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New Fixed Point Results for Generalized Θ-Contraction in Extended G_b-Metric Spaces with an Application

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Highlights

• The paper focuses on defining a new metric space, namely extended G_b-metric space.
• The notion of generalized Geraghty type Θ-berinde contraction mapping is proposed.
• Eventually an application is presented to emphasize the main result.

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Abstract

Through this work, we analyze the structure of extended G_b-metric spaces and show a fundamental lemma for sequence convergence within the same metric. We also propose the new notion of generalized geraghty type Θ-berinde contraction mappings and demonstrate several fixed point theorems for these mappings in the sense of extended G_b-metric spaces. Eventually, the existence result for solutions of a Fredholm integral equation is furnished to show the efficacy of the technique developed.

1. INTRODUCTION

Over the past few centuries, fixed point theory has been one of the increasingly significant fields of research in nonlinear functional analysis. Mustafa and Sims [1] initiated the idea of G-metric spaces, which is subsequently studied and proposed to acquire various types of fixed point theorems, see [2-7]. Based on the ideas of G-metric spaces (GMS) and *b*-metric spaces, Aghajani et al. examined the conception of G_b-metric spaces (G_bMS). Furthermore, in the sense of G_p-metric spaces, Zand and Nezhad [8] presented several fixed-point theorems related to GMS and partial metric spaces. Currently, some Geraghty kind contraction theorems have been explored in different metric spaces, see [9-12].

In this article, we reveal the new sort of contraction in the context of extended G_b-metric spaces (EG_bMS), namely generalized geraghty type Θ-berinde contraction (GGTΘB contraction). We also provide an application for proving an existence result for a fredholm integral equation to illustrate the effectiveness of the research being done.

2. PRELIMINARIES

Bakhtin [13] brought the idea of *b*-metric space in 1989 and then it was utilized broadly by Czerwik [14] and the rest.

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Definition 2.1. Let Ϻ is a non-empty set and ş ≥ 1 be a real number. A function $\widetilde{d_b}$: Ϻ × Ϻ → [0, ∞) is a *b*-metric on Ϻ if for all ῤ, ϰ̃, ṽ ϵ Ϻ, it fulfills:

(1) $\widetilde{d_b}$(ῤ, ϰ̃) = 0 if and only if ῤ = ϰ̃;
(2) $\widetilde{d_b}$(ῤ, ϰ̃) = $\widetilde{d_b}$(ϰ̃, ῤ);
(3) $\widetilde{d_b}$(ῤ, ϰ̃) ≤ ş[$\widetilde{d_b}$(ῤ, ṽ) + $\widetilde{d_b}$(ṽ, ϰ̃)].

The pair (Ϻ, $\widetilde{d_b}$) is named a *b*-metric space.

To the other side, there is a metric called *p*-metric space, was introduced by Parvaneh [15].

Definition 2.2. Let Ϻ is a nonempty set. A function $\widetilde{d_p}$: Ϻ × Ϻ → [0, ∞) is a *p*-metric if there exists a strictly increasing continuous function (abbreviated SIC function hereafter) Ω̃ : [0, ∞) → [0, ∞) with ŧ ≤ Ω̃(ŧ) for ŧ ϵ [0, ∞) such that for all ῤ, ϰ̃, ṽ ϵ Ϻ, it fulfills:

(1) $\widetilde{d_p}$(ῤ, ϰ̃) = 0 if and only if ῤ = ϰ̃;
(2) $\widetilde{d_p}$(ῤ, ϰ̃) = $\widetilde{d_p}$(ϰ̃, ῤ);
(3) $\widetilde{d_p}$(ῤ, ϰ̃) ≤ Ω̃[$\widetilde{d_p}$(ῤ, ṽ) + $\widetilde{d_p}$(ṽ, ϰ̃)].

The pair (Ϻ, $\widetilde{d_p}$) is named a *p*-metric space, or an extended *b*-metric space.

In the year 2006, Zead Mustafa et al. [16] implemented the definition of *G*-metric space.

Definition 2.3. Let Ϻ is a non-empty set and G: Ϻ × Ϻ × Ϻ → [0, ∞) fulfills:

(G1) G(ῤ, ϰ̃, ṽ) = 0 if ῤ = ϰ̃ = ṽ;
(G2) G(ῤ, ῤ, ϰ̃) > 0 for all ῤ, ϰ̃ ϵ Ϻ with ῤ ≠ ϰ̃;
(G3) G(ῤ, ῤ, ϰ̃) ≤ G(ῤ, ϰ̃, ṽ) for all ῤ, ϰ̃, ṽ ϵ Ϻ with ϰ̃ ≠ ṽ;
(G4) G(ῤ, ϰ̃, ṽ) = G(ῤ, ṽ, ϰ̃) = G(ṽ, ϰ̃, ῤ) = . . . , [symmetry in all three variables]
(G5) G(ῤ, ϰ̃, ṽ) ≤ G(ῤ, ã, ã) + G(ã, ϰ̃, ṽ) for all ῤ, ϰ̃, ṽ, ã ϵ Ϻ.

Then the pair (Ϻ, G) is called a GMS.

The framework of G_bMS is described below as a generality of GMS and *b*-metric space.

Definition 2.3. [17] Let Ϻ is a non-empty set and ş ≥ 1 be a real number. Assume that G_b : Ϻ × Ϻ × Ϻ → [0, ∞) fulfills:

(G_b1) G_b(ῤ, ϰ̃, ṽ) = 0 if ῤ = ϰ̃ = ṽ;
(G_b2) G_b(ῤ, ῤ, ϰ̃) > 0 for all ῤ, ϰ̃ ϵ Ϻ with ῤ ≠ ϰ̃;
(G_b3) G_b(ῤ, ῤ, ϰ̃) ≤ G_b(ῤ, ϰ̃, ṽ) for all ῤ, ϰ̃, ṽ ϵ Ϻ with ϰ̃ ≠ ṽ;
(G_b4) G_b(ῤ, ϰ̃, ṽ) = G_b(ῤ, ṽ, ϰ̃) = G_b(ṽ, ϰ̃, ῤ) = . . . , [symmetry in all three variables]
(G_b5) G_b(ῤ, ϰ̃, ṽ) ≤ ş[G_b(ῤ, ã, ã) + G_b(ã, ϰ̃, ṽ)] for all ῤ, ϰ̃, ṽ, ã ϵ Ϻ.

Then the pair (Ϻ, G_b) is called a G_bMS.

Jleli and Samet [18] have implemented a new form of contraction called Θ-contraction, that is relied on the subsequent class of supplementary functions

$$\boldsymbol{\Theta := \{\theta \mid \theta : (0, \infty) \to (1, \infty) \text{ fulfills } (\Theta_1) - (\Theta_4)\}}$$

where

(Θ_1) θ is non-decreasing;
(Θ_2) For every sequence $\{s_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(s_n) = 1 \Leftrightarrow \lim_{n\to\infty} s_n = 0^+$;
(Θ_3) There exists $q \in (0,1)$ and $l \in (0, \infty]$ such that $\lim_{s\to 0^+} \frac{\theta(s)-1}{s^q} = l$;
(Θ_4) θ is continuous.

