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On The Norms of Another Form of $r-$Circulant Matrices with The Hyper-Fibonacci and Lucas Numbers

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Abstract. In this paper, we compute the spectral norms of $r-$ circulant matrices with the hyper-Fibonacci and hyper-Lucas numbers of the forms $F_r = Circ - r(F_k^{(0)}, F_k^{(1)}, \ldots, F_k^{(n-1)})$, $L_r = Circ - r(L_k^{(0)}, L_k^{(1)}, \ldots, L_k^{(n-1)})$ and their Hadamard and Kronecker products. For this, we firstly compute the spectral and Euclidean norms of circulant matrices of the forms $F = Circ(F_k^{(0)}, F_k^{(1)}, \ldots, F_k^{(n-1)})$ and $L = Circ(L_k^{(0)}, L_k^{(1)}, \ldots, L_k^{(n-1)})$. Moreover, we give some examples related to special cases of our results.

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1. Introduction

The circulant matrices and $r-$ circulant matrices are closely related to signal processing, coding theory and many other areas [1, 10, 11]. An $n \times n$ $r-$circulant matrix C_r is of the form

$$C_r = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{bmatrix}.$$

When we take $r = 1$, the matrix $C_1 = C$ is called a circulant matrix. For brevity, we denote the matrices C_r and C_1 as $C_r = Circ - r\,(c_0, c_1, \ldots, c_{n-1})$ and $C = Circ\,(c_0, c_1, \ldots, c_{n-1})$, respectively. If A and B are circulant matrices then they are normal, their inverses (if any), conjugate transposes, sums and products are also circulant [8]. The eigenvalues of C are

$$\lambda_{\substack{m \\ 0\le m\le n-1}} = \sum_{k=0}^{n-1} c_k w^{-mk}$$

where $w = e^{\frac{2\pi i}{n}}$ and $i = \sqrt{-1}$ [8, 14].

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The circulant matrices and $r-$circulant matrices have been scientific research area in the recent past decades. Especially, the norms of circulant matrices with special elements such as Fibonacci or Fibonacci like numbers have been investigated extensively [2, 3, 5, 6, 15–19, 21–25]. Shen and Cen [21] derived upper and lower bounds for the spectral norms of r-circulant matrices in the forms $A = C_r(F_0, F_1, \ldots, F_{n-1})$ and $B = C_r(L_0, L_1, \ldots, L_{n-1})$. Tuğlu and Kızılateş [18] studied norms of circulant and $r-$circulant matrices involving harmonic Fibonacci and hyperharmonic Fibonacci numbers. Türkmen and Gökbaş [24] found some bound estimations for the spectral norm of $r-$circulant matrices with Pell and Pell-Lucas numbers. In [5], the authors computed spectral norms of circulant matrices in the forms $F = Circ(F_0^{(k)}, F_1^{(k)}, \ldots, F_{n-1}^{(k)})$, $L = Circ(L_0^{(k)}, L_1^{(k)}, \ldots, L_{n-1}^{(k)})$ and $r-$circulant matrices in the forms $F_r = Circ - r(F_0^{(k)}, F_1^{(k)}, \ldots, F_{n-1}^{(k)})$, $L_r = Circ - r(L_0^{(k)}, L_1^{(k)}, \ldots, L_{n-1}^{(k)})$, where $F_n^{(k)}$ and $L_n^{(k)}$ denote the hyper-Fibonacci and hyper-Lucas numbers, respectively.

In this research, we establish some bounds for the spectral norms of $r-$circulant matrices with the hyper-Fibonacci and hyper-Lucas numbers of the forms $F_r = Circ-r(F_k^{(0)}, F_k^{(1)}, \ldots, F_k^{(n-1)})$, $L_r = Circ-r(L_k^{(0)}, L_k^{(1)}, \ldots, L_k^{(n-1)})$ and their Hadamard and Kronecker products. For this, we firstly compute the spectral and Euclidean norms of circulant matrices of the forms $F = Circ(F_k^{(0)}, F_k^{(1)}, \ldots, F_k^{(n-1)})$ and $L = Circ(L_k^{(0)}, L_k^{(1)}, \ldots, L_k^{(n-1)})$. We use some relations concerning the spectral norm, Euclidean norm, row norm, column norm. Moreover, we give some examples related to special cases of our results.

