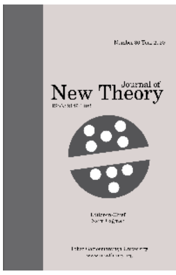
Journal of New Theory
ISSN: 2149-1402

30 (2020) 86-99
Journal of New Theory
http://www.newtheory.org
Open Access



Axiomatic Characterizations of Quadripartitioned Single Valued Neutrosophic Rough Sets

Mohanasundari Mohan[1], Mohana Krishnaswamy[2]

Article History
Received: 27.06.2019
Accepted: 03.02.2020
Published: 27.03.2020
Original Article

Abstract − In this paper, axiomatic characterizations of quadripartitioned single-valued neutrosophic rough sets have been studied and also studied some properties of quadripartitioned single-valued neutrosophic rough sets. A numerical example in medical diagnosis is given, which is based on the similarity measure of quadripartitioned single-valued neutrosophic rough sets.

Keywords − *Quadripartitioned single valued neutrosophic rough sets, similarity measure, axiomatic characterization, quadripartitioned single-valued neutrosophic number*

1. Introduction

Zadeh [1] proposed the concept of fuzzy sets which is very useful to deal the concept of imprecision, uncertainty, and degrees of the truthfulness of values and is represented by membership functions which lie in a unit interval [0,1]. Atanassov [2] developed the concept of intuitionistic fuzzy sets in 1983 which is a generalization of fuzzy sets and is dealing with the concept of vagueness. This concept consists of both membership and non-membership functions. In 1998, Smarandache presented Neutrosophic sets with three components called truth membership function, indeterminacy membership function, and falsity membership function [3,4].

In 1982, Pawlak [5] introduced the concept of rough sets which expresses vagueness in the notions of lower and upper approximations of a set and it employs boundary region of a set. A hybrid structure of rough neutrosophic sets was introduced by Broumi and Smarandache in 2014 [6]. Smarandache [7] and later Wang et al. [8] studied the concept of single-valued neutrosophic sets which is very useful in real scientific and engineering applications. Broumi et al. [9-11] solved the shortest path problem using Bellman algorithm under neutrosophic environment. Then, a new hybrid model of single-valued neutrosophic rough sets was introduced by Hai Long Yang [12].

Smarandache [7] firstly presented the refinement of the neutrosophic set and logic, i.e. the truth value T is refined into types of sub-truths such as T_1, T_2, etc.; similarly indeterminacy I is split/refined into types of sub-indeterminacies I_1, I_2, etc., and the sub-falsehood F is split into F_1, F_2, etc. Based on Belnap's [13] four-valued logic that is (Truth-T, Falsity-F, Unknown-U, Contradiction-C) Smarandache proposed the concept of four numerical valued neutrosophic logic that is quadripartitioned single valued neutrosophic sets. In this set, the indeterminacy is split into two parts known as unknown (neither true nor false) and contradiction (both

[1]mohanasundari@bitsathy.ac.in (Corresponding Author); [2]riyaraju1116@gmail.com
[1]Department of Mathematics, Bannari Amman Institute of Technology, Sathya Mangalam, Tamilnadu, India
[2]Department of Mathematics, Nirmala College for Women, Coimbatore, Tamilnadu, India

true and false). Mohana and Mohanasundari [14] studied the concept of quadripartitioned single-valued neutrosophic relations (QSVNR) and also studied some properties of a quadripartitioned single-valued neutrosophic rough sets. Chatterjee et al. [15] studied the concept of some similarity measures and entropy on quadripartitioned single-valued neutrosophic sets.

This paper is structured in the following ways. Section 1 provides a brief introduction. Section 2 delivered the basic definitions which we need to prove the results in further. Section 3 defined the concepts of empty quadripartitioned single-valued neutrosophic sets (QSVNS), full QSVNS and also singleton and its complement of QSVNS. And also, we have studied some properties of quadripartitioned single-valued neutrosophic rough sets. Section 4 deals the concept of axiomatic characterizations of quadripartitioned single-valued neutrosophic rough sets in detail. Section 5 illustrates an example for quadripartitioned single-valued neutrosophic rough sets in medical diagnosis. Section 6 concludes the paper.

2. Preliminaries

In this section, we recall the basic definitions of rough sets, Neutrosophic sets, QSVNS, and QSVNR, which will be used in proving the rest of the paper.

Definition 2.1. [5] Let U be any non-empty set. Suppose R is an equivalence relation over U. For any non-null subset X of U, the sets $A_1(x) = \{X: [x]_R \subseteq X\}$ and $A_2(x) = \{X: [x]_R \cap X \neq \varphi\}$ are called the lower approximation and upper approximation respectively of X where the pair $S = (U, R)$ is called an approximation space. This equivalence relation R is called indiscernibility relation. The pair $A(X) = (A_1(X), A_2(X))$ is called the rough set of X in S. Here $[x]_R$ denotes the equivalence class of R containing X.

Definition 2.2. [4] Let X be a universe of discourse, with a generic element in X denoted by x, a neutrosophic set (NS) is an object having the form,

$$A = \{< x: \mu_A(x), \nu_A(x), \omega_A(x) >, x \in X\}$$

where the functions $\mu, \nu, \omega: X \to]^{-}0, 1^{+}[$ define the degree of membership (or truth) respectively, the degree of indeterminacy, and the degree of non-membership (or falsehood) of the element $x \in X$ to the set A with the condition, $^{-}0 \leq \mu_A(x) + \nu_A(x) + \omega_A(x) \leq 3^{+}$.

Definition 2.3. [15] Let X be a non-empty set. A quadripartitioned single-valued neutrosophic set (QSVNS) A over X characterizes each element x in X by a truth-membership function T_A , a contradiction membership function C_A, an ignorance membership function U_A and a falsity membership function F_A such that for each, $x \in X, T_A, C_A, U_A, F_A \in [0,1]$ and $0 \leq T_A(x) + C_A(x) + U_A(x) + F_A(x) \leq 4$ when X is discrete, A is represented as, $A = \sum_{i=1}^{n} \langle T_A(x_i), C_A(x_i), U_A(x_i), F_A(x_i)\rangle / x_i, x_i \in X$. However, when the universe of discourse is continuous, A is represented as, $A = \int_X \langle T_A(x), C_A(x), U_A(x), F_A(x)\rangle / x, x \in X$

Definition 2.4. [14] A QSVNS R in $U \times U$ is called a quadripartitioned single-valued neutrosophic relation (QSVNR) in U, denoted by,

$$R = \{\langle (x, y), T_R(x, y), C_R(x, y), U_R(x, y), F_R(x, y)\rangle / (x. y) \in U \times U\}$$

where $T_R: U \times U \to [0,1], C_R: U \times U \to [0,1], U_R: U \times U \to [0,1],$ and $F_R: U \times U \to [0,1]$ denote the truth membership function, a contradiction membership function, an ignorance membership function and a falsity membership function of R respectively.

Definition 2.5. [14] Let R be a QSVNR in U, the tuple (U, R) is called a quadripartitioned single-valued neutrosophic approximation space $\forall A \in QSVNS(U)$, the lower and upper approximations of A with respect to (U, R) denoted by $\underline{R}(A)$ and $\overline{R}(A)$ are two QSVNS's whose membership functions are defined as $\forall\ x \in U$,

$$T_{\underline{R}(A)}(x) = \wedge_{y \in U} (F_R(x,y) \vee T_A(y)), \quad T_{\overline{R}(A)}(x) = \vee_{y \in U} (T_R(x,y) \wedge T_A(y))$$
$$C_{\underline{R}(A)}(x) = \wedge_{y \in U} (U_R(x,y) \vee C_A(y)), \quad C_{\overline{R}(A)}(x) = \vee_{y \in U} (C_R(x,y) \wedge C_A(y))$$
$$U_{\underline{R}(A)}(x) = \vee_{y \in U} (C_R(x,y) \wedge U_A(y)), \quad U_{\overline{R}(A)}(x) = \wedge_{y \in U} (U_R(x,y) \vee U_A(y))$$
$$F_{\underline{R}(A)}(x) = \vee_{y \in U} (T_R(x,y) \wedge F_A(y)), \quad F_{\overline{R}(A)}(x) = \wedge_{y \in U} (F_R(x,y) \vee F_A(y)).$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called the quadripartitioned single-valued neutrosophic rough set of A with respect to (U,R). $\underline{R}$ and $\overline{R}$ are referred to as the quadripartitioned single-valued neutrosophic lower and upper approximation operators, respectively.

