)=-\sin(x)\frac{t^{\beta}}{\Gamma(\beta+1)},\tag{4.9}$$

$$V_2(x,t)=\sin(x)\frac{t^{2\beta}}{\Gamma(2\beta+1)},\tag{4.10}$$

$$V_3(x,t)=-\sin(x)\frac{t^{3\beta}}{\Gamma(3\beta+1)},\tag{4.11}$$

$$\vdots\tag{4.12}$$

Accumulating the terms, we have the approximate solutions of unknown functions $U(x,t)$, $V(x,t)$ as

$$\begin{aligned} U(x,t) &= \sin(x) - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\sin(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin(x) - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\sin(x) + ... \\ &= \sin(x)E_{\alpha}(-t^{\alpha}) \end{aligned} \tag{4.13}$$

and similarly

$$V(x,t) = sin(x)[\sum_{k=0}^{\infty}(-1)^k \frac{t^{k\beta}}{\Gamma(k\beta+1)}] = sin(x)E_{\beta}(-t^{\beta}). \tag{4.14}$$

Now if we put $\alpha = \beta = 1$ in (4.13) and (4.14), we get exact solution as given in [34] which are $U(x,t) = \sin(x)e^{-t}$, $V(x,t) = \sin(x)e^{-t}$.

At $\alpha = 1$ and $\beta = 1$ the exact and approximate solutions are in close agreement with each other. When α, β are tending to 1 the approximate solutions approaches to exact solutions. This phenomenon stats that our method is in close agreement with exact solution.

Example 5. *Consider the following coupled system of nonlinear FPDES of the form*

$$\begin{aligned} &\mathcal{D}_t^{\alpha}U(x,y,t) - \mathcal{D}_xV(x,y,t)\mathcal{D}_yW(x,y,t) - \mathcal{D}_xV(x,y,t)\mathcal{D}_xW(x,y,t) = -U(x,y,t), \quad 0 < \alpha \leq 1, \\ &\mathcal{D}_t^{\beta}V(x,y,t) + \mathcal{D}_xW(x,y,t)\mathcal{D}_yU(x,y,t) - \mathcal{D}_yU(x,y,t)\mathcal{D}_xW(x,y,t) = V(x,y,t), \quad 0 < \beta \leq 1, \\ &\mathcal{D}_t^{\gamma}W(x,y,t) - \mathcal{D}_xU(x,y,t)\mathcal{D}_yV(x,t) - \mathcal{D}_xU(x,y,t)\mathcal{D}_xV(x,y,t) = W(x,y,t), \quad 0 < \gamma \leq 1, \\ &\textit{subject to the initial conditions} \\ &U(x,y,0) = e^{x+y},\ V(x,y,0) = e^{x-y},\ W(x,y,0) = e^{y-x}. \end{aligned} \tag{4.15}$$

We solve the above system with the help of proposed method and get the terms of required series solutions as

$$\begin{aligned} U_0(x,y,t) &= e^{x-y}, \\ U_1(x,y,t) &= -e^{x+y}\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ U_2(x,y,t) &= e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ U_3(x,y,t) &= -e^{x+y}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\vdots \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} V_0(x,y,t) &= e^{x-y}, \\ V_1(x,y,t) &= e^{x-y}\frac{t^{\beta}}{\Gamma(\beta+1)}, \\ V_2(x,y,t) &== e^{x-y}\frac{t^{2\beta}}{\Gamma(2\beta+1)}, \\ V_3(x,y,t) &= e^{x-y}\frac{t^{3\beta}}{\Gamma(3\beta+1)}, \\ V_4(x,y,t) &= e^{x-y}\frac{t^{4\beta}}{\Gamma(4\beta+1)}, \\ &\vdots \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} W_0(x,y,t) &= e^{x-y} \\ W_1(x,y,t) &= e^{y-x}\frac{t^{\gamma}}{\Gamma(\gamma+1)}, \\ W_2(x,y,t) &= e^{y-x}\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}, \\ W_3(x,y,t) &= e^{y-x}\frac{t^{3\gamma}}{\Gamma(3\gamma+1)}, \\ &\vdots \end{aligned} \tag{4.18}$$

After accumulating the terms, we have

$$\begin{aligned} U(x,y,t) &= U_0 + U_1 + U_2 + \cdots \\ &= e^{x+y} - e^{x+y}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \\ &= e^{x+y}E_{\alpha}(-t^{\alpha}) \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} V(x,y,t) &= V_0 + V_1 + V_2 + \cdots \\ &= e^{x-y}E_{\beta}(t^{\beta}) \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} W(x,y,t) &= W_0 + W_1 + W_2 + \cdots \\ &= e^{y-x}E_{\gamma}(t^{\gamma}). \end{aligned} \tag{4.21}$$

If we put $\alpha = \beta = \gamma = 1$ in (4.19), (4.20) and (4.21), we have

$$U(x,y,t) = e^{y+x-t},\ V(x,y,t) = e^{x-y+t},\ \text{and } W(x,y,t) = e^{y-x+t}$$

which is the exact solution obtained in [20] by using ADM and [34] using NDTM. At $\alpha = 1, \beta = 1$ and $\gamma = 1$ the exact and approximate solutions are in close agreement with each other. When $\alpha, \beta,\ \gamma$ are tending to 1 the approximate solutions approaches to exact solutions. This phenomenon stats that our method is in close agreement with exact solution.

5. Conclusions

we have applied successfully the newly developed method Natural Decomposition transform method to nonlinear coupled system of fractional order partial differential equations. This new proposed method established in this article is highly accurate. With the help of this method we found exact solution of nonlinear coupled system of FPDES. We also compared our results with exact solution obtained by using other method such as ADM, HAM and LDM. We found that our results completely agree with the results of aforementioned method. Also the applicability of the proposed method is shown by images of exact and approximate solutions for the considered problems. Hence Natural decomposition transform method is applicable to a variety of problem both linear and non-linear fractional order partial differential equations and fractional order ordinary differential equations.

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