Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (2) (2020), 494 – 497
DOI : 10.15672/hujms.712019

Research Article

Statistical cluster point and statistical limit point sets of subsequences of a given sequence

Harry I. Miller, Leila Miller-Van Wieren*

Faculty of Engineering and Natural Sciences, International University of Sarajevo, Sarajevo, 71000, Bosnia-Herzegovina

Abstract

J.A. Fridy [Statistical limit points, Proc. Amer. Math. Soc., 1993] considered statistical cluster points and statistical limit points of a given sequence x. Here we show that almost all subsequences of x have the same statistical cluster point set as x. Also, we show an analogous result for the statistical limit points of x.

Mathematics Subject Classification (2010). 40D25, 40G99, 28A12

Keywords. sequences, subsequences, statistical cluster points, statistical limit points

1. Introduction

Fridy [1] has proven that Γ_x, the set of statistical cluster points of $x = (x_n)$, is always a closed set and Γ_x is non-empty if x is bounded. However Λ_x, the set of statistical limit points of x, need not be closed. In [2] H.I. Miller studied statistical convergence and relations between statistical convergence of a sequence x and statistical convergence of the subsequences of x. In particular, in [2], it is shown that if L is the statistical limit of x, then almost all subsequences of x have L as their statistical limit. Here we combine two notions, statistical cluster points and subsequences, showing that Γ_x is equal to the statistical cluster point set of almost all subsequences of x. This is a continuation of the results in [3] that also combine statistical cluster points and subsequences. Namely, in [3] it is shown that if $\Gamma_x \neq \emptyset$ and F is a non-empty closed subset of Γ_x, then there exists a subsequence y of x such that $\Gamma_y = F$. Additionally we show that Λ_x is equal to the statistical limit point set of almost all subsequences of x. This is a continuation of the results in [4] that also combine statistical limit points and subsequences.

2. Preliminaries

If $t \in (0, 1]$, then t has a unique binary expansion $t = \sum_{n=1}^{\infty} \frac{e_n}{2^n}, e_n \in \{0, 1\}$, with infinitely many ones. Next if $x = (x_n)$ is a sequence of reals, for each $t \in (0, 1]$, let $x(t)$ denote the subsequence of x obtained by the following rule: x_n is in the subsequence if and only if $e_n = 1$. Clearly the mapping $t \to x(t)$ is a one-to-one onto mapping between $(0, 1]$ and the collection of all subsequences of x.

*Corresponding Author.
Email addresses: himiller@hotmail.com (H.I. Miller), lmiller@ius.edu.ba (L.M. Wieren)
Received: 10.07.2016; Accepted: 06.10.2016

If K is a subset of the positive integers N, then following Fridy [1], K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the number of elements in K_n. The natural density of K (see [5]) is given by $\delta(K) = \lim_{n\to\infty} n^{-1}|K_n|$, provided this limit exists. In the case that $\delta(K) = 0$ we say that K is thin, and otherwise we say that K is non-thin.

Statistical convergence of a sequence is defined as follows.

We say that L is the statistical limit of the sequence x, if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$

Statistical convergence and its connection to subsequences is studied in [2].

Statistical limit points and statistical cluster points of a sequence x are defined as follows.

We say that a number λ is a statistical limit point of a sequence of reals $x = (x_n)$ if $\lim_{k\to\infty} x_{n_k} = \lambda$ for some non-thin subsequence of (x_n).

We say that a number γ is a statistical cluster point of a sequence of reals (x_n) if for every $\epsilon > 0$ the set $\{k \in N : |x_k - \gamma| < \epsilon\}$ is non-thin.

In [1], given a sequence x, three sets are considered. L_x, the set of limit points of x; Λ_x, the set of statistical limit points of x, and Γ_x, the set of statistical cluster points of x. Also, if x is bounded, then Γ_x is closed and non-empty.

In this paper we want to examine, Γ_x and its relation to $\Gamma_{x(t)}$. Additionally we also consider Λ_x and its relation to $\Lambda_{x(t)}$.

3. Results

Our main result is the following.

Theorem 3.1. If $x = (x_n)$ is a bounded sequence, then $\Gamma_x = \Gamma_{x(t)}$ for almost all $t \in (0, 1]$ (in the sense of Lebesgue measure).

Proof. Since Γ_x is closed, it is either finite or separable, i.e. there is a countable subset of Γ_x, $\{l_n : n \in N\}$ such that its closure is Γ_x. We consider only the second case, the proof in the first case is much simpler.