Theorem 2.1. [14] Let (X,R) be a quadripartitioned single-valued neutrosophic approximation space. The quadripartitioned single-valued neutrosophic lower and upper approximation operators defined in 3.4 have the following properties. $\forall A, B \in QSVNS(X)$,

i. $\underline{R}(X)=X$, $\overline{R}(\varphi) = \varphi$
ii. If $A \subseteq B$ then $\underline{R}(A) \subseteq \underline{R}(B)$ and $\overline{R}(A) \subseteq \overline{R}(B)$
iii. $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B), \overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$
iv. $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B), \overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$
v. $\underline{R}(A^C) = (\overline{R}(A))^C, \overline{R}(A^C) = (\underline{R}(A))^C$

3. The Properties of Quadripartitioned Single-Valued Neutrosophic Rough Sets

In this paper, QSVNS(X) will denote the family of all QSVNSs in X.

Definition 3.1. Let A be a QSVNS in X. If $\forall x \in X, T_A(x) = 0, C_A(x) = 0$ and $U_A(x) = 1, F_A(x) = 1$then A is called an empty QSVNS, denoted by ϕ. If $\forall x \in X, T_A(x) = 1, C_A(x) = 1$ and $U_A(x) = 0, F_A(x) = 0$ then A is called a full QSVNS, denoted by X.

$\forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1], \alpha_{1,}\widehat{\alpha_{2,}\alpha_3}{}_{,}\alpha_4$ denotes a constant QSVNS satisfying,

$$T_{\alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4}(x) = \alpha_1, C_{\alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4}(x) = \alpha_2, U_{\alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4}(x) = \alpha_3, \text{ and } F_{\alpha_{1,}\widehat{\alpha_{2,}\alpha_{3,}}\alpha_4}(x) = \alpha_4$$

Definition 3.2. For any $y \in X$, a quadripartitioned single-valued neutrosophic singleton set 1_y and its complement $1_{X-\{y\}}$ are defined as $\forall x \in X$,

$$T_{1_y}(x) = C_{1_y}(x) = \begin{cases} 1, x = y \\ 0, x \neq y \end{cases}$$
$$U_{1_y}(x) = F_{1_y}(x) = \begin{cases} 0, x = y \\ 1, x \neq y \end{cases}$$
$$T_{1_{X-\{y\}}}(x) = C_{1_{X-\{y\}}}(x) = \begin{cases} 0, x = y \\ 1, x \neq y \end{cases}$$

and

$$U_{1_{X-\{y\}}}(x) = F_{1_{X-\{y\}}}(x) = \begin{cases} 1, x = y \\ 0, x \neq y \end{cases}$$

Definition 3.3. If $\forall x \in X, \vee_{y \in X} T_R(x,y) = 1$, $\vee_{y \in X} C_R(x,y) = 1$, $\wedge_{y \in X} U_R(x,y) = 0$, and $\wedge_{y \in X} F_R(x,y) = 0$, then R is called a serial QSVNR where " $\vee$ " and " $\wedge$ " denote maximum and minimum respectively.

Theorem 3.1. Let (X,R) be a quadripartitioned single-valued neutrosophic approximation space. The quadripartitioned single-valued neutrosophic lower and upper approximation operators defined in 2.5 have the following properties. $\forall A, B \in QSVNS(X), \forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$,

(1) $\underline{R}\left(A \cup \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4\right) = \underline{R}(A) \cup \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4,$ and $\overline{R}(A \cap \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4) = \overline{R}(A) \cap \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4;$

(2) $\underline{R}\left(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4\right) = \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4 \Leftrightarrow \underline{R}(\phi) = \phi,$ and $\overline{R}(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4) = \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4 \Leftrightarrow \overline{R}(U) = U$

Proof. By definition 2.5, $\forall x \in U$, we have

$$\begin{aligned}
T_{\underline{R}(A\cup\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4)}(x) &= \bigwedge_{y\in X} \left(F_R(x,y) \vee T_{A\cup\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)\right) \\
&= \bigwedge_{y\in X} \left(F_R(x,y) \vee \left(T_A(y) \vee T_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)\right)\right) \\
&= \bigwedge_{y\in X} \left(F_R(x,y) \vee T_A(y) \vee \alpha_1\right) \\
&= \left(\bigwedge_{y\in X} \left(F_R(x,y) \vee T_A(y)\right)\right) \vee \alpha_1 \\
&= T_{\underline{R}(A)}(x) \vee \alpha_1
\end{aligned}$$

$$\begin{aligned}
C_{\underline{R}(A\cup\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4)}(x) &= \bigwedge_{y\in X} \left(U_R(x,y) \vee C_{A\cup\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)\right) \\
&= \bigwedge_{y\in X} \left(U_R(x,y) \vee \left(C_A(y) \vee C_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)\right)\right) \\
&= \bigwedge_{y\in X} \left(U_R(x,y) \vee C_A(y) \vee \alpha_2\right) \\
&= \left(\bigwedge_{y\in X} \left(U_R(x,y) \vee C_A(y)\right)\right) \vee \alpha_2 \\
&= C_{\underline{R}(A)}(x) \vee \alpha_2
\end{aligned}$$

$$\begin{aligned}
U_{\underline{R}(A\cup\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4)}(x) &= \bigvee_{y\in X} \left(C_R(x,y) \wedge U_{A\cup\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)\right) \\
&= \bigvee_{y\in X} \left(C_R(x,y) \wedge \left(U_A(y) \wedge U_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)\right)\right) \\
&= \bigvee_{y\in X} \left(C_R(x,y) \wedge U_A(y) \wedge \alpha_3\right) \\
&= \left(\bigvee_{y\in X} \left(C_R(x,y) \wedge U_A(y)\right)\right) \wedge \alpha_3 \\
&= U_{\underline{R}(A)}(x) \wedge \alpha_3
\end{aligned}$$

$$\begin{aligned}
F_{\underline{R}(A\cup\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4)}(x) &= \bigvee_{y\in X} \left(T_R(x,y) \wedge F_{A\cup\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)\right) \\
&= \bigvee_{y\in X} \left(T_R(x,y) \wedge \left(F_A(y) \wedge F_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)\right)\right) \\
&= \bigvee_{y\in X} \left(T_R(x,y) \wedge F_A(y) \wedge \alpha_4\right) \\
&= \left(\bigvee_{y\in X} \left(T_R(x,y) \wedge F_A(y)\right)\right) \wedge \alpha_4 \\
&= F_{\underline{R}(A)}(x) \wedge \alpha_4
\end{aligned}$$

So, $\underline{R}(A \cup \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4) = \underline{R}(A) \cup \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4$.

Similarly, we can show that $\overline{R}(A \cap \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4) = \overline{R}(A) \cap \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4$.

(2) If $\underline{R}(\phi) = \phi$, then we have,

$$\underline{R}(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4) = \underline{R}(\phi \cup \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4) = \underline{R}(\phi) \cup \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4 = \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4 \text{ by (1)}$$

Conversely, if $\forall \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4 \in [0,1]$,

$$\begin{aligned}
&\underline{R}(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4) = \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4, take \alpha_1 = \alpha_2 = 0 and \alpha_3 = \alpha_4 = 1 \\
&i.e., \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4 = \phi, then we have \underline{R}(\phi) = \phi
\end{aligned}$$

Similarly, we can show $\overline{R}(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4) = \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4 \Leftrightarrow \overline{R}(U) = U$.