Many authors used this idea to yield fixed-point theorems; see, for instance, [19-22].

Definition 2.5. [23] Let Ψ represent the functions $\psi : [0, \infty) \rightarrow [0, \infty)$ so that ψ is continuous and $\psi(\mathfrak{t}) = 0 \Leftrightarrow \mathfrak{t} = 0$.

Definition 2.6. [23] Let Φ signify the functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ so that φ is non-decreasing, continuous and $\varphi(\mathfrak{t}) = 0 \Leftrightarrow \mathfrak{t} = 0$.

In [24], the specific class of functions were implemented by Zead Mustafa et al. as follows.

Definition 2.7. Let $(\mathrm{M}, \tilde{r})$ be an extended rectangular *b*-metric space with nontrivial function $\widetilde{\Omega}$ (i.e., $\widetilde{\Omega}(\mathfrak{t}) \neq \mathfrak{t}$) and $F_{\widetilde{\Omega}}$ represents the class of all functions $\eta : [0, \infty) \rightarrow [0, \widetilde{\Omega}^{-1}(1))$.

3. MAIN RESULTS

We commence this section, by presenting our first and significant definition.

Definition 3.1. Let M is a non-empty set and $\widetilde{\Omega} : [0, \infty) \rightarrow [0, \infty)$ be a SIC function with $\mathfrak{t} \leq \widetilde{\Omega}(\mathfrak{t})$ for all $\mathfrak{t} > 0$ and $\widetilde{\Omega}(0) = 0$. We say that a function $\widetilde{G_b} : \mathrm{M} \times \mathrm{M} \times \mathrm{M} \rightarrow [0, \infty)$ is called an EG$_b$M if it fulfills:

- ($\widetilde{G_b}1$) $\widetilde{G_b}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}) = 0$ if $\tilde{\rho} = \tilde{\kappa} = \tilde{v}$;
- ($\widetilde{G_b}2$) $\widetilde{G_b}(\tilde{\rho}, \tilde{\rho}, \tilde{\kappa}) > 0$ for all $\tilde{\rho}, \tilde{\kappa} \in \mathrm{M}$ with $\tilde{\rho} \neq \tilde{\kappa}$
- ($\widetilde{G_b}3$) $\widetilde{G_b}(\tilde{\rho}, \tilde{\rho}, \tilde{\kappa}) \leq \widetilde{G_b}(\tilde{\rho}, \tilde{\kappa}, \tilde{v})$ for all $\tilde{\rho}, \tilde{\kappa}, \tilde{v} \in \mathrm{M}$ with $\tilde{\kappa} \neq \tilde{v}$;
- ($\widetilde{G_b}4$) $\widetilde{G_b}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}) = \widetilde{G_b}(\tilde{\rho}, \tilde{v}, \tilde{\kappa}) = \widetilde{G_b}(\tilde{v}, \tilde{\kappa}, \tilde{\rho}) = \ldots$, [symmetry in all three variables]
- ($\widetilde{G_b}5$) $\widetilde{G_b}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}) \leq \widetilde{\Omega}[\widetilde{G_b}(\tilde{\rho}, \tilde{a}, \tilde{a}) + \widetilde{G_b}(\tilde{a}, \tilde{\kappa}, \tilde{v})]$ for all for all $\tilde{\rho}, \tilde{\kappa}, \tilde{v}, \tilde{a} \in \mathrm{M}$.

Then $(\mathrm{M},\widetilde{G_b})$ is called an extended G_b-metric space or simply EG$_b$MS.

Remark 3.2. Notice that each EG$_b$MS is a G$_b$MS with $\widetilde{\Omega}(\mathfrak{t}) = \varsigma\mathfrak{t}$, $\varsigma \geq 1$.

Example 3.3. Let (M,G_b) be a G$_b$MS and $\xi : [0, \infty) \rightarrow [0, \infty)$ be a SIC function so that $\mathfrak{t} \leq \xi(\mathfrak{t})$ and $\xi(0) = 0$. Let $\widetilde{G_b}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}) = \xi(G_b(\tilde{\rho}, \tilde{\kappa}, \tilde{v}))$. Obviously, for all $\tilde{a} \in \mathrm{M}$ and for three distinct points $\tilde{\rho}, \tilde{\kappa}, \tilde{v} \in \mathrm{M}$ that vary from $\tilde{a}$, we acquire

$$\begin{aligned}\widetilde{G_b}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}) &= \xi(G_b(\tilde{\rho}, \tilde{\kappa}, \tilde{v}))\\ &\leq \xi(\varsigma G_b(\tilde{\rho}, \tilde{a}, \tilde{a}) + \varsigma G_b(\tilde{a}, \tilde{\kappa}, \tilde{v}))\\ &\leq \xi(\varsigma\xi(G_b(\tilde{\rho}, \tilde{a}, \tilde{a})) + \varsigma\xi(G_b(\tilde{a}, \tilde{\kappa}, \tilde{v})))\\ &= \widetilde{\Omega}(\varsigma\widetilde{G_b}(\tilde{\rho}, \tilde{a}, \tilde{a}) + \varsigma\widetilde{G_b}(\tilde{a}, \tilde{\kappa}, \tilde{v})).\end{aligned}$$

Therefore $(\mathrm{M}, \widetilde{G_b})$ is an EG$_b$MS with $\widetilde{\Omega}(\mathfrak{t}) = \xi(\varsigma\mathfrak{t})$.

Example 3.4. Let $\widetilde{G_b}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}) = \tanh(G_b(\tilde{\rho}, \tilde{\kappa}, \tilde{v}))$, where G_b is a G_b-metric space defined by $G_b(\tilde{\rho}, \tilde{\kappa}, \tilde{v}) = \frac{1}{9}(|\tilde{\rho} - \tilde{\kappa}| + |\tilde{\kappa} - \tilde{v}| + |\tilde{\rho} - \tilde{v}|)^2$ with $\mathrm{M} = \mathbb{R}$ and $\varsigma = 2$. We will now show that $\widetilde{G_b}$ is an EG$_b$MS with $\widetilde{\Omega}(\mathfrak{t}) = 2\tanh(20\mathfrak{t})$.

Evidently, conditions ($\widetilde{G_b}1$)-($\widetilde{G_b}4$) of Definition 3.1 are satisfied. For every $\tilde{\rho}, \tilde{\kappa}, \tilde{v} \in \mathrm{M}$, the following holds:

$$\begin{aligned}\widetilde{G_b}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}) &= \tanh(G_b(\tilde{\rho}, \tilde{\kappa}, \tilde{v}))\\ &\leq \tanh(2[G_b(\tilde{\rho}, \tilde{a}, \tilde{a}) + G_b(\tilde{a}, \tilde{\kappa}, \tilde{v})])\\ &= \frac{2\tanh(G_b(\tilde{\rho},\tilde{a},\tilde{a}) + G_b(\tilde{a},\tilde{\kappa},\tilde{v}))}{1 + \tanh 2(G_b(\tilde{\rho},\tilde{a},\tilde{a}) + G_b(\tilde{a},\tilde{\kappa},\tilde{v}))}\end{aligned}$$

$$\leq 2\tanh(G_b(\tilde{\rho}, \tilde{a}, \tilde{a})) + G_b(\tilde{a}, \tilde{\kappa}, \tilde{v}))$$
$$\leq 2\tanh(20\tanh(G_b(\tilde{\rho}, \tilde{a}, \tilde{a}) + 20\tanh(G_b(\tilde{a}, \tilde{\kappa}, \tilde{v}))$$
$$= 2\tanh(20\,\widetilde{G_b}(\tilde{\rho}, \tilde{a}, \tilde{a}) + 20\,\widetilde{G_b}(\tilde{a}, \tilde{\kappa}, \tilde{v})).$$

Hence condition ($\widetilde{G_b}$5) of Definition 3.1 is satisfied. Therefore $\widetilde{G_b}$ is an EG$_b$M on M.

