TURKISH JOURNAL OF SCIENCE
VOLUME 5, ISSUE 2, 110–117
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

A New View on Topological Polygroups

Gulay Oguz[a]

[a]*Department of Mathematics, Faculty of Arts and Sciences, Harran University, Şanlıurfa, TURKEY*

Abstract. Soft set theory, defined by Molodtsov as a novel mathematical tool modeling uncertainty, has been combined with many different discipline fields. In this article, the concept of soft topological polygroups is proposed by examining polygroups, a special class of hypergroups, with a soft topological approach. Also, several results have been obtained by establishing important characterizations related to this concept. In last, by presenting the definition of soft topological subpolygroups, some of their properties are examined.

1. Introduction

Hyperstructure theory, as a generalization of classical algebraic theory, was initiated by F. Marty at the eighth congress of Scandinavian Mathematicians in 1934 [2]. Although it does not have a long history, this theory has been used successfully in both applied and theoretical branches of mathematics. A special subclass of hypergroups, one of the most important hyperstructures, is polygroups. Polygroups studied by many researchers were defined by Ioulidis in 1981 [17]. Some algebraic and topological properties were investigated in detail. Davvaz and Poursalavati in [16] described matrix representations of polygroups over hyperrings. Subsequently, Davvaz introduced permutation polygroups and notions related to it [15]. Also, by examining the topological properties of this concept, the concept of topological polygroups was presented by Heidari et al. as a generalization of topological groups [19].

Another important theory in the basis of this study is soft set theory. In 1999, soft set theory was proposed by Molodtsov to resolve some complex problems involving uncertain data in engineering, medical science, economics, environment science [1]. This theory, which is a powerful mathematical approach for modeling uncertainties, has been studied algebraically and topologically by many mathematicians. Aktas and cagman presented the definition of soft groups [3]. Later on, Jun defined the notion of soft ideals on BCK/ BCI-algebras [8]. By defining the actions of soft groups, Oguz *et al*. examined the relation between the soft action and soft symmetric group [9]. Also, topological studies on soft sets were introduced by Shabir and Naz [6]. By proposing the definition of a soft topological space, they studied the separation axioms in a soft topological space. Aygunoglu and Aygun described soft product topologies and soft compactness [11]. Oguz *et al*. defined soft topological categories and obtained some important properties [7]. After that, Oguz proposed the concept of soft topological transformation groups [10]. On the other hand, soft hyperstructures are introduced by applying soft set theory to hyperstructures. Leoreanu-Fotea and Corsini [13] defined the concept of soft hypergroups. Yamak *et. al.* [12] introduced the notion of soft hypergroupoids. Morever, soft polygroups were studied by Wanga *et. al*. [14].

Corresponding author: GO, mail address: gulay.oguz@harran.edu.tr ORCID: https://orcid.org/0000-0003-4302-8401
Received: 29 March 2020; Accepted: 18 October 2020; Puplised: 31 October 2020
Keywords. Soft set, soft topological group, topological polygroup, soft topological polygroup.
2010 *Mathematics Subject Classification*. 03E99, 20N20, 22A05
Cited this article as: Oğuz G. A New View on Topological Polygroups. Turkish Journal of Science. 2020, 5(2), 110-117.

The main purpose of this study is to introduce the notion of soft topological polygroups by applying soft set theory to topological polygroups. In addition, some important properties of soft topological polygroups are examined and soft topological subpolygroups are studied.

2. Preliminaries

In this section, we review some fundamental notions and properties of soft sets and topological polygroups for the sake of completeness. See [1-4, 18].

Assume that X is an initial universe set and E is a set of parameters. Also, $P(X)$ denotes the power set of X and $A \subset E$. Then, Molodtsov defined the soft set follow as:

Definition 2.1. [1] *A pair $(\mathcal{F}, A)$ is said to be a soft set over X, where $\mathcal{F}$ is a mapping defined by*

$$\mathcal{F} : A \longrightarrow P(X)$$

Clearly, a soft set over X can be regarded as a parametrized family of subsets of the universe X.

Definition 2.2. [4] *Let $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ be two soft sets over the common universe X. Then, $(\mathcal{F}, A)$ is said to be a soft subset of $(\mathcal{G}, B)$ if*
i) *$A \subseteq B$,*
ii) *$\mathcal{F}(a)$ and $\mathcal{G}(a)$ are identical approximations for all $\alpha \in A$.*
We denote it as $(\mathcal{F}, A)\widetilde{\subset}(\mathcal{G}, B)$.

Definition 2.3. [4] *A soft set $(\mathcal{F}, A)$ over X is is said to be a null soft set denoted by Φ, if $\mathcal{F}(\alpha) = \emptyset$ for all $\alpha \in A$.*

Definition 2.4. [4] *A soft set $(\mathcal{F}, A)$ over X is is said to be an absolute soft set denoted by $\tilde{A}$, if $\mathcal{F}(\alpha) = X$ for all $\alpha \in A$.*

From an general perspective, the following notions are presented for the nonempty family $\{(\mathcal{F}_i, A_i)|i \in I\}$ of soft sets over the common universe X

Definition 2.5. [5] *The restricted intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft set $(\mathcal{F}, A) = \widetilde{\bigcap}_{i\in I}(\mathcal{F}_i, A_i)$ such that $A = \bigcap_{i\in I} A_i \neq \emptyset$ and $\mathcal{F}(a) = \bigcap_{i\in I} \mathcal{F}_i(a)$ for all $a \in A_i$.*

Definition 2.6. [5] *The restricted union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft set $(\mathcal{F}, A) = (\bigcup_{\mathcal{R}})_{i\in I}(\mathcal{F}_i, A_i)$ such that $A = \bigcap_{i\in I} A_i \neq \emptyset$ and $\mathcal{F}(a) = \bigcup_{i\in I} \mathcal{F}_i(a)$ for all $a \in A_i$.*