2. Preliminaries

The Fibonacci numbers are defined by the recurrence relation: $F_{n+1} = F_n + F_{n-1}$ $(n \geq 1)$, $F_0 = 0$ and $F_1 = 1$. Similarly, the Lucas numbers are defined by $L_{n+1} = L_n + L_{n-1}$ $(n \geq 1)$, $L_0 = 2$ and $L_1 = 1$. Fibonacci and Lucas numbers have many generalizations [7, 9, 20]. In [9], Dil and Mezö introduced two concepts as hyper - Fibonacci numbers and hyper - Lucas numbers. These concepts are defined as

$$F_n^{(k)} = \sum_{s=0}^{n} F_s^{(k-1)}, \text{ with } F_n^{(0)} = F_n,\ F_0^{(k)} = 0 \text{ and } F_1^{(k)} = 1$$

and

$$L_n^{(k)} = \sum_{s=0}^{n} L_s^{(k-1)}, \text{ with } L_n^{(0)} = L_n,\ L_0^{(k)} = 2,\ L_1^{(k)} = 2k+1.$$

The hyper-Fibonacci and the hyper-Lucas numbers have the recurrence relations $F_n^{(k)} = F_{n-1}^{(k)} + F_n^{(k-1)}$ and $L_n^{(k)} = L_{n-1}^{(k)} + L_n^{(k-1)}$, respectively. Also, $F_n^{(k)}$ and $L_n^{(k)}$ have the following more explicit forms when $k = 1, 2, 3$ or $n = 2, 3$.

$$F_n^{(1)} = F_{n+2} - 1,\ F_n^{(2)} = F_{n+4} - n - 3 \text{ and } F_n^{(3)} = F_{n+6} - \frac{n^2 + 7n + 16}{2},$$

$$L_n^{(1)} = L_{n+2} - 1,\quad L_n^{(2)} = L_{n+4} - n - 5 \text{ and } L_n^{(3)} = L_{n+6} - \frac{n^2 + 11n + 32}{2}.$$

$$F_2^{(n)} = n + 1,\ F_3^{(n)} = \frac{n^2 + 3n + 4}{2} \text{ and } L_2^{(n)} = n^2 + 2n + 3. \tag{2.1}$$

In [4], the authors defined hyper-Horadam numbers and studied their some properties. Also, they gave the following formulas related to sums of hyper - Fibonacci and hyper - Lucas numbers

$$\sum_{s=0}^{r} F_n^{(s)} = F_{n+1}^{(r)} - F_{n-1} \tag{2.2}$$

and

$$\sum_{s=0}^{s} L_n^{(s)} = L_{n+1}^{(r)} - L_{n-1}.$$

For more information related to hyper - Fibonacci numbers see [4, 7, 9].

Now we give some definitions and lemmas related to our study.

Definition 2.1. Let $A = \left(a_{ij}\right)$ be any $m \times n$ matrix. The *Euclidean norm* of A is

$$\|A\|_E = \sqrt{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left|a_{ij}\right|^2\right)}.$$

Definition 2.2. Let $A = \left(a_{ij}\right)$ be any $m \times n$ matrix. The *spectral norm* of A is

$$\|A\|_2 = \sqrt{\max_i \lambda_i\left(A^H A\right)},$$

where $\lambda_i\left(A^H A\right)$ are eigenvalues of $A^H A$ and A^H is conjugate transpose of A.