Proposition 3.5. Let (M, $\widetilde{G_b}$) be an EG$_b$MS then for each $\tilde{\rho}, \tilde{\kappa}, \tilde{v} \in$ M, we have

1) If $\widetilde{G_b}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}) = 0$, then $\tilde{\rho} = \tilde{\kappa} = \tilde{v}$;
2) $\widetilde{G_b}(\tilde{\rho}, \tilde{\kappa}, \tilde{\kappa}) \leq \tilde{\Omega}(2\widetilde{G_b}(\tilde{\kappa}, \tilde{\rho}, \tilde{\rho}))$.

Proof.

1) Let $\tilde{\rho} \neq \tilde{\kappa} \neq \tilde{v}$, then $0 = \widetilde{G_b}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}) \geq \widetilde{G_b}(\tilde{\rho}, \tilde{\rho}, \tilde{\kappa}) > 0$, which is impossible. Further, $0 = \widetilde{G_b}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}) \geq \widetilde{G_b}(\tilde{\kappa}, \tilde{\kappa}, \tilde{v}) > 0$, when $\tilde{\rho} = \tilde{\kappa}$ and $\tilde{\kappa} \neq \tilde{v}$.

2)
$$\widetilde{G_b}(\tilde{\rho}, \tilde{\kappa}, \tilde{\kappa}) = \widetilde{G_b}(\tilde{\kappa}, \tilde{\rho}, \tilde{\kappa})$$
$$\leq \tilde{\Omega}[\widetilde{G_b}(\tilde{\kappa}, \tilde{\rho}, \tilde{\rho}) + \widetilde{G_b}(\tilde{\rho}, \tilde{\rho}, \tilde{\kappa})]$$
$$= \tilde{\Omega}[\widetilde{G_b}(\tilde{\kappa}, \tilde{\rho}, \tilde{\rho}) + \widetilde{G_b}(\tilde{\kappa}, \tilde{\rho}, \tilde{\rho})] = \tilde{\Omega}(2\widetilde{G_b}(\tilde{\kappa}, \tilde{\rho}, \tilde{\rho})).$$

Definition 3.6. Let (M,$\widetilde{G_b}$) be an EG$_b$MS. Let $\{a_n\}$ be a sequence in M. Then one can assert that

1) $\{a_n\}$ is $\widetilde{G_b}$-Cauchy, if $\lim_{n,m,l\to\infty} \widetilde{G_b}(a_n, a_m, a_l) = 0$.
2) $\{a_n\}$ is $\widetilde{G_b}$-convergent, if $\lim_{n,m\to\infty} \widetilde{G_b}(a_n, a_m, a) = 0$.

We are now presenting the following propositions.

Proposition 3.7. Let (M,$\widetilde{G_b}$) be an EG$_b$MS. The preceding are then equivalent:

(i) The sequence $\{a_n\}$ is $\widetilde{G_b}$-Cauchy:
(ii) For every $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that $\widetilde{G_b}(a_n, a_m, a_m) < \epsilon, \forall m, n \geq n_0$.

Proof. (i) ⇒ (ii). In (1) of Definition 3.6, we put l = m.

(ii) ⇒ (i) Let $\epsilon > 0$ and choose $\tilde{\Omega}(2\epsilon_1) = \frac{\epsilon}{4}$. By (ii), $\exists\, n_0 \in \mathbb{N}$ such that $\widetilde{G_b}(a_n, a_m, a_m) < \epsilon_1$, for all $m, n \geq n_0$. Thus

$$\widetilde{G_b}(a_n, a_m, a_l) \leq \tilde{\Omega}[\widetilde{G_b}(a_n, a_m, a_m) + \widetilde{G_b}(a_m, a_m, a_l)]$$
$$< \tilde{\Omega}(2\epsilon_1) = \frac{\epsilon}{4} < \epsilon, \quad \forall m, n, l \geq n_0.$$

Proposition 3.8. Let (M,$\widetilde{G_b}$) be an EG$_b$MS. The preceding are then equivalent:

(i) $\{a_n\}$is $\widetilde{G_b}$-convergent to a;
(ii) $\widetilde{G_b}(a_n, a_n, a) \to 0$ as $n \to +\infty$;
(iii) $\widetilde{G_b}(a_n, a, a) \to 0$ as $n \to +\infty$.

Proof. (i) ⇒ (ii) when $m = n$.

(ii) ⇒ (iii). Let $\epsilon > 0$ and choose $\tilde{\Omega}(2\epsilon_1) = \frac{\epsilon}{4}$. By (ii), $\exists\, n_0 \in \mathbb{N}$ such that $\widetilde{G_b}(a_n, a_n, a) < \epsilon_1$, for all $n \geq n_0$. Then by (2) of Proposition 3.5, we have

$$\widetilde{G_b}(a_n, a, a) \leq \tilde{\Omega}[2\widetilde{G_b}(a_n, a_n, a)]$$
$$< \tilde{\Omega}(2\epsilon_1) = \frac{\epsilon}{4} < \epsilon, \forall m, n, l \geq n_0.$$

(iii) ⇒ (i). Let $\epsilon > 0$ and choose $\tilde{\Omega}(2\epsilon_1) = \frac{\epsilon}{4}$. By (iii), we get $\widetilde{G_b}(a_n, a, a) < \epsilon_1$, for all $n \geq n_0$, where $n_0 \in \mathbb{N}$.

$$\widetilde{G_b}(a_n, a_m, a) \leq \widetilde{\Omega}[\widetilde{G_b}(a_n, a, a) + \widetilde{G_b}(a, a_m, a)] < \widetilde{\Omega}(2\epsilon_1) = \frac{\epsilon}{4} < \epsilon, \ \forall\ m, n \geq n_0.$$

Definition 3.9. We say that $(\mathrm{M}, \widetilde{G_b})$ is $\widetilde{G_b}$-complete if every $\widetilde{G_b}$-Cauchy sequence is $\widetilde{G_b}$-convergent.