Definition 2.7. [5] *The extended union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft set $(\mathcal{F}, A) = \widetilde{\bigcup}_{i\in I}(\mathcal{F}_i, A_i)$ such that $A = \bigcup_{i\in I} A_i$ and $\mathcal{F}(a) = \bigcup_{i\in I(a)} \mathcal{F}_i(a)$, $I(a) = \{i \in I : a \in A_i\}$ for all $a \in A_i$.*

Definition 2.8. [5] *The extended intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft set $(\mathcal{F}, A) = (\bigcap_{\mathcal{E}})_{i\in I}(\mathcal{F}_i, A_i)$ such that $A = \bigcup_{i\in I} A_i$ and $\mathcal{F}(a) = \bigcap_{i\in I(a)} \mathcal{F}_i(a)$, $I(a) = \{i \in I : a \in A_i\}$ for all $a \in A_i$*

Definition 2.9. [5] *The $\wedge-$intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft set $(\mathcal{F}, A) = \widetilde{\bigwedge}_{i\in I}(\mathcal{F}_i, A_i)$ such that $A = \Pi_{i\in I} A_i$ and $\mathcal{F}((a_i)_{i\in I}) = \bigcap_{i\in I} \mathcal{F}_i(a_i)$ for all $(a_i)_{i\in I} \in A_i$.*

Definition 2.10. [5] *The $\vee-$intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft set $(\mathcal{F}, A) = \widetilde{\bigvee}_{i\in I}(\mathcal{F}_i, A_i)$ such that $A = \Pi_{i\in I} A_i$ and $\mathcal{F}((a_i)_{i\in I}) = \bigcup_{i\in I} \mathcal{F}_i(a_i)$ for all $(a_i)_{i\in I} \in A_i$.*

Now, we recall the definitions of polygroup and topological polygroup. Assume $P^*(P)$ be the set of all non-empty subsets of P.

Definition 2.11. [18] *A polygroup is a multi-valued system* $\mathcal{P} =< P, \circ, e,^{-1} >$, *where* $\circ : P \times P \longrightarrow P^*(P)$, $e \in P$, $^{-1}$ *is a unitary operation on* P *and the following conditions hold for all* $x, y, z \in P$: **i.** $(x \circ y) \circ z = x \circ (y \circ z)$, **ii.** $e \circ x = x \circ e = x$, **iii.** $x \in y \circ z$ *implies* $y \in x \circ z^{-1}$ *and* $z \in y^{-1} \circ x$.

Definition 2.12. [14] *Let* $\mathcal{P} =< P, \circ, e,^{-1} >$ *be a polygroup and* K *be a non-empty subset of* P. *Then* K *is said to be a subpolygroup if* $< K, \circ, e,^{-1} >$ *is itself a polygroup.*

The concept of polygroup is examined with the soft set theory and the concept of soft polygroup is defined as follows:

Definition 2.13. [14] *For a non-null soft set* $(\mathcal{F}, A)$ *over the polygroup* $\mathcal{P} =< P, \circ, e,^{-1} >$, $(\mathcal{F}, A)$ *is said to be a soft polygroup over* $\mathcal{P}$ *if and only if* $\mathcal{F}(a)$ *is a subpolygroup of* $\mathcal{P}$ *for all* $a \in Supp(F, A)$.

Definition 2.14. [19] *Let* $\mathcal{P} =< P, \circ, e,^{-1} >$ *be a polygroup and* (P, τ) *be a topological space. Then multi-valued system* $\mathcal{P} =< P, \circ, e,^{-1}, \tau >$ *is said to be a topological polygroup if the mappings* $^{-1} : P \longrightarrow P$ *and* $\circ : P \times P \longrightarrow P^*(P)$ *are continuous with respect to the the product topology on* $\tau \times \tau$ *and the topology* τ^* *on* $P^*(P)$ *which is generated by* $\mathfrak{B} = \{S_V | V \in \tau\}$, *where* $S_V = \{U \in P^*(P) | U \subseteq V, U \in \tau\}$.

Definition 2.15. [19] *Let* $\mathcal{P} =< P, \circ, e,^{-1}, \tau >$ *and* $\mathcal{P}' =< P', \circ', e',^{-1}, \tau' >$ *be two topological polygroups. A mapping* $\theta : P \longrightarrow P'$ *is called a* ***good topological homomorphism*** *if the following conditions are satisfied for all* $x, y \in P$:
i. $\theta(e) = e'$
ii. $\theta(x \circ y) = \theta(x) \circ' \theta(y)$
iii. θ *is continuous and open.*

Note that a good topological homomorphism is a topological isomorphism if the mapping θ is one to one and onto.

3. Soft Topological Polygroups

In this section, we define soft topological polygroups and present some of their features. From now on, $\mathcal{P}^*$ denotes the set of all subpolygroups of a polygroup $\mathcal{P} =< P, \circ, e,^{-1} >$ and $P^*(P)$ denotes the set of all non-empty subsets of P.

Definition 3.1. *Let* τ *be a topology on the polygroup* $\mathcal{P} =< P, \circ, e,^{-1} >$ *such that and* τ^* *be a topology on* P^*, *which is generated by* $\mathfrak{B} = \{S_V | V \in \tau\}$, *where* $S_V = \{U \in \mathcal{P}^* | U \subseteq V, U \in \tau\}$. *Let* $(\mathcal{F}, A)$ *be a non-null soft set over* $\mathcal{P}$. *The pair* $(\mathcal{F}, A)$ *is said to be a soft topological polygroup over* $\mathcal{P}$ *with the topology* τ *if the following axioms hold:*
i. $F(a)$ *is a subhpolygroup of* $\mathcal{P}$ *for all* $a \in Supp(F, A)$.
ii. *The mappings* $\circ : \mathcal{F}(a) \times \mathcal{F}(a) \longrightarrow P^*(\mathcal{F}(a))$ *and* $^{-1} : \mathcal{F}(a) \longrightarrow \mathcal{F}(a)$ *are continuous with respect to the topologies induced by* $\tau \times \tau$ *and* τ^* *for all* $a \in Supp(F, A)$.