There are two well known relations between Euclidean norm and spectral norm as the following:

$$\frac{1}{\sqrt{n}}\|A\|_E \le \|A\|_2 \le \|A\|_E \tag{2.3}$$

$$\|A\|_2 \le \|A\|_E \le \sqrt{n}\,\|A\|_2\,. \tag{2.4}$$

Definition 2.3 ([13]). Let $A = \left(a_{ij}\right)$ and $B = \left(b_{ij}\right)$ be $m \times n$ matrices.Then their Hadamard product A $\circ$ B is defined

$$A \circ B = \left[a_{ij}b_{ij}\right].$$

Definition 2.4 ([13]). Let $A = \left(a_{ij}\right)$ and $B = \left(b_{ij}\right)$ be $m \times n$ and $p \times r$ matrices, respectively.Then their Kronecker product A $\otimes$ B is defined

$$A \otimes B = \left[a_{ij}B\right].$$

Lemma 2.5 ([13]). *Let A and B be two $m \times n$ matrices. Then we have*

$$\|A \circ B\|_2 \le \|A\|_2\,\|B\|_2\,.$$

Lemma 2.6 ([13]). *Let A and B be two $m \times n$ matrices. Then we have*

$$\|A \circ B\|_2 \le r_1(A)\,c_1(B)$$

where $r_1(A) = \max\limits_{1\le i\le m}\sqrt{\sum\limits_{j=1}^{n}\left|a_{ij}\right|^2}$ *and* $c_1(B) = \max\limits_{1\le j\le n}\sqrt{\sum\limits_{i=1}^{m}\left|b_{ij}\right|^2}$.

Lemma 2.7 ([13]). *Let A and B be two $m \times n$ matrices. Then we have*

$$\|A \otimes B\|_2 = \|A\|_2\,\|B\|_2\,.$$

Lemma 2.8 ([12]). *Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then, A is a normal matrix if and only if the eigenvalues of $A^H A$ are $|\lambda_1|^2, |\lambda_2|^2, \ldots, |\lambda_n|^2$.*

3. Main Results

Theorem 3.1. *The spectral norm of the matrix $F = Circ(F_k^{(0)}, F_k^{(1)}, \ldots, F_k^{(n-1)})$ is*

$$\|F\|_2 = F_{k+1}^{(n-1)} - F_{k-1}.$$

Proof. Since the circulant matrix F is normal, its spectral norm is equal to its spectral radius. Furthermore, by considering F is irreducible and its entries are nonnegative, we have that the spectral radius (or spectral norm) of the matrix F is equal to its Perron root. We select an n-dimensional column vector $v = (1, 1, \ldots, 1)^T$, then

$$Fv = \left(\sum_{s=0}^{n-1}F_k^{(s)}\right)v.$$

Obviously, $\sum\limits_{s=0}^{n-1}F_k^{(s)}$ is an eigenvalue of F associated with v and it is the Perron root of F. Hence, by (2.2) we have

$$\|F\|_2 = \sum_{s=0}^{n-1}F_k^{(s)} = F_{k+1}^{(n-1)} - F_{k-1}.$$

This completes the proof. □

Example 3.2. Theorem 3.1 and the equations in (2.1) yield

$$\|F\|_2 = \begin{cases} 0, & \text{if } k = 0, \\ n, & \text{if } k = 1, \\ \frac{n^2+n}{2}, & \text{if } k = 2. \end{cases}$$

Corollary 3.3. *Euclidean norm of the matrix $F = Circ(F_k^{(0)}, F_k^{(1)}, \ldots, F_k^{(n-1)})$ holds*

$$F_{k+1}^{(n-1)} - F_{k-1} \leq \|F\|_E \leq \sqrt{n}(F_{k+1}^{(n-1)} - F_{k-1}).$$

Proof. The proof is trivial from Theorem 3.1 and the relation between spectral norm and Euclidean norm in (2.4). □

Corollary 3.4. *We have*

$$\frac{1}{\sqrt{n}}(F_{k+1}^{(n-1)} - F_{k-1}) \leq \sqrt{\sum_{s=0}^{n-1}\left(F_k^{(s)}\right)^2} \leq F_{k+1}^{(n-1)} - F_{k-1}. \tag{3.1}$$

Proof. This follows from the definition of Euclidean norm and Corollary 3.3. □

Theorem 3.5. *The spectral norm of the matrix $L = Circ(L_k^{(0)}, L_k^{(1)}, \ldots, L_k^{(n-1)})$ is*

$$\|L\|_2 = L_{k+1}^{(n-1)} - L_{k-1}.$$

Proof. This theorem can be proved by using a similar method to method of the proof of Theorem 3.1. □