Definition 3.10. Let $(\mathrm{M}, \widetilde{G_b})$ be an EG_bMS with nontrivial function Ω. A mapping $T : \mathrm{M} \to \mathrm{M}$ is called a GGTΘB contraction on M, if there exists $\eta \in F_{\widetilde{\Omega}}$, $\theta \in \Theta$, $\delta \in \Delta$, $\varphi \in \Phi$, $\psi \in \Psi$ and $L \geq 0$ such that for all $\tilde{\rho}$, $\tilde{\kappa}$, $\tilde{v} \in \mathrm{M}$, it satisfies

$$\theta(\Omega(\widetilde{G_b}(T\tilde{\rho}, T\tilde{\kappa}, T\tilde{v}))) \leq \delta(\varphi(\widetilde{M}(\tilde{\rho}, \tilde{\kappa}, \tilde{v})))\theta(\widetilde{M}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}))^{\eta(\widetilde{G_b}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}))} + L\, \widetilde{N}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}) - \psi(\widetilde{M}(\tilde{\rho}, \tilde{\kappa}, \tilde{v})) \quad (1)$$

where

$$\widetilde{M}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}) = \max\{\widetilde{G_b}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}), \widetilde{G_b}(\tilde{\rho}, T\tilde{\rho}, T\tilde{\rho}), \widetilde{G_b}(\tilde{\kappa}, T\tilde{\kappa}, T\tilde{\kappa}), \widetilde{G_b}(\tilde{v}, T\tilde{v}, T\tilde{v})\}$$

and

$$\widetilde{N}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}) = \min \widetilde{G_b}(\tilde{\rho}, T\tilde{\kappa}, T\tilde{\kappa}), \widetilde{G_b}(\tilde{\kappa}, T\tilde{v}, T\tilde{v}), \widetilde{G_b}(\tilde{v}, T\tilde{\rho}, T\tilde{\rho})\}.$$

Theorem 3.11. Let $(\mathrm{M}, \preccurlyeq, \widetilde{G_b})$ be a complete ordered EG_bMS and $T : \mathrm{M} \to \mathrm{M}$ is a GGTΘB contraction. If T is an increasing mapping (I.M) with respect to $\preccurlyeq$ such that there exists a comparable element $a_0 \in \mathrm{M}$ with $a_0 \preccurlyeq Ta_0$, then T has a fixed point in M.

Proof. Let $a_0 \in \mathrm{M}$ be arbitrary and $a_n = T^n a_0$. Beyond lack of generality, we presume now that $a_n \neq a_{n+1}$, $\forall\, n \in \mathbb{N}$. Since $a_0 \preccurlyeq Ta_0$ and T is an I.M, we attain by induction that

$$a_0 \preccurlyeq Ta_0 \preccurlyeq T^2 a_0 \preccurlyeq \ldots T^n a_0 \preccurlyeq \ldots.$$

We will now attempt to prove that $\lim_{n \to \infty} \widetilde{G_b}(a_n, a_{n+1}, a_{n+1}) = 0$. As of $a_n \preccurlyeq a_{n+1}$, for every $n \in \mathbb{N}$. By inequality (1), we have

$$\begin{aligned}\theta(\widetilde{G_b}(a_n, a_{n+1}, a_{n+1})) &= \theta(\widetilde{G_b}(Ta_{n-1}, Ta_n, Ta_n)) \\ &\leq \theta(\widetilde{\Omega}^2(\widetilde{G_b}(Ta_{n-1}, Ta_n, Ta_n))) \\ &\leq \delta(\varphi(\widetilde{M}(a_{n-1}, a_n, a_n)))\theta(\widetilde{M}(a_{n-1}, a_n, a_n))^{\eta(\widetilde{G_b}(a_{n-1}, a_n, a_n))} \\ &\quad + L\, \widetilde{N}(a_{n-1}, a_n, a_n) - \psi(\widetilde{M}(a_{n-1}, a_n, a_n)) \qquad (2)\end{aligned}$$

where

$$\widetilde{M}(a_{n-1}, a_n, a_n) = \max\{\widetilde{G_b}(a_{n-1}, a_n, a_n), \widetilde{G_b}(a_{n-1}, Ta_{n-1}, Ta_{n-1}), \widetilde{G_b}(a_n, Ta_n, Ta_n), \widetilde{G_b}(a_n, Ta_n, Ta_n)\}$$

and

$$\widetilde{N}(a_{n-1}, a_n, a_n) = \min\{\widetilde{G_b}(a_{n-1}, Ta_n, Ta_n), \widetilde{G_b}(a_n, Ta_n, Ta_n), \widetilde{G_b}(a_n, Ta_n, Ta_n)\} = 0.$$

If $\max\{\widetilde{G_b}(a_{n-1}, a_n, a_n), \widetilde{G_b}(a_n, a_{n+1}, a_{n+1})\} = \widetilde{G_b}(a_n, a_{n+1}, a_{n+1})$. Then by Equation (2), we acquire

$$\theta(\widetilde{G_b}(a_n, a_{n+1}, a_{n+1})) \leq \delta(\varphi(\widetilde{G_b}(a_n, a_{n+1}, a_{n+1})))\theta(\widetilde{G_b}(a_n, a_{n+1}, a_{n+1}))^{\eta(\widetilde{G_b}(a_{n-1}, a_n, a_n))} - \psi\left(\widetilde{G_b}(a_n, a_{n+1}, a_{n+1})\right)$$

$$< \theta\left(\widetilde{G_b}(a_n, a_{n+1}, a_{n+1})\right)^{\widetilde{\Omega}^{-1}(1)} - \psi\left(\widetilde{G_b}(a_n, a_{n+1}, a_{n+1})\right) \tag{3}$$

$$< \theta\left(\widetilde{G_b}(a_n, a_{n+1}, a_{n+1})\right),$$

which is a contradiction. Hence, $\max\{\widetilde{G_b}(a_{n-1}, a_n, a_n), \widetilde{G_b}(a_n, a_{n+1}, a_{n+1})\} = \widetilde{G_b}(a_{n-1}, a_n, a_n)$. Again from the conditions of φ and ψ, and from inequality (2), we find that

$$\theta(\widetilde{G_b}(a_n, a_{n+1}, a_{n+1})) \leq \theta(\widetilde{G_b}(a_{n-1}, a_n, a_n))^{\eta(\widetilde{G_b}(a_{n-1}, a_n, a_n))} - \psi\left(\widetilde{G_b}(a_{n-1}, a_n, a_n)\right)$$

$$\leq \theta\left(\widetilde{G_b}(a_{n-2}, a_{n-1}, a_{n-1})\right)^{\eta\left(\widetilde{G_b}(a_{n-1}, a_n, a_n)\right)\eta\left(\widetilde{G_b}(a_{n-2}, a_{n-1}, a_{n-1})\right)}$$

$$\vdots$$

$$\leq \theta\left(\widetilde{G_b}(a_0, a_1, a_1)\right)^{\prod_{i=1}^{n}\eta\left(\widetilde{G_b}(a_{i-1}, a_i, a_i)\right)}.$$

We get by definition of ɳ that $\eta(\mathfrak{t}) < \widetilde{\Omega}^{-1}(1) \leq 1$, $\forall\, \mathfrak{t} \in [0, \infty)$. Hence

$$\lim_{n\to\infty}\prod_{i=1}^{n}\eta\left(\widetilde{G_b}(a_{i-1}, a_i, a_i)\right) = 0$$

which yields

$$\lim_{n\to\infty}\theta(\widetilde{G_b}(a_n, a_{n+1}, a_{n+1})) = 1 \text{ i.e., } \lim_{n\to\infty}\widetilde{G_b}(a_n, a_{n+1}, a_{n+1}) = 0. \tag{4}$$