It is to be noted that if $\mathcal{P}$ *is a topological polygroup, it is sufficient that only the first condition of the above definition is satisfied in order to the pair* $(\mathcal{F}, A)$ *to be defined as a soft topological polygroup. Namely, the soft topological polygroup* $(\mathcal{F}, A)$ *can be considered as a parameterized family of subpolygroups of the topological polygroup* $\mathcal{P}$.

Theorem 3.2. *Every soft polygroup on a topological polygroup is a soft topological polygroup.*

Proof. Let $\mathcal{P}$ be a topological polygroup and let $(\mathcal{F}, A)$ be a soft polygroup over $\mathcal{P}$ with the topology τ. Then $\mathcal{F}(a)$ is a subpolygroup of $\mathcal{P}$ for all $a \in A$. Hence, $\mathcal{F}(a)$ is a topological subpolygroup of $\mathcal{P}$ with recpect to the topologies induced by τ and τ^* for all $a \in A$. Therefore, $(\mathcal{F}, A)$ is also a soft topological polygroup over $\mathcal{P}$. □

Remark 3.3. *Each soft polygroup* $\mathcal{P}$ *can be transformed into a soft topological polygroup by equipping both* $\mathcal{P}$ *and* $P^*(P)$ *with discrete or indiscrete topology. However, every soft polygroup over a polygroup is not a soft topological polygroup.*

Theorem 3.4. *Let $\{(\mathcal{F}_i, A_i)|i \in I\}$ be a non-empty family of soft topological polygroups over $\mathcal{P}$ with the topology τ.*
i. *The restricted intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ with $\bigcap_{i\in I} A_i \neq \emptyset$ is a soft topological polygroup over $\mathcal{P}$ if $\widetilde{\bigcap}_{i\in I}(\mathcal{F}_i, A_i) \neq \emptyset$*
ii. *The extended intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological polygroup over $\mathcal{P}$ if $(\bigcap_{\mathcal{E}})_{i\in I}(\mathcal{F}_i, A_i) \neq \emptyset$*

Proof. **i**. The restricted intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ with $\bigcap_{i\in I} A_i \neq \emptyset$ defined as the soft set $\widetilde{\bigcap}_{i\in I}(\mathcal{F}_i, A_i) = (\mathcal{F}, A)$ such that $\bigcap_{i\in I} \mathcal{F}_i(a)$ for all $a \in A$. Choose $a \in Supp(F, A)$. Suppose $\bigcap_{i\in I} F_i(a) \neq \emptyset$ so that $\mathcal{F}_i(a) \neq \emptyset$ for all $i \in I$. Since $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a non-empty family of soft topological polygroup over $\mathcal{P}$ with the topology τ, $\mathcal{F}_i(a)$ is a topological polygroup of $\mathcal{P}$ for all $i \in I$. Then, $\bigcap_{i\in I} \mathcal{F}_i(a)$ is a topological subpolygroup of $\mathcal{P}$. Thus, $(\mathcal{F}, A)$ is a soft topological polygroup over $\mathcal{P}$ with the topology τ.
ii. The proof is similar to **i**. □

Theorem 3.5. *Let $\{(\mathcal{F}_i, A_i)|i \in I\}$ be a non-empty family of soft topological polygroups over $\mathcal{P}$ with the topology τ.*
i. *The extended union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological polygroup over $\mathcal{P}$ if $\mathcal{F}_i(x) \subseteq \mathcal{F}_j(x)$ or $\mathcal{F}_j(x) \subseteq \mathcal{F}_i(x)$ for all $i, j \in I$, $x \in \bigcup_{i\in I} A_i$*
ii. *The restricted union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological polygroup over $\mathcal{P}$ if $\mathcal{F}_i(x) \subseteq \mathcal{F}_j(x)$ or $\mathcal{F}_j(x) \subseteq \mathcal{F}_i(x)$ for all $i, j \in I$, $x \in \cap_{i\in I} A_i$ with $\bigcap_{i\in I} A_i \neq \emptyset$.*

Proof. **i**. Assume $(\mathcal{F}, A) = \widetilde{\bigcup}_{i\in I}(\mathcal{F}_i, A_i)$ as the extended union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ with $\bigcap_{i\in I} A_i \neq \emptyset$. Let $\mathcal{F}_i(x) \subseteq \mathcal{F}_j(x)$ or $\mathcal{F}_j(x) \subseteq \mathcal{F}_i(x)$ for all $i, j \in I$, $x \in \bigcup_{i\in I} A_i$. Choose $a \in Supp(\mathcal{F}, A)$. Since each $(\mathcal{F}_i, A_i)$ is non-null soft sets over $\mathcal{P}$, then $\bigcup_{i\in I}(\mathcal{F}_i, A_i)$ is also a non-null soft set over $\mathcal{P}$ for all $i \in I$. By the hypothesis, $\mathcal{F}_i(x) \subseteq \mathcal{F}_j(x)$ or $\mathcal{F}_j(x) \subseteq \mathcal{F}_i(x)$ for all $i, j \in I$, $x \in \bigcap_{i\in I} A_i$ with $\bigcap_{i\in I} A_i \neq \emptyset$ such that $\mathcal{F}_i(x)$ and $\mathcal{F}_j(x)$ are the topological subpolygroups of $\mathcal{P}$ and thus their union must be non-null too. Therefore, $\mathcal{F}(x)$ is a topological subpolygroup of $\mathcal{P}$. Hence, $(\mathcal{F}, A)$ is a soft topological polygroup over $\mathcal{P}$ with the topology τ.
ii. The proof is similar to that of **i**. □