Theorem 3.2. Let (X,R) be a quadripartitioned single-valued neutrosophic approximation space. $\underline{R}(A)$ and $\overline{R}(A)$ are the lower and upper approximation in Definition 2.5 then we have,

(1) R is serial $\Leftrightarrow \underline{R}(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4) = \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4, \forall \alpha_1,\alpha_2,\alpha_3,\alpha_4 \in [0,1]$

$\Leftrightarrow \underline{R}(\phi) = \phi$

$\Leftrightarrow \overline{R}(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4) = \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4, \forall \alpha_1,\alpha_2,\alpha_3,\alpha_4 \in [0,1]$

$\Leftrightarrow \overline{R}(U) = U$

$\Leftrightarrow \underline{R}(A) \subset \overline{R}(A), \forall A \in QSVNS(X)$

(2) R is reflexive $\Leftrightarrow \underline{R}(A) \subset A, \forall A \in QSVNS(X)$,

$\Leftrightarrow A \subset \overline{R}(A), \forall A \in QSVNS(X)$

(3) R is symmetric $\Leftrightarrow \underline{R}\left(1_{X-\{x\}}\right)(y) = \underline{R}\left(1_{X-\{y\}}\right)(x), \forall x,y \in X$

$\Leftrightarrow \overline{R}(1_x)(y) = \overline{R}\left(1_y\right)(x), \forall x,y \in X$

(4) R is transitive $\Leftrightarrow \underline{R}(A) \subset \underline{R}(\underline{R}(A)), \forall A \in QSVNS(X)$

$\Leftrightarrow \overline{R}(\overline{R}(A)) \subset \overline{R}(A), \forall A \in QSVNS(X)$

PROOF. Since quadripartitioned single-valued neutrosophic approximation operators satisfy the duality property, it is enough to show us the properties for upper quadripartitioned single-valued neutrosophic approximation operator.

Based on Theorem 2.1(1), 3.1(2) we only need to show

$$R \text{ is serial} \Leftrightarrow \underline{R}(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4) = \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4, \forall \alpha_1,\alpha_2,\alpha_3,\alpha_4 \in [0,1],$$
$$\Leftrightarrow \underline{R}(A) \subset \overline{R}(A), \forall A \in QSVNS(X)$$

(I) We first prove

$$R \text{ is serial} \Leftrightarrow \underline{R}(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4) = \alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4, \forall \alpha_1,\alpha_2,\alpha_3,\alpha_4 \in [0,1].$$

First assume that $Risserial$, then $\forall x \in X$,

$\vee_{y\in X} T_R(x,y) = 1, \vee_{y\in X} C_R(x,y) = 1$ and $\wedge_{y\in X} U_R(x,y) = 0, \wedge_{y\in X} F_R(x,y) = 0 \forall \alpha_1,\alpha_2,\alpha_3,\alpha_4 \in [0,1]$

By Definition 2.5, $\forall x \in X$,

$$\begin{aligned} T_{\underline{R}(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4)}(x) &= \bigwedge_{y\in X} (F_R(x,y) \vee T_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)) \\ &= \bigwedge_{y\in X} (F_R(x,y) \vee \alpha_1) \\ &= \alpha_1 \vee \bigwedge_{y\in X} F_R(x,y) = \alpha_1 \vee 0 = \alpha_1 \end{aligned}$$

$$\begin{aligned} C_{\underline{R}(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4)}(x) &= \bigwedge_{y\in X} (U_R(x,y) \vee C_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)) \\ &= \bigwedge_{y\in X} (U_R(x,y) \vee \alpha_2) \\ &= \alpha_2 \vee \bigwedge_{y\in X} U_R(x,y) = \alpha_2 \vee 0 = \alpha_2 \end{aligned}$$

$$\begin{aligned} U_{\underline{R}(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4)}(x) &= \bigvee_{y\in X} (C_R(x,y) \wedge U_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)) \\ &= \bigvee_{y\in X} (C_R(x,y) \wedge \alpha_3) \\ &= \alpha_3 \wedge \bigvee_{y\in X} C_R(x,y) = \alpha_3 \wedge 1 = \alpha_3 \end{aligned}$$

$$\begin{aligned} F_{\underline{R}(\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4)}(x) &= \bigvee_{y\in X} (T_R(x,y) \wedge F_{\alpha_1,\widehat{\alpha_2,\alpha_3},\alpha_4}(y)) \\ &= \bigvee_{y\in X} (T_R(x,y) \wedge \alpha_4) \\ &= \alpha_4 \wedge \bigvee_{y\in X} T_R(x,y) = \alpha_4 \wedge 1 = \alpha_4. \end{aligned}$$

Therefore, $\forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$, $\underline{R}(\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4) = \alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4$.

Conversely, assume that $\forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1], \underline{R}(\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4) = \alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4$.

Take $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \alpha_4 = 1$, then by Definition 2.5, $\forall x \in X$,

$$0 = T_{\underline{R}(0,\widehat{0,1},1)}(x) = \bigwedge_{y\in X} (F_R(x,y) \vee T_{0,\widehat{0,1},1}(y)) = \bigwedge_{y\in X} (F_R(x,y) \vee 0) = \bigwedge_{y\in X} F_R(x,y)$$
$$0 = C_{\underline{R}(0,\widehat{0,1},1)}(x) = \bigwedge_{y\in X} (U_R(x,y) \vee C_{0,\widehat{0,1},1}(y)) = \bigwedge_{y\in X} (U_R(x,y) \vee 0) = \bigwedge_{y\in X} U_R(x,y)$$
$$1 = U_{\underline{R}(0,\widehat{0,1},1)}(x) = \bigvee_{y\in X} (C_R(x,y) \wedge U_{0,\widehat{0,1},1}(y)) = \bigvee_{y\in X} (C_R(x,y) \wedge 1) = \bigvee_{y\in X} C_R(x,y)$$
$$1 = F_{\underline{R}(0,\widehat{0,1},1)}(x) = \bigvee_{y\in X} (T_R(x,y) \wedge F_{0,\widehat{0,1},1}(y)) = \bigvee_{y\in X} (T_R(x,y) \wedge 1) = \bigvee_{y\in X} T_R(x,y)$$

Then, R is serial.

Hence, R is serial $\Leftrightarrow \underline{R}(\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4) = \alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4, \forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$.

(i) Next, we prove that R is serial $\Leftrightarrow \underline{R}(A) \subset \overline{R}(A), \forall A \in QSVNS(X)$.

First, assume that R is serial. Since X is finite, there exists $z \in X$ such that $T_R(x,z) = C_R(x,z) = 1$ and $U_R(x,z) = F_R(x,z) = 0$. Then by Definition 2.5, $\forall x \in X$,

$$\begin{aligned} T_{\underline{R}(A)}(x) = \bigwedge_{y\in X} (F_R(x,y) \vee T_A(y)) &= \bigwedge_{y\in X-\{z\}} (F_R(x,y) \vee T_A(y)) \wedge (F_R(x,z) \vee T_A(z)) \\ &= \bigwedge_{y\in X-\{z\}} (F_R(x,y) \vee T_A(y)) \wedge T_A(z) \le T_A(z) \end{aligned}$$

$$\begin{aligned} T_{\overline{R}(A)}(x) = \bigvee_{y\in X} (T_R(x,y) \wedge T_A(y)) &= \bigvee_{y\in X-\{z\}} (T_R(x,y) \wedge T_A(y)) \vee (T_R(x,z) \wedge T_A(z)) \\ &= \bigvee_{y\in X-\{z\}} (T_R(x,y) \wedge T_A(y)) \vee T_A(z) \ge T_A(z) \end{aligned}$$

Then, $T_{\underline{R}(A)}(x) \le T_{\overline{R}(A)}(x)$.