Now, we will demonstrate that $a_n \neq a_m$ for $n \neq m$. Suppose that $a_n = a_m$ for some $n > m$, thus we have $a_{n+1} = Ta_n = Ta_m = a_{m+1}$. By proceeding with this procedure, we observe that $a_{n+k} = a_{m+k}$ for all $k \in \mathbb{N}$. Then from inequality (1), we obtain

$$\theta(\widetilde{G_b}(a_m, a_{m+1}, a_{m+1})) = \theta(\widetilde{G_b}(a_n, a_{n+1}, a_{n+1}))$$

$$\leq \theta(\widetilde{\Omega}^2(\widetilde{G_b}(a_n, a_{n+1}, a_{n+1})))$$

$$\leq \delta(\varphi(\widetilde{M}(a_{n-1}, a_n, a_n)))\theta(\widetilde{M}(a_{n-1}, a_n, a_n))^{\eta(\widetilde{G_b}(a_{n-1}, a_n, a_n))}$$

$$+ L\,\widetilde{N}(a_{n-1}, a_n, a_n) - \psi(\widetilde{G_b}(a_{n-1}, a_n, a_n)) \tag{5}$$

$$< (\theta(\max\{\widetilde{G_b}(a_{n-1}, a_n, a_n), \widetilde{G_b}(a_n, a_{n+1}, a_{n+1})\}))^{\eta(\widetilde{G_b}(a_{n-1}, a_n, a_n))}.$$

If $\max\{\widetilde{G_b}(a_{n-1}, a_n, a_n), \widetilde{G_b}(a_n, a_{n+1}, a_{n+1})\} = \widetilde{G_b}(a_n, a_{n+1}, a_{n+1})$. Then

$$\theta(\widetilde{G_b}(a_m, a_{m+1}, a_{m+1})) \leq \theta(\widetilde{G_b}(a_n, a_{n+1}, a_{n+1}))^{\eta(\widetilde{G_b}(a_{n-1}, a_n, a_n))} - \psi\left(\widetilde{G_b}(a_n, a_{n+1}, a_{n+1})\right)$$

$$< \theta\left(\widetilde{G_b}(a_n, a_{n+1}, a_{n+1})\right)$$

which is impossible. If $\max\{\widetilde{G_b}(a_{n-1}, a_n, a_n), \widetilde{G_b}(a_n, a_{n+1}, a_{n+1})\} = \widetilde{G_b}(a_{n-1}, a_n, a_n)$, then the inequality above implies that

$$\theta(\widetilde{G_b}(a_m, a_{m+1}, a_{m+1})) \leq \theta(\widetilde{G_b}(a_{n-1}, a_n, a_n))$$

$$\leq \theta(\widetilde{M}(a_{n-2}, a_{n-1}, a_{n-1}))^{\eta(\widetilde{G_b}(a_{n-2}, a_{n-1}, a_{n-1}))}$$

$$\leq \theta(\max\{\widetilde{G_b}(a_{n-2}, a_{n-1}, a_{n-1}), \widetilde{G_b}(a_{n-1}, a_n, a_n)\}) \tag{6}$$

$$\vdots$$

$$< \theta(\widetilde{G_b}(a_m, a_{m+1}, a_{m+1})),$$

a contradiction. Hence $a_n \neq a_m$ for $n \neq m$.

The following step is to affirm that $\{a_n\}$ is $\widetilde{G_b}$-Cauchy sequence. Conversely, claim that there is an $\epsilon > 0$ where we can consider two subsequences $\{a_{m_i}\}$and $\{a_{n_i}\}$of $\{a_n\}$, so that n_i is the least factor where

$$n_i > m_i > \mathrm{i} \text{ and } \widetilde{G_b}(a_{m_i}, a_{n_i}, a_{n_i}) \geq \epsilon. \tag{7}$$

This implies

$$\widetilde{G_b}(a_{m_i}, a_{n_i-2}, a_{n_i-2}), \widetilde{G_b}(a_{m_i}, a_{n_i-1}, a_{n_i-1}) < \epsilon. \tag{8}$$

From (7) and ($\widetilde{G_b}$5), we get

$$\epsilon \leq \widetilde{G_b}(a_{m_i}, a_{n_i}, a_{n_i})$$

$$\leq \widetilde{\Omega}[\widetilde{G_b}(a_{m_i}, a_{n_i-1}, a_{n_i-1}) + \widetilde{G_b}(a_{n_i-1}, a_{n_i}, a_{n_i})].$$

Taking upper limit i → ∞ and applying Equation (4), the latter inequality becomes

$$\widetilde{\Omega}^{-1}(\epsilon) \leq \limsup_{i\to\infty} \widetilde{G_b}(a_{m_i}, a_{n_i-1}, a_{n_i-1}). \tag{9}$$

Consider

$$\theta\left(\widetilde{\Omega}^2\left(\widetilde{G_b}(a_{m_i}, a_{n_i-1}, a_{n_i-1})\right)\right) \leq \delta(\varphi(\widetilde{M}(a_{m_i-1}, a_{n_i-2}, a_{n_i-2})))$$

$$\theta(\widetilde{M}(a_{m_i-1}, a_{n_i-2}, a_{n_i-2}))^{\eta(\widetilde{G_b}(a_{m_i-1}, a_{n_i-2}, a_{n_i-2}))} +$$

$$L\, \widetilde{N}(a_{m_i-1}, a_{n_i-2}, a_{n_i-2}) - \psi(\widetilde{M}(a_{m_i-1}, a_{n_i-2}, a_{n_i-2}))$$

where

$$\widetilde{M}(a_{m_i-1}, a_{n_i-2}, a_{n_i-2})) =$$

$$\max\{\widetilde{G_b}(a_{m_i-1}, a_{n_i-2}, a_{n_i-2}), \widetilde{G_b}(a_{m_i-1}, a_{m_i}, a_{m_i}), \widetilde{G_b}(a_{n_i-2}, a_{n_i-1}, a_{n_i-1}), \widetilde{G_b}(a_{n_i-2}, a_{n_i-2}, a_{n_i-1})\}$$

$$\leq \max\{\widetilde{\Omega}[\widetilde{G_b}(a_{m_i-1}, a_{m_i}, a_{m_i})$$

$$+ \widetilde{G_b}(a_{m_i}, a_{n_i-2}, a_{n_i-2})], \widetilde{G_b}(a_{m_i-1}, a_{m_i}, a_{m_i}), \widetilde{G_b}(a_{n_i-2}, a_{n_i-1}, a_{n_i-1}), \widetilde{G_b}(, a_{n_i-2}, a_{n_i-1}, a_{n_i-1})\}$$

and

$$\widetilde{N}(a_{m_i-1}, a_{n_i-2}, a_{n_i-2}) = \min\{\widetilde{G_b}(a_{m_i-1}, a_{n_i-1}, a_{n_i-1}), \widetilde{G_b}(a_{n_i-2}, a_{n_i-1}, a_{n_i-1}), \widetilde{G_b}(, a_{n_i-2}, a_{m_i}, a_{m_i})\}.$$