From the above proposition, the following resultf is easily obtained:

Corollary 3.6. *Let $\{(\mathcal{F}_i, A_i)|i \in I\}$ be a non-empty family of soft topological polygroups over $\mathcal{P}$ with the topology τ. Then the extended union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological polygroup over $\mathcal{P}$ with the topology τ if $A_i \cap A_j \neq \emptyset$ for all $i, j \in I$, $i \neq j$.*

Theorem 3.7. *Let $\{(\mathcal{F}_i, A_i)|i \in I\}$ be a non-empty family of soft topological polygroups over $\mathcal{P}$ with the topology τ.*
i. *The $\wedge-$intersection $\widetilde{\bigwedge}_{i\in I}(\mathcal{F}_i, A_i)$ is a soft topological polygroup over $\mathcal{P}$ if it is non-null.*
ii. *The $\vee-$union $\widetilde{\bigvee}_{i\in I}(\mathcal{F}_i, A_i)$ is a soft topological polygroup over $\mathcal{P}$ if $\mathcal{F}_i(x_i) \subseteq \mathcal{F}_j(x_j)$ or $\mathcal{F}_j(x_j) \subseteq \mathcal{F}_i(x_i)$ for all $i, j \in I$, $x_i \in A_i$.*

Proof. **i**. Write $(\mathcal{F}, A) = \widetilde{\bigwedge}_{i\in I}(\mathcal{F}_i, A_i)$ for a non-empty family $\{(\mathcal{F}_i, A_i)|i \in I\}$ of soft topological polygroups over $\mathcal{P}$ with the topology τ. Let $a \in Supp(F, A)$. By the assumption, $\bigcap_{i\in I} \mathcal{F}_i(a_i) \neq \emptyset$ so that $\mathcal{F}_i(a_i) \neq \emptyset$ for all $i \in I$ and $(a_i)_{i\in I} \in A_i$. Hence, $\mathcal{F}_i(a_i)$ is a topological subpolygroup of $\mathcal{P}$ for all $i \in I$ so that their intersection must be a topological subpolygroup of $\mathcal{P}$ too. Thus, $(\mathcal{F}, A)$ is a soft topological polygroup over H with the topology τ.
□

Definition 3.8. *Let $\{(\mathcal{F}_i, A_i)|i \in I\}$ be a non-empty family of soft topological polygroups over $\mathcal{P}_i$ with the topologies τ_i. Then the cartesian product of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ over $\Pi_{i\in I} H_i$ with the product topology $\prod_{i\in I} \tau_i$ is denoted by $\Pi_{i\in I}(\mathcal{F}_i, A_i)$, is defined as $\Pi_{i\in I}(\mathcal{F}_i, A_i) = (\mathcal{F}, A)$ where $A = \Pi_{i\in I} A_i$ and $\mathcal{F}(x_i) = \Pi_{i\in I}\mathcal{F}_i(x_i)$ for all $(x_i)_{i\in I} \in A$.*

Theorem 3.9. *The cartesian product of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological polygroup over $\Pi_{i\in I} H_i$ with the product topology $\Pi_{i\in I}\tau_i$.*

Proof. Assume that $(\mathcal{F}_i, A_i)$ is a soft topological polygroup over $\mathcal{P}_i$ with the topology τ_i for all $i \in I$. Then, $\mathcal{F}_i(a) \neq \emptyset$ and $\mathcal{F}_i(a_i)$ a topological subpolygroup of $\mathcal{P}_i$ for all $(a_i)_{i\in I} \in Supp(\mathcal{F}_i, A_i)$. Thus, $\Pi_{i\in I}\mathcal{F}_i(a_i) \neq \emptyset$ and $\Pi_{i\in I}\mathcal{F}_i(a_i)$ a topological subpolygroup of $\Pi_{i\in I}\mathcal{P}_i$ with the product topology $\Pi_{i\in I}\tau_i$. Therefore, $\Pi_{i\in I}(Fi, Ai)$ is a soft topological polygroup over $\Pi_{i\in I}\mathcal{P}_i$. □

3.1. Soft Topological Polygroup Homomorphisms

Definition 3.10. *Let $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ be soft topological polygroups over $\mathcal{P}$ and $\mathcal{P}'$ with the topologies τ and τ', respectively. Let $\varphi : A \longrightarrow B$ and $\psi : \mathcal{P} \longrightarrow \mathcal{P}'$ be two mappings. Then, the pair (ψ, φ) is said to be a soft topological homomorphism if the following axioms hold:*
i. *ψ is a good homomorphism.*
ii. *$\psi(\mathcal{F}(a)) = \mathcal{K}(\varphi(a))$ for all $a \in Supp(\mathcal{F}, A)$.*
ii. *$\psi_a : (\mathcal{F}(a), \tau_{\mathcal{F}(a)}) \longrightarrow (\mathcal{K}(\varphi(a)), \tau'_{\mathcal{K}(\varphi(a))})$ is continuous and open for all $a \in Supp(\mathcal{F}, A)$.*

In this perspective,, it follows that a soft topological homomorphism (ψ, φ) is a mapping of soft topological polygroups. Therefore, we define a new category whose objects are soft topological polygroups and whose arrows are soft topological homomorphisms.