First we show that $\Gamma_x \subseteq \Gamma_{x(t)}$ for almost all t. It is sufficient to show that $m(B_n) = 1$ for $n = 1, 2, \dots$ where $B_n = \{t \in (0, 1] : l_n \in \Gamma_{x(t)}\}$. This is true since in that case $m(B) = 1$ for $B = \bigcap_{n=1}^{\infty} B_n$ and then $\{l_n : n \in N\} \subseteq \Gamma_{x(t)}$ for all $t \in B$ and consequently $\Gamma_x \subseteq \Gamma_{x(t)}$ for all $t \in B$.

Since $l_n \in \Gamma_x$, then for every $\epsilon > 0$, $\{k \in N : |x_k - l_n| < \epsilon\}$ is non-thin . If $\epsilon = \frac{1}{j}$ we can denote the above set by $\{k_1^j, k_2^j, k_3^j, \dots\}$. Then, since it is non-thin there exists $\delta_j > 0$ such that

$$\frac{1}{p}|\{i : k_i^j \leq p\}| > \delta_j$$

for infinitely many p. We can assume that $p = k_M^j$ for infinitely many sufficiently large M. Now for each j, by the Law of Large Numbers, the limiting frequency of $x_{k_i^j}$ $i = 1, 2, \dots$ among the sequence $x(t)$ is $\frac{1}{2}$ for almost all $t \in (0, 1]$, i.e. if $t = \sum_{m=1}^{\infty} \frac{e_m}{2^m}$, then $\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} t_{k_i}^j = \frac{1}{2}$ for almost all $t \in (0, 1]$. That is, $m(D_j) = 1$, where

$$D_j = \{t \in (0, 1] : \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} t_{k_i}^j = \frac{1}{2}\} \tag{3.1}$$

for all j. Hence if $D = \bigcap_{j=1}^{\infty} D_j$, $m(D) = 1$. Now we will check that l_n is a statistical cluster point for each t in D.

To see this we will show that $\{i \in N : |x(t)_i - l_n| < \frac{1}{j}\}$ is non-thin for every $j \in N$ and every $t \in D_j$.

Consider the earlier mentioned $p = k_M^j$ for M large enough. Then the number of such $i \leq p$, with $|x_i - l_n| < \frac{1}{j}$ is greater than $p\delta_j$. Now take $t \in D_j$. By (3.1), $\lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^m t_{k_i}^j = \frac{1}{2}$. So for large M, $p = k_M^j$, we have

$$\frac{1}{p}|\{i \leq p : |x(t)_i - l_n| < \frac{1}{j}\}| > \frac{\delta_j}{4},$$

i.e. this holds for infinitely many p, i.e. $\{i \in N : |x(t)_i - l_n| < \frac{1}{j}\}$ is non-thin for every $j \in N$ and every $t \in D_j$. Hence l_n is a statistical cluster point for every $t \in D$. This completes the proof that $\Gamma_x \subseteq \Gamma_{x(t)}$ for almost all t.

Next we show that $\Gamma_{x(t)} \subseteq \Gamma_x$ for almost all t. We will show that this inclusion holds for all normal $t \in (0, 1]$, i.e. for all $t = \sum_{n=1}^\infty \frac{e_n}{2^n}$ for which $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n e_i = \frac{1}{2}$. It is well known that almost all $t \in (0, 1]$ are normal (see [5]).

Suppose that l is a statistical cluster point of $x(t)$ for some normal t. Then for any $\epsilon > 0$, $\{i : |(x(t))_i - l| < \epsilon\}$ is non-thin, i.e. there exists $\delta_\epsilon > 0$ such that

$$\frac{1}{n}|i \leq n : |(x(t))_i - l| < \epsilon| > 2\delta_\epsilon$$

for infinitely many n. This implies that

$$\frac{1}{n}|i \leq n : |x_i - l| < \epsilon| > \frac{1}{2}\delta_\epsilon$$

for infinitely many n, and hence l is a statistical cluster point of x. Therefore $\Gamma_{x(t)} \subseteq \Gamma_x$ for all normal t, and consequently for almost all $t \in (0, 1]$. Therefore we conclude that $\Gamma_{x(t)} = \Gamma_x$ for almost all $t \in (0, 1]$. □

Next, we will prove an analogous result for the set of statistical limit points of x and its subsequences. The set Λ_x is not necessarily closed (see [4]). However the following useful theorem was proved by Kostyrko, Mačaj, Šalat and Strauch [4].