Taking upper limit i → ∞ in the latter two equations, we get

$$\limsup_{i\to\infty} \widetilde{M}(a_{m_i-1}, a_{n_i-2}, a_{n_i-2}) \leq \widetilde{\Omega}(\epsilon) \tag{10}$$

and

$$\limsup_{i\to\infty} \widetilde{N}(a_{m_i-1}, a_{n_i-2}, a_{n_i-2}) = 0. \tag{11}$$

Thus, we have

$$1 < \theta(\widetilde{\Omega}(\epsilon)) = \theta(\widetilde{\Omega}^2(\widetilde{\Omega}^{-1}(\epsilon))$$
$$\leq \theta(\widetilde{\Omega}^2(\limsup_{i\to\infty} \widetilde{G_b}(a_{m_i}, a_{n_i-1}, a_{n_i-1})))$$
$$\leq \theta(\limsup_{i\to\infty} \widetilde{M}(a_{m_i-1}, a_{n_i-2}, a_{n_i-2}))^{\eta(\widetilde{G_b}(a_{m_i-1}, a_{n_i-2}, a_{n_i-2}))}$$
$$< \theta(\widetilde{\Omega}(\epsilon))^{\widetilde{\Omega}^{-1}(1)} \leq \theta(\widetilde{\Omega}(\epsilon)),$$

which is wrong. Appropriately $\{a_n\}$ is a $\widetilde{G_b}$-Cauchy sequence in X. The sequence $\{a_n\}$ is therefore $\widetilde{G_b}$-converges to some $a \in M$ i.e., $\lim_{n,m\to\infty} \widetilde{G_b}(a_n, a_m, a) = 0$. Now, we demonstrate that a is a fixed point of T. Assume that $Ta \neq a$. So it follows that a_n differs from both Ta and a for sufficiently large n and $a_n \preccurlyeq a$. Therefore,

$$\theta(\widetilde{G_b}(a_{n+1}, Ta, Ta)) \leq \delta(\varphi(\widetilde{M}(a_n, a, a)))\theta(\widetilde{M}(a_n, a, a))^{\eta(\widetilde{G_b}(a_n, a, a))} - \psi\left(\widetilde{M}(a_n, a, a)\right)$$
$$\leq \theta\left(\widetilde{M}(a_n, a, a)\right)^{\eta\left(\widetilde{G_b}(a_n, a, a)\right)}$$
$$= \theta(\max\{\widetilde{G_b}(a_n, a, a), \widetilde{G_b}(a_n, Ta_n, Ta_n), \widetilde{G_b}(a, Ta, Ta), \widetilde{G_b}(a, Ta, Ta)\})^{\eta(\widetilde{G_b}(a_n, a, a))}.$$

Thus

$$\theta(\widetilde{G_b}(a, Ta, Ta)) \leq \liminf_{n\to\infty} \theta(\widetilde{G_b}(a_{n+1}, Ta, Ta)) \leq \theta(\widetilde{G_b}(\liminf_{n\to\infty} a_{n+1}, Ta, Ta))$$
$$\leq \theta(\limsup_{n\to\infty} (\max\{\widetilde{G_b}(a_n, a, a), \widetilde{G_b}(a_n, Ta_n, Ta_n), \widetilde{G_b}(a, Ta, Ta), \widetilde{G_b}(a, Ta, Ta)\}))^{\limsup_{n\to\infty} \eta(\widetilde{G_b}(a_n, a, a))}$$
$$\leq \theta(\widetilde{G_b}(a, Ta, Ta))^{\limsup_{n\to\infty} \eta(\widetilde{G_b}(a_n, a, a))}$$

$< \theta(\widetilde{G_b}(a, Ta, Ta))^{\widetilde{\Omega}^{-1}(1)} \leq \theta(\widetilde{G_b}(a, Ta, Ta))$, a contradiction. Therefore 'a' is a fixed point of T .

By selecting $0 < \eta(\widetilde{G_b}(\tilde{\rho}, \tilde{\kappa}, \tilde{v})) = r < \widetilde{\Omega}^{-1}(1)$ in Theorem 3.11, the corollary we acquire is as follows:

Corollary 3.12. Theorem 3.11 remains true, if we supplant the supposition, by the following (Apart from maintaining the other hypotheses)

$$\theta(\widetilde{\Omega}^2(\widetilde{G_b}(T\tilde{\rho}, T\tilde{\kappa}, T\tilde{v}))) \leq \delta(\varphi(\widetilde{M}(\tilde{\rho}, \tilde{\kappa}, \tilde{v})))\theta(\widetilde{M}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}))^r + L\,\widetilde{N}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}) - \psi(\widetilde{M}(\tilde{\rho}, \tilde{\kappa}, \tilde{v})) \quad (12)$$

for some $\theta \in \Theta$, $\delta \in \Delta$, $\varphi \in \Phi$, $\psi \in \Psi$, $L \geq 0$ and for all comparable elements $\tilde{\rho}, \tilde{\kappa}, \tilde{v} \in M$.

Taking L = 0 in Corollary 3.12, we have a corollary below.

Corollary 3.13. Theorem 3.11 remains true, if we supplant the supposition, by the following (Apart from maintaining the other hypotheses)

$$\theta(\widetilde{\Omega}^2(\widetilde{G_b}(T\tilde{\rho}, T\tilde{\kappa}, T\tilde{v}))) \leq \delta(\varphi(\widetilde{M}(\tilde{\rho}, \tilde{\kappa}, \tilde{v})))\theta(\widetilde{M}(\tilde{\rho}, \tilde{\kappa}, \tilde{v}))^r - \psi(\widetilde{M}(\tilde{\rho}, \tilde{\kappa}, \tilde{v})) \quad (13)$$

for some $\theta \in \Theta, \delta \in \Delta, \varphi \in \Phi, \psi \in \Psi$ and for all comparable elements $\tilde{\rho}, \tilde{\kappa}, \tilde{v} \in M$.

Further by putting $\varphi(t) = t$ in Corollary 3.13, we have the preceeding corollary as a result of Theorem 3.11.

Corollary 3.14. Let $(M, \preccurlyeq, \widetilde{G_b})$ be a complete ordered EG_bMS and $T : M \to M$ be an I.M with respect to $\preccurlyeq$ such that there exists a comparable element $\tilde{\rho}_0 \in X$ with $\tilde{\rho}_0 \preccurlyeq T\tilde{\rho}_0$. Suppose that

$$\theta(\widetilde{\Omega}^2(\widetilde{G_b}(T\tilde{\rho},T\widetilde{\kappa},T\tilde{v}))) \le \delta(\widetilde{M}(\tilde{\rho},\widetilde{\kappa},\tilde{v}))\theta(\widetilde{M}(\tilde{\rho},\widetilde{\kappa},\tilde{v}))^r - \psi(\widetilde{M}(\tilde{\rho},\widetilde{\kappa},\tilde{v})) \quad (14)$$

for some $\theta \in \Theta, \delta \in \Delta, \psi \in \Psi$ and for all comparable elements $\tilde{\rho},\widetilde{\kappa},\tilde{v} \in M$, where

$$\widetilde{M}(\tilde{\rho},\widetilde{\kappa},\tilde{v}) = \max\{\widetilde{G_b}(\tilde{\rho},\widetilde{\kappa},\tilde{v}),\widetilde{G_b}(\tilde{\rho},\widetilde{T\rho},T\tilde{\rho}),\widetilde{G_b}(\widetilde{\kappa},\widetilde{T\kappa},T\widetilde{\kappa}),\widetilde{G_b}(\tilde{v},\widetilde{Tv},T\tilde{v})\}.$$

Then T has a fixed point in M.