Similarly, we can prove that $C_{\underline{R}(A)}(x) \le C_{\overline{R}(A)}(x), U_{\underline{R}(A)}(x) \ge U_{\overline{R}(A)}(x)$, and $F_{\underline{R}(A)}(x) \ge F_{\overline{R}(A)}(x)$. Therefore $\underline{R}(A) \subset \overline{R}(A)$.

Conversely, assume that $\underline{R}(A) \subset \overline{R}(A), \forall A \in QSVNS(X)$. Take $A = X$, then by Theorem 2.1(1) and Definition 2.5, then we have

$$1 = T_X(x) = T_{\underline{R}(X)}(x) \le \overline{R}(X)(x) = \vee_{y\in X} (T_R(x,y) \wedge T_X(y)) = \vee_{y\in X} T_R(x,y)$$

which means $\vee_{y\in X} T_R(x,y) = 1$. Similarly, we can prove that

$$\vee_{y\in X} C_R(x,y) = 1, \wedge_{y\in X} U_R(x,y) = 0, \text{ and } \wedge_{y\in X} F_R(x,y) = 0$$

Hence, R is serial.

(1) $\Rightarrow R$ is reflexive, then $\forall x \in X$, we have

$$T_R(x,x) = C_R(x,x) = 1 \text{ and } U_R(x,x) = F_R(x,x) = 0$$

By Definition 2.5, $\forall A \in QSVNS(X), \forall x \in X$,

$$T_{\underline{R}(A)}(x) = \bigwedge_{y\in X} (F_R(x,y) \vee T_A(y)) \le F_R(x,x) \vee T_A(x) = 0 \vee T_A(x) = T_A(x)$$

$$C_{\underline{R}(A)}(x) = \bigwedge_{y\in X} (U_R(x,y) \vee C_A(y)) \le U_R(x,x) \vee C_A(x) = 0 \vee C_A(x) = C_A(x)$$

$$U_{\underline{R}(A)}(x) = \vee_{y\in X} (C_R(x,y) \wedge U_A(y)) \geq C_R(x,x) \wedge U_A(x) = 1 \wedge U_A(x) = U_A(x)$$

$$F_{\underline{R}(A)}(x) = \vee_{y\in X} (T_R(x,y) \wedge F_A(y)) \geq T_R(x,x) \wedge F_A(x) = 1 \wedge F_A(x) = F_A(x)$$

So, $\underline{R}(A) \subset A$.

" $\Leftarrow$ " Now assume that $\forall A \in QSVNS(X)$, $\underline{R}(A) \subset A$.

$\forall x \in X$, take $A = 1_{X-\{x\}}$, then we have

$$\begin{aligned} 0 = T_{1_{X-\{x\}}}(x) \geq T_{\underline{R}(1_{X-\{x\}})}(x) &= \wedge_{y\in X} (F_R(x,y) \vee T_{1_{X-\{x\}}}(y)) \\ &= (F_R(x,x) \vee T_{1_{X-\{x\}}}(x)) \wedge \wedge_{y\in X-\{x\}} (F_R(x,y) \vee T_{1_{X-\{x\}}}(y)) \\ &= (F_R(x,x) \vee 0) \wedge 1 = F_R(x,x), \text{then} F_R(x,x) = 0 \end{aligned}$$

$$\begin{aligned} 0 = C_{1_{X-\{x\}}}(x) \geq C_{\underline{R}(1_{X-\{x\}})}(x) &= \wedge_{y\in X} (U_R(x,y) \vee C_{1_{X-\{x\}}}(y)) \\ &= (U_R(x,x) \vee C_{1_{X-\{x\}}}(x)) \wedge \wedge_{y\in X-\{x\}} (U_R(x,y) \vee C_{1_{X-\{x\}}}(y)) \\ &= (U_R(x,x) \vee 0) \wedge 1 = U_R(x,x), \text{then } U_R(x,x) = 0 \end{aligned}$$

$$\begin{aligned} 1 = U_{1_{X-\{x\}}}(x) \leq U_{\underline{R}(1_{X-\{x\}})}(x) &= \vee_{y\in X} (C_R(x,y) \wedge U_{1_{X-\{x\}}}(y)) \\ &= (C_R(x,x) \wedge U_{1_{X-\{x\}}}(x)) \vee \vee_{y\in X-\{x\}} (C_R(x,y) \wedge U_{1_{X-\{x\}}}(y)) \\ &= (C_R(x,x) \wedge 1) \vee 0 = C_R(x,x), \text{then } C_R(x,x) = 1 \end{aligned}$$

$$\begin{aligned} 1 = F_{1_{X-\{x\}}}(x) \leq F_{\underline{R}(1_{X-\{x\}})}(x) &= \vee_{y\in X} (T_R(x,y) \wedge F_{1_{X-\{x\}}}(y)) \\ &= (T_R(x,x) \wedge F_{1_{X-\{x\}}}(x)) \vee \vee_{y\in X-\{x\}} (T_R(x,y) \wedge F_{1_{X-\{x\}}}(y)) \\ &= (T_R(x,x) \wedge 1) \vee 0 = T_R(x,x), \text{then } T_R(x,x) = 1 \end{aligned}$$

Thus, R is reflexive. So, R is reflexive $\Leftrightarrow \underline{R}(A) \subset A, \forall A \in QSVNS(X)$.

(2) By Definition 2.5, $\forall x, y \in X$

$$\begin{aligned} T_{\underline{R}(1_{X-\{x\}})}(y) &= \wedge_{z\in X} (F_R(y,z) \vee T_{1_{X-\{x\}}}(z)) \\ &= (F_R(y,x) \vee T_{1_{X-\{x\}}}(x)) \wedge \wedge_{z\in X-\{x\}} (F_R(y,z) \vee T_{1_{X-\{x\}}}(z)) \\ &= (F_R(y,x) \vee 0) \wedge 1 = F_R(y,x) \end{aligned}$$

$$\begin{aligned} T_{\underline{R}(1_{X-\{y\}})}(x) &= \wedge_{z\in X} (F_R(x,z) \vee T_{1_{X-\{y\}}}(z)) \\ &= (F_R(x,y) \vee T_{1_{X-\{y\}}}(y)) \wedge \wedge_{z\in X-\{y\}} (F_R(x,z) \vee T_{1_{X-\{y\}}}(z)) \\ &= (F_R(x,y) \vee 0) \wedge 1 = F_R(x,y) \end{aligned}$$

$$\begin{aligned} C_{\underline{R}(1_{X-\{x\}})}(y) &= \wedge_{z\in X} (U_R(y,z) \vee C_{1_{X-\{x\}}}(z)) \\ &= (U_R(y,x) \vee C_{1_{X-\{x\}}}(x)) \wedge \wedge_{z\in X-\{x\}} (U_R(y,z) \vee C_{1_{X-\{x\}}}(z)) \\ &= (U_R(y,x) \vee 0) \wedge 1 = U_R(y,x) \end{aligned}$$

$$\begin{aligned} C_{\underline{R}(1_{X-\{y\}})}(x) &= \wedge_{z\in X} (U_R(x,z) \vee C_{1_{X-\{y\}}}(z)) \\ &= (U_R(x,y) \vee C_{1_{X-\{y\}}}(y)) \wedge \wedge_{z\in X-\{y\}} (U_R(x,z) \vee C_{1_{X-\{y\}}}(z)) \\ &= (U_R(x,y) \vee 0) \wedge 1 = U_R(x,y) \end{aligned}$$

$$\begin{aligned} U_{\underline{R}(1_{X-\{x\}})}(y) &= \vee_{z\in X} (C_R(y,z) \wedge U_{1_{X-\{x\}}}(z)) \\ &= (C_R(y,x) \wedge U_{1_{X-\{x\}}}(x)) \vee \vee_{z\in X-\{x\}} (C_R(y,z) \wedge U_{1_{X-\{x\}}}(z)) \\ &= (C_R(y,x) \wedge 1) \vee 0 = C_R(y,x) \end{aligned}$$