In addition, it can be said that $(\mathcal{F}, A)$ is soft topologically isomorphic to $(\mathcal{K}, B)$ if the mappings ψ and φ are one to one and onto.

Example 3.11. *Let $(\mathcal{K}, B)$ be a soft topological subpolygroup of $(\mathcal{F}, A)$ over $\mathcal{P}$. Together with the inclusion map $i : B \longrightarrow A$ and the identity map $\mathcal{I} : P \longrightarrow P$, the pair $(\mathcal{I}, i)$ is a soft topological homomorphism from $(\mathcal{K}, B)$ to $(\mathcal{F}, A)$.*

Example 3.12. *Let $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ be the two soft good homomorphic polygroups defined over $\mathcal{P}$ and $\mathcal{P}'$, respectively. Then $(\mathcal{F}, A)$ is soft topological homomorphic to $(\mathcal{K}, B)$ with discrete or anti-discrete topology. Thus, any soft good homomorphic polygroups can be regarded as soft topological homomorphic polygroups in the discrete or anti-discrete topology.*

Theorem 3.13. *Let the pair (ψ, φ) be a soft topological homomorphism from $(\mathcal{F}, A)$ to $(\mathcal{K}, B)$, where $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ are two soft topological polygroups over $\mathcal{P}$ and $\mathcal{P}'$, respectively. Then, $(\psi(\mathcal{F}), B)$ is a soft topological polygroup over $\mathcal{P}'$ if $\varphi : A \longrightarrow B$ be an injective mapping.*

Proof. Let $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ be two soft topological polygroups over $\mathcal{P}$ and $\mathcal{P}'$ with the topologies τ and τ', respectively. Then, $\mathcal{F}(a)$ is a topological subpolygroup of $\mathcal{P}$ for all $a \in Supp(F, A)$. Since $(\psi, \varphi) : (\mathcal{F}, A) \longrightarrow (\mathcal{K}, B)$ is a soft topological homomorphism, we have $\varphi(Supp(\mathcal{F}, A)) = Supp(\psi(\mathcal{F}), B)$. Choose $b \in Supp(\psi(\mathcal{F}), B)$. So there exist $a \in Supp(\mathcal{F}, A)$ such that $\varphi(a) = b$, thus we have $\mathcal{F}(a) \neq \emptyset$. Further, $\mathcal{F}(a)$ is a topological subpolygroup of $\mathcal{P}$ with respect to the topology induced by τ. Since ψ is a good topological homomorphism, then $\psi(\mathcal{F}(x))$ is a topological subpolygroup of $\mathcal{P}'$ with respect to the topology induced by τ'. Therefore, $(\psi(\mathcal{F}), B)$ is a soft topological polygroup over $\mathcal{P}'$ with the topology τ'. □

Theorem 3.14. *Let the pair (ψ, φ) be a soft topological homomorphism from $(\mathcal{F}, A)$ to $(\mathcal{K}, B)$, where $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ are two soft topological polygroups over $\mathcal{P}$ and $\mathcal{P}'$, respectively. Then, $(\psi^{-1}(\mathcal{K}), A)$ is a soft topological polygroup over $\mathcal{P}$ if it is non-null.*

Proof. Assume that $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ are two soft topological polygroups over $\mathcal{P}$ and $\mathcal{P}'$ with the topologies τ and τ', respectively. So for all $b \in Supp(\mathcal{K}, B)$, it is easy to show that $\varphi(Supp(\psi^{-1}(\mathcal{K}), A)) = \varphi^{-1}(Supp(\mathcal{K}, B))$. Let $a \in Supp(\psi^{-1}(\mathcal{K}), A)$, thus $\varphi(a) \in Supp(\mathcal{K}, B)$. Hence, the nonempty set $\mathcal{K}(\varphi(a))$ is a topological subpolygroup of $\mathcal{P}'$ with respect to the topology induced by τ'. Since ψ is a good topological homomorphism, then $\psi^{-1}(\mathcal{K}(\varphi(b))) = \psi^{-1}(\mathcal{K}(a))$ is a topological subpolygroup of $\mathcal{P}$ with respect to the topology induced by τ. Thus, it has been proven that the pair $(\psi^{-1}(\mathcal{K}), A)$ is a soft topological polygroup over $\mathcal{P}$ with the topology τ. □

Theorem 3.15. *Let $(\mathcal{F}, A)$, $(\mathcal{K}, B)$ and $(\mathcal{N}, C)$ be soft topological polygroups over $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{P}''$ with the topologies τ, τ' and τ'', respectively. Then, $(\psi' \circ \psi, \varphi' \circ \varphi) : (\mathcal{F}, A) \longrightarrow (\mathcal{N}, C)$ is a soft topological homomorphism if $(\psi, \varphi) : (\mathcal{F}, A) \longrightarrow (\mathcal{K}, B)$ and $(\psi', \varphi') : (\mathcal{K}, B) \longrightarrow (\mathcal{N}, C)$ are two soft topological homomorphisms.*

Proof. Suppose that $(\psi, \varphi) : (\mathcal{F}, A) \longrightarrow (\mathcal{K}, B)$ and $(\psi', \varphi') : (\mathcal{K}, B) \longrightarrow (\mathcal{N}, C)$ are two soft topological homomorphisms. Then, $\psi : P \longrightarrow P'$ and $\psi' : P' \longrightarrow P''$ are two good topological homomorphisms, and $\varphi : A \longrightarrow B$ and $\varphi' : B \longrightarrow C$ are two mappings such that the equalities $\psi(\mathcal{F}(a)) = \mathcal{K}(\varphi(a))$ and $\psi'(\mathcal{K}(b)) = \mathcal{N}(\varphi'(b))$ hold for all $a \in Supp(\mathcal{F}, A)$, $b \in Supp(\mathcal{K}, B)$. Obviously, $\psi' \circ \psi : \mathcal{P} \longrightarrow \mathcal{P}''$ is also good topological homomorphism and $\varphi' \circ \varphi : A \longrightarrow C$ is a mapping so that the equality

$$(\psi' \circ \psi)(\mathcal{F}(a)) = \psi'(\psi(\mathcal{F}(a))) = \psi'(\mathcal{K}(\varphi(a))) = \mathcal{N}(\varphi'(\varphi(a))) = \mathcal{N}((\varphi' \circ \varphi)(a))$$

holds for all $a \in Supp(\mathcal{F}, A)$. Thus, the pair $(\psi' \circ \psi, \varphi' \circ \varphi)$ is a soft topological homomorphism from $(\mathcal{F}, A)$ to $(\mathcal{N}, C)$. □