Another version of Corollary 3.14 is the Corollary we mention below.

Corollary 3.15. Let $(M, \preccurlyeq, \widetilde{G_b})$ be a complete ordered EG_bMS and $T : M \to M$ is an an I.M with respect to $\preccurlyeq$ such that there exists a comparable element $\tilde{\rho}_0 \in M$ with $\tilde{\rho}_0 \preccurlyeq T\tilde{\rho}_0$. Suppose that

$$\theta(\widetilde{\Omega}^2(\widetilde{G_b}(T\tilde{\rho},T\widetilde{\kappa},T\tilde{v}))) \le \theta(\widetilde{M}(\tilde{\rho},\widetilde{\kappa},\tilde{v}))^r - \psi(\widetilde{M}(\tilde{\rho},\widetilde{\kappa},\tilde{v})) \quad (15)$$

for some $\theta \in \Theta, \psi \in \Psi$ and for all comparable elements $\tilde{\rho},\widetilde{\kappa},\tilde{v} \in M$, where

$\widetilde{M}(\tilde{\rho},\widetilde{\kappa},\tilde{v}) = \max\{\widetilde{G_b}(\tilde{\rho},\widetilde{\kappa},\tilde{v}),\widetilde{G_b}(\tilde{\rho},\widetilde{T\rho},T\tilde{\rho}),\widetilde{G_b}(\widetilde{\kappa},\widetilde{T\kappa},T\widetilde{\kappa}),\widetilde{G_b}(\tilde{v},\widetilde{Tv},T\tilde{v})\}$. Then T has a fixed point in M.

4. EXISTENCE THEOREM FOR SOLUTIONS OF A FREDHOLM INTEGRAL EQUATION

Let $M = C([0, 1], \mathbb{R})$ the set of all continuous real valued functions defined on [0, 1]. We perform partial order for M $\preccurlyeq$given by $\tilde{\rho} \preccurlyeq \widetilde{\kappa} \Leftrightarrow \tilde{\rho}(t) \le \widetilde{\kappa}(t), \forall\, t \in [0, 1]$. The metric G is defined as

$$G(\tilde{\rho},\widetilde{\kappa},\tilde{v}) = \frac{1}{3}\sup_{t\in[0,1]}(|\tilde{\rho}(t) - \widetilde{\kappa}(t)| + |\widetilde{\kappa}(t) - \tilde{v}(t)| + |\tilde{\rho}(t) - \tilde{v}(t)|).$$

Let $\xi : [0,\infty) \to [0,\infty)$ be SIC function with $t \le \xi(t)$ and $\widetilde{G_b}(\tilde{\rho},\widetilde{\kappa},\tilde{v}) = \xi\left(\frac{3}{2}G(\tilde{\rho},\widetilde{\kappa},\tilde{v})\right)$. Consider the fredholm integral equations

$$\tilde{\rho}(t) = f(t) + \int_0^1 \tau(t,s,\tilde{\rho}(s))ds, \quad t,s \in [0,1], \quad (16)$$

where a(t) is an unknown solution, $\tau(t, s, \tilde{\rho}(s))$ is called a smooth function. Presume that the conditions below hold:

(i) The mapping $T : C[0, 1] \to C[0, 1]$ defined by $T\tilde{\rho}(t) = f(t) + \int_0^1 \tau(t,s,\tilde{\rho}(s))ds$ is a continuous mapping and $\tau : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$.

(ii) $\theta : [0, \infty) \to [1, \infty)$ with $\theta(t) < t, \forall\, t > 0$.

(iii) If $\tilde{\rho} \preccurlyeq \widetilde{\kappa}$ then $\tau(t, s, \tilde{\rho}(s)) \le \tau(t, s, \widetilde{\kappa}(s)), \forall\, t, s \in [0, 1]$.

(iv) For all $\tilde{\rho},\widetilde{\kappa},\tilde{v} \in M$ and for all $t \in [0, 1]$

$$\xi(2) + \xi^2(\xi\left(\int_0^1|\tau(t,s,\tilde{\rho}(s)) - \tau(t,s,\widetilde{\kappa}(s))|ds\right) \le 1 + [\theta(|\tilde{\rho}(s) - \widetilde{\kappa}(s)|)]^r.$$

Under assertions (i)-(iv), the Equation (16) has a solution in M, where $M = C([0,1], \mathbb{R})$. Consider

$$2 + \xi^2\left(\xi\left(T\tilde{\rho}(t) - T\widetilde{\kappa}(t)\right)\right) \le \xi(2) + \xi^2(\xi\left(\int_0^1|\tau(t,s,\tilde{\rho}(s)) - \tau(t,s,\widetilde{\kappa}(s))|ds\right)$$

$$\le 1 + [\theta(|\tilde{\rho}(s) - \widetilde{\kappa}(s)|)]^r = 1 + (\theta(\frac{3}{2}G(\tilde{\rho},\widetilde{\kappa},\widetilde{\kappa}))^r$$

$$\le 1 + (\theta(\xi(\frac{3}{2}G(\tilde{\rho},\widetilde{\kappa},\widetilde{\kappa})))^r$$

$$= \ 1 + \ \theta\left(\widetilde{G_b}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa})\right)^r \ \le \ 1 + \ \left(\theta\left(\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa})\right)\right)^r$$

where $\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa}) = \max\{\widetilde{G_b}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa}), \widetilde{G_b}(\tilde{\rho},T\tilde{\rho},T\tilde{\rho}), \widetilde{G_b}(\tilde{\kappa},T\tilde{\kappa},T\tilde{\kappa}), \widetilde{G_b}(\tilde{\kappa},T\tilde{\kappa},T\tilde{\kappa})\}$. Thus

$$\begin{aligned}
2 + \ \xi^2\left(\xi\left(T\tilde{\rho}(\mathfrak{t}) - \ T\tilde{\kappa}(\mathfrak{t})\right)\right) \ &\le \ 1 + \ \left(\theta\left(\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa})\right)\right)^r \\
&= \ 1 + \ \left(\theta\left(\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa})\right)\right)^r + \frac{2\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa}) + \ 1}{\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa}) + \ 1} - \frac{2\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa}) + \ 1}{\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa}) + \ 1} \\
&= \ \left(\theta\left(\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa})\right)\right)^r - \left(\frac{2\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa}) + 1}{\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa}) + 1} - \ 1\right) + \frac{2\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa}) + 1}{\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa}) + 1} \\
&= \left(\theta\left(\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa})\right)\right)^r - \frac{\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa})}{\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa}) + \ 1} + 2 - \frac{1}{\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa}) + \ 1} \\
&< 2 + \left(\theta\left(\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa})\right)\right)^r - \ \frac{\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa})}{\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa}) + 1}.
\end{aligned}$$