Example 3.6. Theorem 3.5 and the equations in (2.1) yield

$$\|L\|_2 = \begin{cases} 2n, & \text{if } k = 0, \\ n^2 + 2n + 1, & \text{if } k = 1. \end{cases}$$

Corollary 3.7. *We have*

$$L_{k+1}^{(n-1)} - L_{k-1} \leq \|L\|_E \leq \sqrt{n}(L_{k+1}^{(n-1)} - L_{k-1}).$$

Proof. Theorem 3.5 and the relation between spectral norm and Euclidean norm in (2.4) immediately yield desired result. □

Corollary 3.8. *Sum of squares of hyper-Lucas numbers holds*

$$\frac{1}{\sqrt{n}}(L_{k+1}^{(n-1)} - L_{k-1}) \leq \sqrt{\sum_{s=0}^{n-1}\left(L_k^{(s)}\right)^2} \leq L_{k+1}^{(n-1)} - L_{k-1}. \tag{3.2}$$

Proof. From the definition of Euclidean norm and Corollary 3.7, desired result is obtained. □

Corollary 3.9. *The spectral norm of the Hadamard product of $F = Circ(F_k^{(0)}, F_k^{(1)}, \ldots, F_k^{(n-1)})$ and $L = Circ(L_k^{(0)}, L_k^{(1)}, \ldots, L_k^{(n-1)})$ satisfies*

$$\|F \circ L\|_2 \leq \left(F_{k+1}^{(n-1)} - F_{k-1}\right)\left(L_{k+1}^{(n-1)} - L_{k-1}\right).$$

Proof. Since $\|F \circ L\|_2 \leq \|F\|_2\,\|L\|_2$ desired result is trivial. □

Corollary 3.10. *The spectral norm of the Kronecker product of $F = Circ(F_k^{(0)}, F_k^{(1)}, \ldots, F_k^{(n-1)})$ and $L = Circ(L_k^{(0)}, L_k^{(1)}, \ldots, L_k^{(n-1)})$ satisfies*

$$\|F \otimes L\|_2 = \left(F_{k+1}^{(n-1)} - F_{k-1}\right)\left(L_{k+1}^{(n-1)} - L_{k-1}\right).$$

Proof. Since $\|F \otimes L\|_2 = \|F\|_2\,\|L\|_2$ we get the desired result. □

Theorem 3.11. *Let $F_r = Circ - r(F_k^{(0)}, F_k^{(1)}, \ldots, F_k^{(n-1)})$ be an $r-$circulant matrix.*
i) If $|r| \geq 1$, then

$$\frac{1}{\sqrt{n}}(F_{k+1}^{(n-1)} - F_{k-1}) \leq \|F_r\|_2 \leq |r|\left(F_{k+1}^{(n-1)} - F_{k-1}\right)^2$$

ii) If $|r| < 1$*, then*

$$\frac{|r|}{\sqrt{n}}(F_{k+1}^{(n-1)} - F_{k-1}) \leq \|F_r\|_2 \leq \sqrt{n}(F_{k+1}^{(n-1)} - F_{k-1}).$$

Proof. Since the matrix F_r is of the form

$$F_r = \begin{bmatrix} F_k^{(0)} & F_k^{(1)} & F_k^{(2)} & \cdots & F_k^{(n-2)} & F_k^{(n-1)} \\ rF_k^{(n-1)} & F_k^{(0)} & F_k^{(1)} & \cdots & F_k^{(n-3)} & F_k^{(n-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rF_k^{(2)} & rF_k^{(3)} & rF_k^{(4)} & \cdots & F_k^{(0)} & F_k^{(1)} \\ rF_k^{(1)} & rF_k^{(2)} & rF_k^{(3)} & \cdots & rF_k^{(n-1)} & F_k^{(0)} \end{bmatrix}$$

and from the definition of Euclidean norm, we have

$$\|F_r\|_E = \sqrt{\sum_{s=0}^{n-1}(n-s)\left(F_k^{(s)}\right)^2 + \sum_{s=0}^{n-1} s\,|r|^2\left(F_k^{(s)}\right)^2}.$$

i) Since $|r| \geq 1$, (3.1) yields

$$\|F_r\|_E \geq \sqrt{\sum_{s=0}^{n-1}(n-s)\left(F_k^{(s)}\right)^2 + \sum_{s=0}^{n-1} s\left(F_k^{(s)}\right)^2} = \sqrt{n\sum_{s=0}^{n-1}\left(F_k^{(s)}\right)^2} \geq F_{k+1}^{(n-1)} - F_{k-1}.$$