$$\begin{aligned} U_{\underline{R}(1_{X-\{y\}})}(x) &= \vee_{z\in X} (C_R(x,z) \wedge U_{1_{X-\{y\}}}(z)) \\ &= (C_R(x,y) \wedge U_{1_{X-\{y\}}}(y)) \vee \vee_{z\in X-\{y\}} (C_R(x,z) \wedge U_{1_{X-\{y\}}}(z)) \\ &= (C_R(x,y) \wedge 1) \vee 0 = C_R(x,y) \end{aligned}$$

$$\begin{aligned} F_{\underline{R}(1_{X-\{x\}})}(y) &= \vee_{z\in X} (T_R(y,z) \wedge F_{1_{X-\{x\}}}(z)) \\ &= (T_R(y,x) \wedge F_{1_{X-\{x\}}}(x)) \vee \vee_{z\in X-\{x\}} (T_R(y,z) \wedge F_{1_{X-\{x\}}}(z)) \\ &= (T_R(y,x) \wedge 1) \vee 0 = T_R(y,x) \end{aligned}$$

$$\begin{aligned} F_{\underline{R}(1_{X-\{y\}})}(x) &= \vee_{z\in X} (T_R(x,z) \wedge F_{1_{X-\{y\}}}(z)) \\ &= (T_R(x,y) \wedge F_{1_{X-\{y\}}}(y)) \vee \vee_{z\in X-\{y\}} (T_R(x,z) \wedge F_{1_{X-\{y\}}}(z)) \\ &= (T_R(x,y) \wedge 1) \vee 0 = T_R(x,y) \end{aligned}$$

R is symmetric iff,

$$\begin{aligned} &\forall x, y \in X, \\ &T_R(x,y) = T_R(y,x), C_R(x,y) = C_R(y,x) \\ &U_R(x,y) = U_R(y,x), F_R(x,y) = F_R(y,x) \end{aligned}$$

Then, R is symmetric iff,

$$\begin{aligned} &\forall x, y \in X, \\ &T_{\underline{R}(1_{X-\{x\}})}(y) = T_{\underline{R}(1_{X-\{y\}})}(x), C_{\underline{R}(1_{X-\{x\}})}(y) = C_{\underline{R}(1_{X-\{y\}})}(x) \\ &U_{\underline{R}(1_{X-\{x\}})}(y) = U_{\underline{R}(1_{X-\{y\}})}(x), F_{\underline{R}(1_{X-\{x\}})}(y) = F_{\underline{R}(1_{X-\{y\}})}(x) \end{aligned}$$

which implies that R is symmetric iff $\forall x, y \in X, \underline{R}(1_{X-\{x\}})(y) = \underline{R}(1_{X-\{y\}})(x)$.

(3) Assume that R is transitive, then

$$\begin{aligned} &\forall x, y, z \in X, \\ &\vee_{y\in X} (T_R(x,y) \wedge T_R(y,z)) \leq T_R(x,z), \vee_{y\in X} (C_R(x,y) \wedge C_R(y,z)) \leq C_R(x,z) \\ &\wedge_{y\in X} (U_R(x,y) \vee U_R(y,z)) \geq U_R(x,z), \wedge_{y\in X} (F_R(x,y) \vee F_R(y,z)) \geq F_R(x,z) \end{aligned}$$

By Definition 2.5, $\forall x \in X$, we have

$$\begin{aligned} T_{\underline{R}(\underline{R}(A))}(x) &= \wedge_{y\in X} (F_R(x,y) \vee T_{\underline{R}(A)}(y)) \\ &= \wedge_{y\in X} (F_R(x,y) \vee \wedge_{z\in X} (F_R(y,z) \vee T_{(A)}(z)) \\ &= \wedge_{z\in X} \wedge_{y\in X} (F_R(x,y) \vee F_R(y,z) \vee T_{(A)}(z)) \\ &= \wedge_{z\in X} \big(\wedge_{y\in X} (F_R(x,y) \vee F_R(y,z)) \vee T_{(A)}(z)\big) \\ &\geq \wedge_{z\in X} (F_R(x,z) \vee T_A(z)) \\ &= T_{\underline{R}(A)}(x) \end{aligned}$$

$$\begin{aligned} C_{\underline{R}(\underline{R}(A))}(x) &= \wedge_{y\in X} (U_R(x,y) \vee C_{\underline{R}(A)}(y)) \\ &= \wedge_{y\in X} (U_R(x,y) \vee \wedge_{z\in X} (U_R(y,z) \vee C_{(A)}(z)) \\ &= \wedge_{z\in X} \wedge_{y\in X} (U_R(x,y) \vee U_R(y,z) \vee C_{(A)}(z)) \\ &= \wedge_{z\in X} \big(\wedge_{y\in X} (U_R(x,y) \vee U_R(y,z)) \vee C_{(A)}(z)\big) \\ &\geq \wedge_{z\in X} (U_R(x,z) \vee C_A(z)) \\ &= C_{\underline{R}(A)}(x) \end{aligned}$$

$$\begin{aligned} U_{\underline{R}(\underline{R}(A))}(x) &= \vee_{y\in X} (C_R(x,y) \wedge U_{\underline{R}(A)}(y)) \\ &= \vee_{y\in X} (C_R(x,y) \wedge \vee_{z\in X} (C_R(y,z) \wedge U_{(A)}(z)) \\ &= \vee_{z\in X}\vee_{y\in X} (C_R(x,y) \wedge C_R(y,z) \wedge U_{(A)}(z)) \\ &= \vee_{z\in X} \left(\vee_{y\in X} (C_R(x,y) \wedge C_R(y,z)) \wedge U_{(A)}(z)\right) \\ &\leq \vee_{z\in X} (C_R(x,z) \wedge U_A(z)) \\ &= U_{\underline{R}(A)}(x) \end{aligned}$$

$$\begin{aligned} F_{\underline{R}(\underline{R}(A))}(x) &= \vee_{y\in X} (T_R(x,y) \wedge F_{\underline{R}(A)}(y)) \\ &= \vee_{y\in X} (T_R(x,y) \wedge \vee_{z\in X} (T_R(y,z) \wedge F_{(A)}(z)) \\ &= \vee_{z\in X}\vee_{y\in X} (T_R(x,y) \wedge T_R(y,z) \wedge F_{(A)}(z)) \\ &= \vee_{z\in X} \left(\vee_{y\in X} (T_R(x,y) \wedge T_R(y,z)) \wedge F_{(A)}(z)\right) \\ &\leq \vee_{z\in X} (T_R(x,z) \wedge F_A(z)) \\ &= F_{\underline{R}(A)}(x) \end{aligned}$$

Hence, $\underline{R}(A) \subset \underline{R}(\underline{R}(A))$.

Conversely, assume that $\forall A \in QSVNS(X)$, $\underline{R}(A) \subset \underline{R}(\underline{R}(A))$.