This yields

$$\xi^2\left(\xi\left(T\tilde{\rho}(\mathfrak{t}) - \ T\tilde{\kappa}(\mathfrak{t})\right)\right) \ \le \ \left(\theta\left(\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa})\right)\right)^r - \frac{\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa})}{\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa}) + \ 1}$$

Hence by taking $\psi(\mathfrak{t}) = \frac{\mathfrak{t}}{\mathfrak{t}+1}$ and $\widetilde{\Omega} = \ \xi$, we acquire that

$$\begin{aligned}
\theta\left(\widetilde{\Omega}^2\left(\widetilde{G_b}(T\tilde{\rho},T\tilde{\kappa},T\tilde{\kappa})\right)\right) \ &\le \ \widetilde{\Omega}^2\left(\widetilde{G_b}(T\tilde{\rho},T\tilde{\kappa},T\tilde{\kappa})\right) \\
&= \ \widetilde{\Omega}^2(\xi\left(\frac{3}{2}G(T\tilde{\rho},T\tilde{\kappa},T\tilde{\kappa})\right)) \\
&= \ \widetilde{\Omega}^2(\xi(|T\tilde{\rho}(t) - T\tilde{\kappa}(t)|)) \\
&\le \ \left(\theta\left(\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa})\right)\right)^r - \ \psi\left(\widetilde{M}(\tilde{\rho},\tilde{\kappa},\tilde{\kappa})\right).
\end{aligned}$$

Therefore all the assumptions of Corollary 3.15 are fulfilled and we deduce the existence of $\tilde{\rho} \epsilon M$ such that $\tilde{\rho} = T(\tilde{\rho})$.

5. RESULTS AND DISCUSSION

Throughout this study, we introduced the class of EG_bMS as an extension of G_bMS, and demonstrated fixed point theorem with GGTΘB contraction on complete ordered EG_bMS. We also acquired some different generalizations of the Banach contraction theory by broadening Jleli and Samet's result, Berinde and Geraghty in [25, 26]. Analyzing the current literature in the light of the newly established EG_bMS would be quite interesting.

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CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Mustafa, Z., Sims, B., "A new approach to generalized metric spaces", Journal of Nonlinear and Convex Analysis, 7(2): 289-297, (2006).

[2] Aydi, H., Felhi, A., Sahmim, S., "Related fixed point results for cyclic contractions on G-metric spaces and application", Filomat, 31(3): 853-869, (2017).

[3] Aydi, H., Postolache, M., Shatanawi, W., "Coupled fixed point results for (ψ-φ)-weakly contractive mappings in ordered G-metric spaces", Computers & Mathematics with Applications, 63(1): 298-309, (2012).

[4] Chandok, S., Mustafa, Z., Postolache, M., "Coupled common fixed point results for mixed g- monotone maps in partially ordered G-metric spaces", Politehnica University of Bucharest Scientific Bulletin Series A Applied Mathematics and Physics, 75(4): 13-26, (2013).

[5] Mustafa, Z., Aydi, H., Karapinar, E., "On common fixed points in G-metric spaces using (E.A) property", Computers & Mathematics with Applications, 64 (6): 1944-1956, (2012).

[6] Popović, B., Shoaib, M., Sarwar, M., "Coupled fixed point theorems for generalized (ψ, φ)-weak contraction in partially ordered G-metric spaces", Journal of Computational Analysis and Applications, 23(1): (2017).

[7] Abbas, M., Kim, J. K., Nazir, T., "Common fixed point of mappings satisfying almost generalized contractive condition in partially ordered G-metric spaces", Journal of Computational Analysis and Applications, 19(6): 928-938, (2015).

[8] Zand, M. R. A., Nezhad, A. D., "A generalization of partial metric spaces", Journal of Contemporary Applied Mathematics, 1(1): 86-93, (2011).

[9] Ameer, E., Arshad, M., Shatanawi, W., "Common fixed point results for generalized α^*-ψ- contraction multivalued mappings in b-metric spaces", Journal of Fixed Point Theory and Applications, 19(4): 3069-3086, (2017).

[10] Dinarvand, M., "Some fixed point results for admissible Geraghty contraction type mappings in fuzzy metric spaces", Iranian Journal of Fuzzy Systems, 14(3): 161-177, (2017).

[11] Liu, X., Zhou, M., Damjanovi´c, B., "Common coupled fixed point theorem for geraghty-type contraction in partially ordered metric spaces", Journal of Function Spaces, Article ID 9063267: 11, (2018).

[12] Aslantas, M., Sahin, H., Turkoglu, D., "Some Caristi type fixed point theorems", The Journal Analysis, (2020), https://doi.org/10.1007/s41478-020-00248-8.

[13] Bakhtin, I. A., "The contraction mapping principle in almost metric space", Annals of Functional Analysis, 30: 26-37, (1989).

[14] Czerwik, C., "Contraction mapping in b-metric spaces", Acta Mathematica et Informatica Universitatis Ostraviensis, 1: 5-11, (1993).

[15] Parvaneh, V., Kadelburg, Z., "Extended partial b-metric spaces and some fixed point results", Filomat, 32(8): 2837-2850, (2018).

[16] Mustafa, Z., Sims, B., "A new approach to generalized metric spaces", J. Nonlinear Convex Anal, 7(2): 289-297, (2006).

[17] Aghajani, A., Abbas, M., Roshan, J. R., "Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces", Filomat, 28(6): 1087- 1101, (2014).

[18] Jleli, M., Samet, B., "A new generalization of the banach contraction principle", Journal of Inequalities and Applications, 38: (2014).

[19] Ahmad, J., Al-Mazrooei, A. E., Altun, I., "Generalized θ-contractive fuzzy mappings", Journal of Intelligent & Fuzzy Systems, 35: 1935-1942, (2018).

[20] Imdad, M., Alfaqih, W. M., Khan, I. A., "Weak θ-contractions and some fixed point results applications to fractal theory", Advances in Difference Equations, 439: (2018).

[21] Mustafa, Z., Jaradat, M., Jaradat, H. M., "A remarks on the paper some fixed point theorems for generalized contractive mappings in complete metric spaces", Journal of Mathematical Analysis, 8: 17-22, (2017).

[22] Abdeljawad, T., Ravi Agarwal, P., Karapinar, E., Sumati Kumari, P., "Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space", Symmetry, 11(686): (2019).

[23] Younis, M., Singh, D., Gopal, D., Goyal, A., Rathore, M.S., "On applications of generalized F-contraction to differential equations", Nonlinear Functional Analysis and Applications, 24(1): 155-174, (2019).

[24] Mustafa, Z., Parvaneh, V., Mohammed Jaradat, M.M., Kadelburg, Z., "Extended rectangular b- metric spaces and some fixed point theorems for contractive mappings", Symmetry, 11(594): (2019).

[25] Berinde, V., "Approximating fixed points of weak φ-contractions using the picard iteration", Fixed Point Theory, 4(2): 131-147, (2003).

[26] Geraghty, M., "On contractive mappings", Proceedings of the American Mathematical Society, 40: 604-608, (1973).