3.2. *Soft Topological Subpolygroups*

Definition 3.16. *Let $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ be soft topological polygroups over $\mathcal{P}$ with the topology τ. Then the pair $(\mathcal{K}, B)$ is said to be a soft topological subpolygroup of $(\mathcal{F}, A)$ if the following axioms hold :*
i. $B \subseteq A$.
ii. *$\mathcal{K}(b)$ is a subpolygroup of $\mathcal{F}(b)$ for all $b \in Supp(K, B)$.*
iii. *The mappings $\cdot : \mathcal{K}(b) \times \mathcal{K}(b) \longrightarrow P^*(\mathcal{K}(b))$ and*

$$^{-1} : \mathcal{K}(b) \longrightarrow \mathcal{K}(b)$$

are continuous for all $b \in Supp(\mathcal{K}, B)$.

Example 3.17. *Take a soft topological polygroup $(\mathcal{F}, A)$ over $\mathcal{P}$ with the topology τ. Then, $(\mathcal{F}|_B, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ if $B \subseteq A$.*

Theorem 3.18. *If $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ and $(\mathcal{N}, C)$ is a soft topological subpolygroup of $(\mathcal{K}, B)$, then $(\mathcal{N}, C)$ is the soft topological subpolygroup of $(\mathcal{F}, A)$.*

Proof. The proof follows from Definition 3.16. □

Theorem 3.19. *Let $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ be two soft topological polygroups over $\mathcal{P}$ with the topology τ. Then, $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ if $(\mathcal{K}, B)$ is a soft subset of $(\mathcal{F}, A)$.*

Proof. Suppose that $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ are two soft topological polygroups over $\mathcal{P}$ with the topology τ. Then, the nonempty sets $\mathcal{F}(x)$ and $\mathcal{K}(x)$ are the topological subpolygroup of $\mathcal{P}$. By the assumption, if $(\mathcal{K}, B)$ is a soft subset of $(\mathcal{F}, A)$, then $B \subseteq A$ and $\mathcal{K}(b) \subseteq \mathcal{F}(b)$ for all $b \in Supp(\mathcal{K}, B)$. So, $\mathcal{K}(b)$ is a topological subpolygroup of $\mathcal{F}(b)$ with respect to the topology induced by τ. From this fact, we conclude that $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ with the topology τ. □

Theorem 3.20. *Let $(\mathcal{F}, A)$ be a soft topological polygroup over $\mathcal{P}$ with the topology τ and $\{(\mathcal{F}_i, A_i)|i \in I\}$ be a non-empty family of soft topological subpolygroups of $(\mathcal{F}, A)$.*
i. *The restricted intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ with $\bigcap_{i\in I} A_i \neq \emptyset$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ if $\tilde{\bigcap}_{i\in I}(\mathcal{F}_i, A_i) \neq \emptyset$*
ii. *The extended intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ if $(\bigcap_{\mathcal{E}})_{i\in I} \neq \emptyset$*

Proof. **i**. The restricted intersection of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ with $\bigcap_{i\in I} A_i \neq \emptyset$ defined as the soft set $\tilde{\bigcap}_{i\in I}(\mathcal{F}_i, A_i) = (\mathcal{F}, A)$ such that $\mathcal{F}(a) = \bigcap_{i\in I} \mathcal{F}_i(a)$ for all $a \in A$. Let $a \in Supp(\mathcal{F}, A)$. Suppose $\bigcap_{i\in I} \mathcal{F}_i(a) \neq \emptyset$, which implies $\mathcal{F}_i(a) \neq \emptyset$ for all $i \in I$. Since $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a non-empty family of soft topological subpolygroups of $(\mathcal{F}, A)$, we get $A_i \subseteq A$ and $\mathcal{F}_i(a)$ is a topological subpolygroup of $\mathcal{F}(a)$ with respect to the topology induced by τ for all $i \in I$. Hence, $\bigcap_{i\in I} A_i \subseteq A$ and $\bigcap_{i\in I} \mathcal{F}_i(a)$ is a topological subpolygroup of $\mathcal{F}(a)$. Consequently, the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological subpolygroup of $(\mathcal{F}, A)$
ii. The proof is similar to **i**. □

Theorem 3.21. *Let $\{(\mathcal{F}_i, A_i)|i \in I\}$ be a non-empty family of soft topological subpolygroups of a soft topological polygroup $(\mathcal{F}, A)$ over $\mathcal{P}$ with the topology τ.*
i. *The extended union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ if $f_i(x) \subseteq f_j(x)$ or $f_j(x) \subseteq f_i(x)$ for all $i, j \in I$, $x \in \bigcup_{i\in I} A_i$*
ii. *The restricted union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ if $f_i(x) \subseteq f_j(x)$ or $f_j(x) \subseteq f_i(x)$ for all $i, j \in I$, $x \in \bigcap_{i\in I} A_i$ with $\bigcap_{i\in I} A_i \neq \emptyset$.*