$\forall x, y, z \in X$, take $A = 1_{X-\{z\}}$, we have

$$\begin{aligned} T_R(x,z) = F_{\underline{R}(1_{X-\{z\}})}(x) &\geq F_{\underline{R}(\underline{R}(1_{X-\{z\}}))}(x) \\ &= \vee_{y\in X} (T_R(x,y) \wedge F_{\underline{R}(1_{X-\{z\}})}(y)) \\ &= \vee_{y\in X} (T_R(x,y) \wedge T_R(y,z)) \end{aligned}$$

$$\begin{aligned} C_R(x,z) = U_{\underline{R}(1_{X-\{z\}})}(x) &\geq U_{\underline{R}(\underline{R}(1_{X-\{z\}}))}(x) \\ &= \vee_{y\in X} (C_R(x,y) \wedge U_{\underline{R}(1_{X-\{z\}})}(y)) \\ &= \vee_{y\in X} (C_R(x,y) \wedge C_R(y,z)) \end{aligned}$$

$$\begin{aligned} U_R(x,z) = C_{\underline{R}(1_{X-\{z\}})}(x) &\leq C_{\underline{R}(\underline{R}(1_{X-\{z\}}))}(x) \\ &= \wedge_{y\in X} (U_R(x,y) \vee C_{\underline{R}(1_{X-\{z\}})}(y)) \\ &= \wedge_{y\in X} (U_R(x,y) \vee U_R(y,z)) \end{aligned}$$

$$\begin{aligned} F_R(x,z) = T_{\underline{R}(1_{X-\{z\}})}(x) &\leq T_{\underline{R}(\underline{R}(1_{X-\{z\}}))}(x) \\ &= \wedge_{y\in X} (F_R(x,y) \vee T_{\underline{R}(1_{X-\{z\}})}(y)) \\ &= \wedge_{y\in X} (F_R(x,y) \vee F_R(y,z)) \end{aligned}$$

So, R is transitive.

4. Axiomatic Characterizations of Quadripartitioned Single-Valued Neutrosophic Rough Sets

This section will provide the axiomatic characterizations of quadripartitioned single-valued neutrosophic rough sets by defining a pair of abstract operators. Consider a system of quadripartitioned single-valued neutrosophic rough sets $(QSVNS(X),\cup,\cap,c,L,H)$ where $L,H\colon QSVNS(X) \to QSVNS(X)$ are two operators from $QSVNS(X)$ to $QSVNS(X)$ Let$T(X), C(X), U(X), F(X)$ denote truth, contradiction, ignorance and falsity membership function respectively.

Define $A \in QSVNS(X), L = (L_T, L_C, L_U, L_F)$ and $H = (H_T, H_C, H_U, H_F)$ where,

$$L_T, H_T\colon T(X) \to T(X), L_C, H_C\colon C(X) \to C(X), L_U, H_U\colon U(X) \to U(X), L_F, H_F\colon F(X) \to F(X)$$

For $A \in QSVNS(X)$, $L(A) = (L_T(T_A), L_C(C_A), L_U(U_A), L_F(F_A))$ which implies that,

$$T_{L(A)} = L_T(T_A), C_{L(A)} = L_C(C_A), U_{L(A)} = L_U(U_A), F_{L(A)} = L_F(F_A)$$

$H(A) = (H_T(T_A), H_C(C_A), H_U(U_A), H_F(F_A))$ which implies that,

$$T_{H(A)} = H_T(T_A), C_{H(A)} = H_C(C_A), U_{H(A)} = H_U(U_A), \text{ and } F_{H(A)} = H_F(F_A).$$

Definition 4.1. Let $L, H: QSVNS(X) \to QSVNS(X)$ be two quadripartitioned single-valued neutrosophic set operators. Then, $\forall A = \{\langle x, T_A(x), C_A(x), U_A(x), F_A(x)\rangle | x \in X\} \in QSVNS(X)$, L and H are known as dual operators if they satisfy the following axioms.

$(QSVNSL1) L(A) = (H(A^c))^c$ i.e., $\forall x \in X$,

i. $L_T(T_A)(x) = H_F(T_A)(x)$
ii. $L_C(C_A)(x) = H_U(C_A)(x)$
iii. $L_U(U_A)(x) = H_C(U_A)(x)$
iv. $L_F(F_A)(x) = H_T(F_A)(x)$

$(QSVNSU1) H(A) = (L(A^c))^c$ i.e., $\forall x \in X$,

i. $H_T(T_A)(x) = L_F(T_A)(x)$
ii. $H_C(C_A)(x) = L_U(C_A)(x)$
iii. $H_U(U_A)(x) = L_C(U_A)(x)$
iv. $H_F(F_A)(x) = L_T(F_A)(x)$

Theorem 4.1 Let $L, H: QSVNS(X) \to QSVNS(X)$ be two dual operators. Then, there exists a QSVNR R in X such that, $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ for all $A \in QSVNS(X)$ iff L satisfies the following axioms (QSVNSL2) and (QSVNSL3), or equivalently, H satisfies axioms (QSVNSU2) and (QSVNSU3):

$\forall A, B \in QSVNS(X), \forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$,
$(QSVNSL2) L(A \cup \alpha_1, \alpha_2, \alpha_3, \alpha_4) = L(A) \cup \alpha_1, \alpha_2, \alpha_3, \alpha_4$, i.e., $\forall x \in X$,

i. $L_T(T_A \cup \bar{\alpha}_1)(x) = L_T(T_A)(x) \vee \alpha_1$
ii. $L_C(C_A \cup \bar{\alpha}_2)(x) = L_C(C_A)(x) \vee \alpha_2$
iii. $L_U(U_A \cap \bar{\alpha}_3)(x) = L_U(U_A)(x) \wedge \alpha_3$
iv. $L_F(F_A \cap \bar{\alpha}_4)(x) = L_F(F_A)(x) \wedge \alpha_4$

where $\bar{\alpha}_i$ is a constant fuzzy set in X satisfying,

$\forall x \in X, \bar{\alpha}_i(x) = \alpha_i (i = 1,2,3,4)$
$(QSVNSL3) L(A \cap B) = L(A) \cap L(B)$ *i.e.*, $\forall x \in X$,

i. $L_T(T_{A\cap B})(x) = L_T(T_A \cap T_B)(x) = L_T(T_A)(x) \wedge L_T(T_B)(x)$
ii. $L_C(C_{A\cap B})(x) = L_C(C_A \cap C_B)(x) = L_C(C_A)(x) \wedge L_C(C_B)(x)$
iii. $L_U(U_{A\cap B})(x) = L_U(U_A \cup U_B)(x) = L_U(U_A)(x) \vee L_U(U_B)(x)$
iv. $L_F(F_{A\cap B})(x) = L_F(F_A \cup F_B)(x) = L_F(F_A)(x) \vee L_F(F_B)(x)$

$(QSVNSU2) H(A \cap \alpha_1, \alpha_2, \alpha_3, \alpha_4) = H(A) \cap \alpha_1, \alpha_2, \alpha_3, \alpha_4$ i.e., $\forall x \in X$,

i. $H_T(T_A \cap \bar{\alpha}_1)(x) = H_T(T_A)(x) \wedge \alpha_1$
ii. $H_C(C_A \cap \bar{\alpha}_2)(x) = H_C(C_A)(x) \wedge \alpha_2$
iii. $H_U(U_A \cup \bar{\alpha}_3)(x) = H_U(U_A)(x) \vee \alpha_3$
iv. $H_F(F_A \cup \bar{\alpha}_4)(x) = H_F(F_A)(x) \vee \alpha_4$
where $\bar{\alpha}_i$ is a constant fuzzy set in X satisfying $\forall x \in X, \bar{\alpha}_i(x) = \alpha_i (i = 1,2,3,4)$

$(QSVNSU3) H(A \cup B) = H(A) \cup L(B)$ i.e., $\forall x \in X$,

i. $H_T(T_{AUB})(x) = H_T(T_A \cup T_B)(x) = H_T(T_A)(x) \vee H_T(T_B)(x)$

ii. $H_C(C_{A\cup B})(x) = H_C(C_A \cup C_B)(x) = H_C(C_A)(x) \vee H_C(C_B)(x)$
iii. $H_U(U_{A\cup B})(x) = H_U(U_A \cap U_B)(x) = H_U(U_A)(x) \wedge H_U(U_B)(x)$
iv. $H_F(F_{A\cup B})(x) = H_F(F_A \cap F_B)(x) = H_F(F_A)(x) \wedge H_F(F_B)(x)$

Proof: " $\Rightarrow$ " It follows immediately from Theorem 2.1,3.1. " $\Leftarrow$ " Suppose that the operator H satisfies axioms (QSVNSU2) and (QSVNSU3). By using H, we can define a QSVNR $R = \{\langle (x,y), T_R(x,y), C_R(x,y), U_R(x,y), F_R(x,y)\rangle | x,y \in X\}$ as follows

$\forall x,y \in X, T_R(x,y) = H_T(T_{1y})(x), C_R(x,y) = H_C(C_{1y})(x), U_R(x,y) = H_U(U_{1y})(x),$ and $F_R(x,y) = H_F(F_{1y})(x)$.