From (2.4)

$$\|F_r\|_2 \geq \frac{1}{\sqrt{n}}\left(F_{k+1}^{(n-1)} - F_{k-1}\right).$$

Let F_r be $F_r = B \circ C$, where

$$B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ rF_k^{(n-1)} & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rF_k^{(2)} & rF_k^{(3)} & rF_k^{(4)} & \cdots & 1 & 1 \\ rF_k^{(1)} & rF_k^{(2)} & rF_k^{(3)} & \cdots & rF_k^{(n-1)} & 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} F_k^{(0)} & F_k^{(1)} & F_k^{(2)} & \cdots & F_k^{(n-2)} & F_k^{(n-1)} \\ 1 & F_k^{(0)} & F_k^{(1)} & \cdots & F_k^{(n-3)} & F_k^{(n-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & F_k^{(0)} & F_k^{(1)} \\ 1 & 1 & 1 & \cdots & 1 & F_k^{(0)} \end{bmatrix}.$$

Then

$$\begin{aligned} r_1(B) &= \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}\left|b_{ij}\right|^2} = \sqrt{\sum_{j=1}^{n}\left|b_{nj}\right|^2} = \sqrt{1 + \sum_{s=1}^{n-1}|r|^2\left(F_k^{(s)}\right)^2} \\ &\leq |r|\sqrt{\sum_{s=0}^{n-1}\left(F_k^{(s)}\right)^2} \end{aligned}$$

and

$$c_1(C) = \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}\left|c_{ij}\right|^2} = \sqrt{\sum_{i=1}^{n}|c_{in}|^2} = \sqrt{\sum_{s=0}^{n-1}\left(F_k^{(s)}\right)^2}.$$

(3.1) and Lemma 2.3 yield

$$\|F_r\|_2 \leq r_1(B)\,c_1(C) \leq |r|\left(F_{k+1}^{(n-1)} - F_{k-1}\right)^2.$$

Thus,

$$\frac{1}{\sqrt{n}}\left(F_{k+1}^{(n-1)}-F_{k-1}\right)\leq\|F_r\|_2\leq|r|\left(F_{k+1}^{(n-1)}-F_{k-1}\right)^2.$$

ii) Since $|r|<1$, (3.1) yields

$$\begin{aligned}\|F_r\|_E &= \sqrt{\sum_{s=0}^{n-1}(n-s)\left(F_k^{(s)}\right)^2+\sum_{s=0}^{n-1}s\,|r|^2\left(F_k^{(s)}\right)^2}\\ &\geq \sqrt{\sum_{s=0}^{n-1}(n-s)\,|r|^2\left(F_k^{(s)}\right)^2+\sum_{s=0}^{n-1}s\,|r|^2\left(F_k^{(s)}\right)^2}\\ &= \sqrt{n\,|r|^2\sum_{s=0}^{n-1}\left(F_k^{(s)}\right)^2}\geq|r|\left(F_{k+1}^{(n-1)}-F_{k-1}\right).\end{aligned}$$

From (2.4)

$$\|F_r\|_2\geq\frac{|r|}{\sqrt{n}}\left(F_{k+1}^{(n-1)}-F_{k-1}\right).$$

Now, let the matrices F_r be $F_r=D\circ E$, where

$$D=\begin{bmatrix}1&1&1&\cdots&1&1\\ r&1&1&\cdots&1&1\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ r&r&r&\cdots&1&1\\ r&r&r&\cdots&r&1\end{bmatrix}$$

and

$$E=\begin{bmatrix}F_k^{(0)}&F_k^{(1)}&F_k^{(2)}&\cdots&F_k^{(n-2)}&F_k^{(n-1)}\\ F_k^{(n-1)}&F_k^{(0)}&F_k^{(1)}&\cdots&F_k^{(n-3)}&F_k^{(n-2)}\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ F_k^{(2)}&F_k^{(3)}&F_k^{(4)}&\cdots&F_k^{(0)}&F_k^{(1)}\\ F_k^{(1)}&F_k^{(2)}&F_k^{(3)}&\cdots&F_k^{(n-1)}&F_k^{(0)}\end{bmatrix}.$$