Proof. **i**. Suppose that $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a non-empty family of soft topological subpolygroups of a soft topological polygroup $(\mathcal{F}, A)$ with $\bigcap_{i\in I} A_i \neq \emptyset$. Let $\mathcal{F}_i(x) \subseteq \mathcal{F}_j(x)$ or $\mathcal{F}_j(x) \subseteq \mathcal{F}_i(x)$ for all $i, j \in I$, $x \in \bigcup_{i\in I} A_i$. Take $a \in Supp(F, A)$. Since each $(\mathcal{F}_i, A_i)$ is non-null soft sets over $\mathcal{P}$, then $\widetilde{\bigcup}_{i\in I}(\mathcal{F}_i, A_i)$ is also a non-null soft set over $\mathcal{P}$ for all $i \in I$. By assumption, $\mathcal{F}_i(a) \subseteq \mathcal{F}_j(a)$ or $\mathcal{F}_j(a) \subseteq \mathcal{F}_i(a)$ for all $i, j \in I$, $a \in \bigcap_{i\in I} A_i$ with $\bigcap_{i\in I} A_i \neq \emptyset$ such that $\mathcal{F}_i(a)$ and $\mathcal{F}_j(a)$ are the topological subpolygroups of $\mathcal{F}(a)$ with respect to the topology induced by τ and so their union must be non-null too. This show that the extended union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ with the topology τ.
ii. The proof is similar to **i**.
□

Corollary 3.22. *Let $\{(\mathcal{F}_i, A_i)|i \in I\}$ be a non-empty family of soft topological subpolygroups of a soft topological polygroup $(\mathcal{F}, A)$ over $\mathcal{P}$ with the topology τ. Then the extended union of the family $\{(\mathcal{F}_i, A_i)|i \in I\}$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ with the topology τ if $A_i \cap A_i \neq \emptyset$ for all $i, j \in I$, $i \neq j$.*

Theorem 3.23. *Let $\{(\mathcal{F}_i, A_i)|i \in I\}$ be a non-empty family of soft topological polygroups over $\mathcal{P}$ with the topology τ and let $(\mathcal{K}_i, B_i)$ be a soft topological subpolygroup of $(\mathcal{F}_i, A_i)$ for all $i \in I$.*
i. *The $\wedge$−intersection $\widetilde{\bigwedge}_{i\in I}(\mathcal{K}_i, B_i)$ is a soft topological subpolygroup of $\widetilde{\bigwedge}_{i\in I}(\mathcal{F}_i, A_i)$ if it is non-null.*
ii. *The $\vee$−union $\widetilde{\bigvee}_{i\in I}(\mathcal{K}_i, B_i)$ is a soft topological subpolygroup of $\widetilde{\bigvee}_{i\in I}(\mathcal{F}_i, A_i)$ if $\mathcal{K}_i(b_i) \subseteq \mathcal{K}_j(b_j)$ or $\mathcal{K}_j(b_j) \subseteq \mathcal{K}_i(b_i)$ for all $i, j \in I$, $b_i \in B_i$.*

Proof. **i**. Consider $\{(\mathcal{F}_i, A_i)|i \in I\}$ as a non-empty family of soft topological polygroups over $\mathcal{P}$ with the topology τ. By 3.5 Theorem (ii), $\widetilde{\bigvee}_{i\in I}(\mathcal{F}_i, A_i)$ is also a soft topological polygroup over $\mathcal{P}$ with the topology τ. Choose $b_i \in Supp(\mathcal{K}_i, B_i)$. By the assumption, $\bigcap_{i\in I} \mathcal{K}_i(b_i) \neq \emptyset$ such that $\mathcal{K}_i(b_i) \neq \emptyset$ for all $i \in I$ and $(b_i)_{i\in I} \in B_i$. Also, $B_i \subseteq A_i$ and $\mathcal{K}_i(b_i)$ is a topological subpolygroup of $\mathcal{F}_i(b_i)$ with respect to the topology induced by τ for all $i \in I$ so that $\bigcap_{i\in I} B_i \subseteq \bigcap_{i\in I} A_i$ and $\bigvee_{i\in I}(\mathcal{K}_i(b_i))$ must be a topological subpolygroup of $\bigvee_{i\in I}(\mathcal{F}_i(b_i))$ too. So, $\widetilde{\bigwedge}_{i\in I}(\mathcal{K}_i, B_i)$ is a soft topological subpolygroup of $\widetilde{\bigwedge}_{i\in I}(\mathcal{F}_i, A_i)$ with the topology τ
ii. The proof is similar to **i**. □

Theorem 3.24. *Let $(\mathcal{F}, A)$ be a soft topological polygroup over $\mathcal{P}$ with the topology τ and $(\mathcal{K}, B)$ be a soft topological subpolygroup of $(\mathcal{F}, A)$.*
i. *The restricted intersection of $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ if it is non-null.*
ii. *The restricted union of $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ if it is non-null.*

Proof. **i**. Assume that $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ over $\mathcal{P}$ with the topology τ. If it is non-null, it follows that $B \subseteq A$ and $\mathcal{K}(b)$ is a topological subpolygroup of $\mathcal{F}(b)$ with respect to the topology induced by τ for all $b \in Supp(\mathcal{K}, B)$. Thus, it is easy to see that $A \cap B \subseteq A$ and $\mathcal{K}(b) \cap \mathcal{F}(b)$ is also a topological subhypergroupoid of $\mathcal{F}(b)$ with respect to the topology induced by τ for all $b \in Supp(\mathcal{K}, B)$. Therefore, the restricted intersection $(\mathcal{F}, A)\tilde{\cap}(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ with the topology τ.
ii. The proof is similar to **i**. □