Clearly, $\forall A \in QSVNS(X)$, we have,

$$T_A = \bigcup_{y\in X}(T_{1y} \cap \overline{T_A(y)}), C_A = \bigcup_{y\in X}(C_{1y} \cap \overline{C_A(y)}), U_A = \bigcap_{y\in X}(U_{1y} \cup \overline{U_A(y)}), F_A = \bigcap_{y\in X}(F_{1y} \cup \overline{F_A(y)}).$$

By definition 2.5, (QSVNSU2) and (QSVNSU3) we have

$$\begin{aligned} T_{\bar{R}(A)}(x) = \bigvee_{y\in X}(T_R(x,y) \wedge T_A(y)) &= \bigvee_{y\in X}(H_T(T_{1y})(x) \wedge T_A(y)) \\ &= \bigvee_{y\in X} H_T(T_{1y} \cap \overline{T_A(y)})(x) \\ &= H_T\left(\bigcup_{y\in X}(T_{1y} \cap \overline{T_A(y)})\right)(x) \\ &= H_T(T_A)(x) = T_{H(A)}(x) \end{aligned}$$

$$\begin{aligned} C_{\bar{R}(A)}(x) = \bigvee_{y\in X}(C_R(x,y) \wedge C_A(y)) &= \bigvee_{y\in X}(H_C(C_{1y})(x) \wedge C_A(y)) \\ &= \bigvee_{y\in X} H_C(C_{1y} \cap \overline{C_A(y)})(x) \\ &= H_C\left(\bigcup_{y\in X}(C_{1y} \cap \overline{C_A(y)})\right)(x) \\ &= H_C(C_A)(x) = C_{H(A)}(x), \end{aligned}$$

$$\begin{aligned} U_{\bar{R}(A)}(x) = \bigwedge_{y\in X}(U_R(x,y) \vee U_A(y)) &= \bigwedge_{y\in X}(H_U(U_{1y})(x) \vee U_A(y)) \\ &= \bigwedge_{y\in X} H_U(U_{1y} \cup \overline{U_A(y)})(x) \\ &= H_U\left(\bigcap_{y\in X}(U_{1y} \cup \overline{U_A(y)})\right)(x) \\ &= H_U(U_A)(x) = U_{H(A)}(x) \end{aligned}$$

$$\begin{aligned} F_{\bar{R}(A)}(x) = \bigwedge_{y\in X}(F_R(x,y) \vee F_A(y)) &= \bigwedge_{y\in X}(H_F(F_{1y})(x) \vee F_A(y)) \\ &= \bigwedge_{y\in X} H_F(F_{1y} \cup \overline{F_A(y)})(x) \\ &= H_F\left(\bigcap_{y\in X}(F_{1y} \cup \overline{F_A(y)})\right)(x) \\ &= H_F(F_A)(x) = F_{H(A)}(x) \end{aligned}$$

$H(A) = \overline{R}(A)$. Since L and H are dual operators and $H(A) = \overline{R}(A)$, we can easily show that $L(A) = \underline{R}(A)$.

From Theorem 4.1, it follows that axioms (QSVNSU1), (QSVNSL1) – (QSVNSL3), or equivalently, axioms (QSVNSL1), (QSVNSU1) – (QSVNSU3) are the basic axioms of quadripartitioned single-valued neutrosophic approximation operators. Then we have the following definition.

Definition 4.2. Let $L, H: QSVNS(X) \to QSVNS(X)$ be two dual operators. If L satisfies axioms (QSVNSL2) and (QSVNSL3) or equivalently H satisfies axioms (QSVNSU2) and (QSVNSU3), then the system $(QSVNS(X), \cup, \cap, c, L, H)$ is known as quadripartitioned single-valued neutrosophic rough set algebra, and L and H are called quadripartitioned single-valued neutrosophic lower and upper approximation operators respectively.

Next, we study axiomatic characterizations of some special classes of quadripartitioned single-valued neutrosophic approximation operators.

Theorem 4.2. Let $L, H: QSVNS(X) \to QSVNS(X)$ be two dual operators, then there exists a serial QSVNR R in X such that $\forall A \in QSVNS(X)$, $L(A) = \underline{R}(A), H(A) = \overline{R}(A)$ if and only if L satisfies axioms (QSVNSL2), (QSVNSL3) and one of the following equivalent axioms, or equivalently H satisfies axioms (QSVNSU2), (QSVNSU3) and one of the following equivalent axioms:

i. $(QSVNSL\,4) L(\phi) = \phi$
ii. $(QSVNSU\,4) H(U) = U$
iii. $(QSVNSL\,5) L(\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4) = \alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4$
iv. $(QSVNSU\,5) H(\alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4) = \alpha_1, \widehat{\alpha_2, \alpha_3}, \alpha_4$
v. $(QSVNSLU\,5) L(A) \subset H(A), \forall A \in QSVNS(X)$

Proof. It follows from Theorem 3.2(1) and 4.1.

Theorem 4.3. Let $L, H: QSVNS(X) \to QSVNS(X)$ be two dual operators, then there exists a reflexive QSVNR R in X such that $\forall A \in QSVNS(X)$, $L(A) = \underline{R}(A), H(A) = \overline{R}(A)$ if and only if L satisfies axioms (QSVNSL2), (QSVNSL3) and one of the following equivalent axioms, or equivalently H satisfies axioms (QSVNSU2), (QSVNSU3) and one of the following equivalent axioms:

i. $(QSVNSL6) L(A) \subset A, \forall A \in QSVNS(X)$
ii. $(QSVNSU6)\, A \subset H(A), \forall A \in QSVNS(X)$

Proof. It follows from Theorem 3.2(2) and 4.1

Theorem 4.4 Let $L, H: QSVNS(X) \to QSVNS(X)$ be two dual operators, then there exists a symmetric QSVNR R in X such that $\forall A \in QSVNS(X)$, $L(A) = \underline{R}(A), H(A) = \overline{R}(A)$ if and only if L satisfies axioms (QSVNSL2), (QSVNSL3) and one of the following equivalent axioms, or equivalently H satisfies axioms (QSVNSU2), (QSVNSU3) and one of the following equivalent axioms:

i. $(QSVNSL7)\, L(1_{X-\{x\}})(y) = L(1_{X-\{y\}})(x), \forall x, y \in X$
ii. $(QSVNSU7)\, H(1_x)(y) = H(1_y)(x), \forall x, y \in X$

Proof. It follows from Theorem 3.2(3) and 4.1

Theorem 4.5. Let $L, H: QSVNS(X) \to QSVNS(X)$ be two dual operators, then there exists a transitive QSVNR R in X such that $\forall A \in QSVNS(X)$, $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ if and only if L satisfies axioms (QSVNSL2), (QSVNSL3) and one of the following equivalent axioms, or equivalently H satisfies axioms (QSVNSU2), (QSVNSU3) and one of the following equivalent axioms:

i. $(QSVNSL8) L(A) \subset L(L(A)), \forall A \in QSVNS(X)$
ii. $(QSVNSU8) H(H(A)) \subset H(A), \forall A \in QSVNS(X)$