Then we compute $r_1(D)$ and $c_1(E)$ as

$$r_1(D)=\max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}\left|d_{ij}\right|^2}=\sqrt{n}$$

and

$$c_1(E)=\max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}\left|e_{ij}\right|^2}=\sqrt{\sum_{s=0}^{n-1}\left(F_k^{(s)}\right)^2}.$$

Hence, from (3.1) and Lemma 2.3, we have

$$\|F_r\|_2\leq r_1(B)\,c_1(E)\leq\sqrt{n}\left(F_{k+1}^{(n-1)}-F_{k-1}\right).$$

Thus,

$$\frac{|r|}{\sqrt{n}}\left(F_{k+1}^{(n-1)}-F_{k-1}\right)\leq\|F_r\|_2\leq\sqrt{n}\left(F_{k+1}^{(n-1)}-F_{k-1}\right).$$

This completes the proof. □

Example 3.12. By using Theorem 3.11 and the equations in (2.1), if $|r|\geq1$, we have

$$\|F_r\|_2=0,\quad if\quad k=0,$$

$$\sqrt{n} \leq \|F_r\|_2 \leq |r|\, n^2, \quad if \ \ k=1,$$
$$\frac{1}{\sqrt{n}}\left(\frac{n^2+n}{2}\right) \leq \|F_r\|_2 \leq |r|\left(\frac{n^2+n}{2}\right)^2, \quad if \ \ k=2,$$

and if $|r|<1$*, we have*

$$\|F_r\|_2 = 0, \quad if \ \ k=0,$$

$$|r|\ \sqrt{n} \leq \|F_r\|_2 \leq n^2\sqrt{n}, \quad if \ \ k=1,$$

$$\frac{|r|}{\sqrt{n}}\left(\frac{n^2+n}{2}\right) \leq \|F_r\|_2 \leq \ \sqrt{n}\left(\frac{n^2+n}{2}\right), \quad if \ \ k=2.$$

Theorem 3.13. *Let* $L_r = Circ - r(L_k^{(0)}, L_k^{(1)}, \ldots, L_k^{(n-1)})$ *be an* $r-$*circulant matrix.*
i) If $|r| \geq 1$*, then*

$$\frac{1}{\sqrt{n}}\left(L_{k+1}^{(n-1)} - L_{k-1}\right) \leq \|L_r\|_2 \leq |r|\left(L_{k+1}^{(n-1)} - L_{k-1}\right)^2.$$

ii) If $|r|<1$*, then*

$$\frac{|r|}{\sqrt{n}}\left(L_{k+1}^{(n-1)} - L_{k-1}\right) \leq \|L_r\|_2 \leq \ \sqrt{n}\left(L_{k+1}^{(n-1)} - L_{k-1}\right).$$

Proof. Since the matrix L_r is of the form

$$L_r = \begin{bmatrix} L_k^{(0)} & L_k^{(1)} & L_k^{(2)} & \cdots & L_k^{(n-2)} & L_k^{(n-1)} \\ rL_k^{(n-1)} & L_k^{(0)} & L_k^{(1)} & \cdots & L_k^{(n-3)} & L_k^{(n-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rL_k^{(2)} & rL_k^{(3)} & rL_k^{(4)} & \cdots & L_k^{(0)} & L_k^{(1)} \\ rL_k^{(1)} & rL_k^{(2)} & rL_k^{(3)} & \cdots & rL_k^{(n-1)} & L_k^{(0)} \end{bmatrix}$$