Theorem 3.2. For every bounded sequence x, the set Λ_x is an F_σ-set in R.

In the proof of the above theorem, the authors show that

$$\Lambda_x = \bigcup_{j=1}^{\infty} \Lambda(x, \frac{1}{j})$$

where $\Lambda(x, \frac{1}{j}) = \{l, \exists k_i, i = 1, 2\ldots, \lim_{i\to\infty} x_{k_i} = l, \overline{\delta}(\{k_i\}) \geq \frac{1}{j}\}$ where $\overline{\delta}$ denotes the upper statistical density (i.e. $\overline{\delta}(\{k_i\}) = \limsup_{i\to\infty} \frac{i}{k_i}$) and $\Lambda(x, \frac{1}{j})$ is closed for all j.

Here is our second result.

Theorem 3.3. If $x = (x_n)$ is a bounded sequence, then $\Lambda_x = \Lambda_{x(t)}$ for almost all $t \in (0, 1]$ (in the sense of Lebesgue measure).

Proof. We proceed in a similar manner as in the proof of Theorem 3.1.

First we show that $\Lambda_x \subseteq \Lambda_{x(t)}$ for almost all t.

As mentioned earlier, $\Lambda_x = \bigcup_{j=1}^\infty T_j$, where

$$T_j = \Lambda(x, \frac{1}{j}) = \{l, \exists k_i, i = 1, 2\ldots, \lim_{i\to\infty} x_{k_i} = l, \overline{\delta}(\{k_i\}) \geq \frac{1}{j}\}.$$

Suppose $j \in N$ is fixed. Using the above notation (from [4]), T_j is closed and separable so there exists a set $\{l_{ij} : i \in N\}$ such that its closure is T_j. Let $i \in N$. If $l = l_{ij}$, then by the Law of Large Numbers, $l \in \Lambda(x(t), \frac{1}{4j})$, for all $t \in B_{ij}$, where $m(B_{ij}) = 1$. Let $B_j = \bigcap_{i=1}^\infty B_{ij}$. Then $m(B_j) = 1$. Hence $\{l_{ij} : i \in N\} \subseteq \Lambda(x(t), \frac{1}{4j})$ for every $t \in B_j$. Now since T_j and $\Lambda(x(t), \frac{1}{4j})$ are both closed we get that $T_j \subseteq \Lambda(x(t), \frac{1}{4j})$ for every $t \in B_j$.

Therefore $\Lambda_x = \bigcup_{j=1}^{\infty} T_j \subseteq \bigcup_{j=1}^{\infty} \Lambda(x(t), \frac{1}{4j}) = \Lambda_{x(t)}$ for all $t \in \bigcap_{j=1}^{\infty} B_j$. Since $m(\bigcap_{j=1}^{\infty} B_j) = 1$, we have shown that $\Lambda_x \subseteq \Lambda_{x(t)}$ for almost all t.

Next we show that $\Lambda_{x(t)} \subseteq \Lambda_x$ for almost all t. Again we show that this inclusion holds for all normal $t \in (0,1]$. Suppose that l is a statistical limit point of $x(t)$ for some normal t. Then $x(t)$ has a non-thin subsequence that converges to l (in the normal sense). It is easy to see that this subsequence $x(t)_i = x_{k_i}$ is then also a non-thin subsequence of x and therefore l is also a statistical limit point of x. This completes the proof. □

4. Concluding remarks

We mentioned that $m(\nu) = 1$, where ν is the set of normal numbers in $(0,1]$. However ν is a set of first Baire category. In light of this we suspect that a category analogue of our Theorem 3.1 is not true.

Also, one could examine possible analogues of our results using permutations rather than subsequences.

References

[1] J.A. Fridy, *Statistical limit points*, Proc. Amer. Math. Soc. **118** 1187–1192, 1993.

[2] H.I. Miller, *Measure theoretical subsequence characterization of statistical convergence*, Trans. Amer. Math. Soc. **347** (5) 1811–1819, 1995.

[3] H.I. Miller and L. Miller-Van Wieren, *Some statistical cluster point theorems*, Hacet. J. Math. Stat. **44** (6) 1405–1409, 2015.

[4] P. Kostyrko, M. Mačaj, T. Šalat, and O. Strauch, *On statistical limit points*, Proc. Amer. Math. Soc. **129** (9), 2647–2654, 2000.

[5] J.C. Oxtoby, *Measure and Category: A survey of the analogies between topological and measure spaces*. Second edition, Springer-Verlag, New York-Berlin,1980.