Proof. It follows from Theorem 3.2(4) and 4.1

5. An application of Quadripartitioned Single-Valued Neutrosophic Rough Sets

Definition 5.1. Let $n = (T_n, C_n, U_n, F_n)$ be a quadripartitioned single-valued neutrosophic number, $n^* = (T_{n^*}, C_{n^*}, U_{n^*}, F_{n^*}) = (1,1,0,0)$ be an ideal quadripartitioned single-valued neutrosophic number, then the cosine similarity measure between n and n^* is defined as follows.

$$S(n,n^*) = \frac{T_n T_{n^*} + C_n C_{n^*} + U_n U_{n^*} + F_n F_{n^*}}{\sqrt{T_n^2 + C_n^2 + U_n^2 + F_n^2}\sqrt{(T_{n^*})^2 + (C_{n^*})^2 + (U_{n^*})^2 + (F_{n^*})^2}}$$

Definition 5.2. Let A and B be two QSVNSs in X. We define the sum of A and B as

$$A \oplus B = \{\langle x, A(x) \oplus B(x) | x \in X\rangle\}; \text{ i.e.}$$

$$A \oplus B = \left\langle \begin{matrix} T_A(x) + T_B(x) - T_A(x)T_B(x), C_A(x) + C_B(x) - C_A(x)\,C_B(x), \\ U_A(x) + U_B(x) - U_A(x)\,U_B(x), F_A(x) + F_B(x) - F_A(x)F_B(x) \end{matrix} \right\rangle$$

Example 5.2. Consider the medical diagnosis decision procedure based on quadripartitioned single-valued neutrosophic rough sets on two universes. Let us consider the two universes. $U = \{x_1, x_2, x_3\}$ which denotes the set of diseases viral fever, common cold and stomach problem and $V = \{y_1, y_2, y_3\}$ be the set of symptoms tired, dry cough and stomach pain respectively. Let $R \in QSVNR(U \times V)$ be a QSVNR from U to V, where $\forall (x_i, y_j) \in U \times V, R(x_i, y_j)$ denotes the degree that the disease $x_i (x_i \in U)$ has the symptom $y_j (y_j \in V)$. According to medical knowledge statistic data, we can obtain the relation R.

Table 1. QSVNR R

R	x_1	x_2	x_3
x_1	(0,0.3,0.5,0.4)	(1,0.7,0.5,0.4)	(0.3,0.1,0.6,0.2)
x_2	(0,0.9,0.8,0.5)	(0.5,0,0.3,0.4)	(0.3,0.2,0.6,0.8)
x_3	(1,0.2,0.5,0.6)	(0.6,0.2,0.3,0.5)	(0,0.3,0.7,1)

Let $A = \{< x_1, (0.3,0.6,0.7,0.5) >, < x_2, (0,0.2,0.5,0.3) >, < x_3, (0.4,0.9,0.7,0.6) >\}$. By the Definition 2.5 the lower and upper approximations are calculated and hence given in detail below,

$$\underline{R}(A)(x_1) = (0.4,0.5,0.5,0.3), \overline{R}(A)(x_1) = (0.3,0.3,0.5,0.4)$$

$$\underline{R}(A)(x_2) = (0.4,0.3,0.7,0.3), \overline{R}(A)(x_2) = (0.3,0.6,0.5,0.4)$$

$$\underline{R}(A)(x_3) = (0.5,0.3,0.3,0.5), \overline{R}(A)(x_3) = (0.3,0.3,0.5,0.5)$$

By Definition 5.2,

$$\underline{R}(A) \oplus \overline{R}(A = \{\langle x_1, 0.58,0.65,0.75,0.58\rangle, \langle x_2, 0.58,0.72,0.85,0.58\rangle, \langle x_3, 0.65,0.51,0.65,0.75\rangle\}$$

By Definition 5.1,

$$S(n,n^*) = \frac{T_n T_{n^*} + C_n C_{n^*} + U_n U_{n^*} + F_n F_{n^*}}{\sqrt{T_n^2 + C_n^2 + U_n^2 + F_n^2}\sqrt{(T_{n^*})^2 + (C_{n^*})^2 + (U_{n^*})^2 + (F_{n^*})^2}}$$

$$S(n_{x_1}, n^*) = \frac{0.58 + 0.65}{\sqrt{0.58^2 + 0.65^2 + 0.75^2 + 0.58^2}\sqrt{1^2 + 1^2}} = 0.675$$

Similarly, we can obtain,

$$S(n_{x_2}, n^*) = 0.665, \; S(n_{x_3}, n^*) = 0.636$$

Here $S(n_{x_1}, n^*) > S(n_{x_2}, n^*) > S(n_{x_3}, n^*)$. So, the optimal decision is to select x_1. That is the patient A is suffering from viral fever x_1.

6. Conclusion

In this paper, we studied the framework of quadripartitioned single-valued neutrosophic rough sets through its axiomatic characterizations. And also, we have studied the properties of quadripartitioned single-valued neutrosophic rough sets. We also illustrate a numerical example in medical diagnosis to show the usefulness of quadripartitioned single-valued neutrosophic rough sets on two-universes.

References

[1] L. Zadeh, *Fuzzy Sets*, Information and Control 8(3) (1965) 87-96.

[2] K. Atanasov, *Intuitionistic Fuzzy Sets*, Fuzzy Sets and Systems 20 (1986) 87-96.

[3] F. Smarandache, *Neutrosophy, Neutrosophic Probability, Set, and Logic*, ProQuest Information & Learning, Ann Arbor, Michigan, USA, (1998) p 105.

[4] F. Smarandache, *A Unifying Field in Logics, Neutrosophy: Neutrosophic Probability, Set and Logic*, American Research Press, (1999) Rehoboth.

[5] Z. Pawlak, *Rough Sets*, International Journal of Computer and Information Sciences 11(5) (1982) 341-356.

[6] S. Broumi, F. Smarandache, *Rough Neutrosophic Sets*. Italian Journal of Pure and Applied Mathematics 32 (2014) 493-502.

[7] F. Smarandache, *n-Valued Refined Neutrosophic Logic and Its Applications in Physics*, Progress in Physics, 4 (2013) 143-146.

[8] H. Wang, F. Smarandache, Y. Q. Zhang, R. Sunderraman, *Single Valued Neutrosophic Sets*, Multispace Multistruct 4 (2010) 410-413.

[9] S Broumi, A. Dey, M. Talea, A. Bakali, F. Smarandache, D. Nagarajan, M. Lathamaheswari, R. Kumar, *Shortest Path Problem using Bellman Algorithm under Neutrosophic Environment*, Complex and Intelligent Systems, (2019) 1-8.

[10] S. Broumi, M. Talea, A. Bakali, F. Smarandache, D. Nagarajan, M. Lathamaheswari, M. Parimala, *Shortest Path Problem in Fuzzy, Intuitionistic Fuzzy and Neutrosophic Environment: An Overview*, Complex and Intelligent Systems 5 (2019) 371–378.

[11] S. Broumi, D. Nagarajan, A. Bakali, M. Talea, F. Smarandache, M. Lathamaheswari, *The Shortest Path Problem in Interval Valued Trapezoidal and Triangular Neutrosophic Environment*, Complex and Intelligent Systems 5 (2019) 391–402.

[12] H. L. Yang, *A Hybrid Model of Single-Valued Neutrosophic Sets and Rough Sets: Single-Valued Neutrosophic Rough Set Model*, Soft Computing 21 (2017) 6253-6267.

[13] N. D. Belnap, *A Useful Four-Valued Logic*, Modern Uses of Multiple-Valued Logic, (1977) 5-37.

[14] K. Mohana, M. Mohanasundari, *Quadripartitioned Single Valued Neutrosophic Rough Sets*, Nirmala Annual Research Congress (NARC-2018), ISBN-978-93-5321-859-1, 3 (2018) p. 165.

[15] R. Chatterjee, P. Majumdar, S. K. Samanta, *On Some Similarity Measures and Entropy on Quadripartitioned Single-Valued Neutrosophic Sets*, Journal of Intelligent and Fuzzy Systems 30 (2016) 2475-2485.