Theorem 3.25. *Let $f : P \longrightarrow P'$ be a good homomorphism of topological polygroups with the topologies τ and τ', respectively, and let $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ be two soft topological polygroups over $\mathcal{P}'$. Then, $(f^{-1}(\mathcal{K}), B)$ is a soft topological subpolygroup of $(f^{-1}(\mathcal{F}), A)$ if $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ with the topology τ.*

Proof. Assume $(\mathcal{K}, B)$ be a soft topological subpolygroup of $(\mathcal{F}, A)$ over $\mathcal{P}$ with the topology τ'. Take $b \in Supp(f^{-1}(\mathcal{K}), B)$. Since $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$, it follows that $B \subseteq A$ and $(\mathcal{K}(b))$ is a topological subpolygroup of $(\mathcal{F}(b)$ with respect to the topology induced by τ' for all $b \in Supp(f^{-1}(\mathcal{K}), B)$. Morever, since $f : \mathcal{P} \longrightarrow \mathcal{P}'$ be a good topological homomorphism, then $f^{-1}(\mathcal{F})(b) = f^{-1}(\mathcal{F}(b))$ is a topological subpolygroup of $f^{-1}(\mathcal{K})(b) = f^{-1}(\mathcal{K}(b))$ with respect to the topology induced by τ for all $b \in Supp(f(\mathcal{K}), B)$. This proves that $(f^{-1}(\mathcal{K}), B)$ is a soft topological subpolygroup of $(f^{-1}(\mathcal{F}), A)$ with the topology τ.
□

Theorem 3.26. *Let $f : P \longrightarrow P'$ be a good homomorphism of topological polygroups with the topologies τ and τ', respectively, and let $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ be two soft topological polygroups over $\mathcal{P}$. Then, $(f(\mathcal{K}), B)$ is a soft topological subpolygroup of $(f(\mathcal{F}), A)$ over P' with the topology τ' if $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ with the topology τ.*

Proof. Suppose that $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$ over $\mathcal{P}$ with the topology τ. If $(\mathcal{K}, B)$ is a soft topological subpolygroup of $(\mathcal{F}, A)$, it follows that $B \subseteq A$ and $(\mathcal{K}(b))$ is a topological subpolygroup of $(\mathcal{F}(b)$ with respect to the topology induced by τ for all $b \in Supp(\mathcal{K}, B)$. Furthermore, since $f : P \longrightarrow P'$ be a good topological homomorphism, so $f(\mathcal{F})(b) = f(\mathcal{F}(b))$ is a topological subpolygroup of $f(\mathcal{K})(b) = f(\mathcal{K}(b))$ with respect to the topology induced by τ' for all $b \in Supp(f(\mathcal{K}), B)$. Therefore, $(f(\mathcal{K}), B)$ is a soft topological subpolygroup of $(f(\mathcal{F}), A)$. □

References

[1] Molodtsov, D. A.(1999). Soft set theory-First results. *Comput. Math. Appl.*, 37(4-5), 19-31.
[2] Marty, F. (1934). Sur une Generalisation de la Notion de Groupe. 8th Congress Mathematiciens Scandinaves, Stockholm, pp. 45–49.
[3] Aktas, H., Cagman, N.(2007). Soft sets and soft groups. *Inform. Sci.*, 77(13), 2726–2735.
[4] Maji, P. K., Biswas, R., Roy, A. R.(2003). Soft set theory. *Comput. Math. Appl.*, 45(4-5), 555–562.
[5] Kazanci, O., Yilmaz, S., Yamak, S.(2010). Soft sets and soft BCH-algebras. *Hacettepe J. Math. Stat.*, 39, 205–217.
[6] Shabir, M., Naz, M.(2011). On soft topological spaces. *Comput. Math. Appl.*, 61(7), 1786-1799.
[7] Oguz, G., Gursoy, M.H., Icen, I.(2019). On soft topological categories. *Hacet. J. Math. Stat., 48*(6), 1675–1681.
[8] Jun, Y.B. (2008). Soft BCK/BCI-algebras. *Comput. Math. Appl.*, 56, 1408–1413.
[9] Oguz, G., Icen, I., Gursoy, M.H.(2019). Actions of soft groups.*Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 68(1), 1163–1174.
[10] Oguz, G. (2020). Soft Topological Transformation Groups. *Mathematics*, 8(9), 1545.
[11] Aygunoglu, A., Aygun, A.(2012). Some notes on soft topological spaces. *Neural Comput. Appl.*, 22(1), 113–119.
[12] Yamak, S., Kazanci, O., Davvaz, B.(2011). Soft hyperstructure. *Comput. Math. with Appl.*, 62(2), 797–803.
[13] Leoreanu Fotea, V., Corsini, P.(2010) Soft hypergroups. Critical Review. A publication of Society for Mathematics of Uncertainty, Creighton University - USA, July , vol. IV, 81–97
[14] Wanga, J., Yin, M., Gua, W.(2011) Soft polygroups. *Comp. Math. Appl.*, 62, 3529–3537.
[15] Davvaz, B.(2000) On polygroups and permutation polygroups. *Math. Balkanica (N.S.)*, 14 (1-2), 41–58.
[16] Davvaz, B., Poursalavati N. S.(1999). On polygroup hyperrings and representations of polygroups. *J. Korean Math. Soc.*, 36 (6), 1021–1031.
[17] Ioulidis, S.(1981). Polygroups et certain de leurs properties. *Bull. Greek Math. Soc.,* 22, 95-–104.
[18] Comer, S.D.(1984). Polygroups derived from cogroups. *J. Algebra*, 89, 397—405
[19] Heidari, D., Davvaz, B., Modarres, S. M. S. (2016). Topological polygroups. *B. Malays. Math. Sci. Soc.*, 39(2), 707–721.