and from the definition of Euclidean norm, we have

$$\|L_r\|_E = \sqrt{\sum_{s=0}^{n-1}(n-s)\left(L_k^{(s)}\right)^2 + \sum_{s=0}^{n-1} s\,|r|^2\left(L_k^{(s)}\right)^2}.$$

i) Since $|r| \geq 1$, (3.2) yields

$$\|L_r\|_E \geq \sqrt{\sum_{s=0}^{n-1}(n-s)\left(L_k^{(s)}\right)^2 + \sum_{s=0}^{n-1} s\left(L_k^{(s)}\right)^2} = \sqrt{n\sum_{s=0}^{n-1}\left(L_k^{(s)}\right)^2} \geq L_{k+1}^{(n-1)} - L_{k-1}.$$

From (2.3)

$$\|L_r\|_2 \geq \frac{1}{\sqrt{n}}\left(L_{k+1}^{(n-1)} - L_{k-1}\right).$$

Now, let the matrices L_r be $L_r = B \circ C$, where

$$B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ rL_k^{(n-1)} & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rL_k^{(2)} & rL_k^{(3)} & rL_k^{(4)} & \cdots & 1 & 1 \\ rL_k^{(1)} & rL_k^{(2)} & rL_k^{(3)} & \cdots & rL_k^{(n-1)} & 1 \end{bmatrix}$$

and

$$C=\begin{bmatrix} L_k^{(0)} & L_k^{(1)} & L_k^{(2)} & \cdots & L_k^{(n-2)} & L_k^{(n-1)} \\ 1 & L_k^{(0)} & L_k^{(1)} & \cdots & L_k^{(n-3)} & L_k^{(n-2)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & L_k^{(0)} & L_k^{(1)} \\ 1 & 1 & 1 & \cdots & 1 & L_k^{(0)} \end{bmatrix}.$$

Then we have

$$\begin{aligned} r_1(B) &= \max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}\left|b_{ij}\right|^2}=\sqrt{\sum_{j=1}^{n}\left|b_{nj}\right|^2}=\sqrt{1+\sum_{s=1}^{n-1}|r|^2\left(L_k^{(s)}\right)^2} \\ &\le |r|\sqrt{\sum_{s=0}^{n-1}\left(L_k^{(s)}\right)^2} \end{aligned}$$

and

$$c_1(C)=\max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}\left|c_{ij}\right|^2}=\sqrt{\sum_{i=1}^{n}|c_{in}|^2}=\sqrt{\sum_{s=0}^{n-1}\left(L_k^{(s)}\right)^2}.$$

Hence, from (3.2) and Lemma 2.3, we have

$$\|L_r\|_2\le r_1(B)\,c_1(C)\le |r|\left(L_{k+1}^{(n-1)}-L_{k-1}\right)^2.$$

Thus, we write

$$\frac{1}{\sqrt{n}}\left(L_{k+1}^{(n-1)}-L_{k-1}\right)\le\|L_r\|_2\le|r|\left(L_{k+1}^{(n-1)}-L_{k-1}\right)^2.$$

ii) Since $|r|<1$, (3.2) yields

$$\begin{aligned} \|L_r\|_E &= \sqrt{\sum_{s=0}^{n-1}(n-s)\left(L_k^{(s)}\right)^2+\sum_{s=0}^{n-1}s\,|r|^2\left(L_k^{(s)}\right)^2} \\ &\ge \sqrt{\sum_{s=0}^{n-1}(n-s)\,|r|^2\left(L_k^{(s)}\right)^2+\sum_{s=0}^{n-1}s\,|r|^2\left(L_k^{(s)}\right)^2} \\ &= \sqrt{n\,|r|^2\sum_{s=0}^{n-1}\left(L_k^{(s)}\right)^2}\ge|r|\left(L_{k+1}^{(n-1)}-L_{k-1}\right). \end{aligned}$$

From (2.3)

$$\|L_r\|_2\ge\frac{|r|}{\sqrt{n}}\left(L_{k+1}^{(n-1)}-L_{k-1}\right).$$

Now, let the matrices B and C be as

$$D=\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ r & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & r & \cdots & 1 & 1 \\ r & r & r & \cdots & r & 1 \end{bmatrix}$$

and

$$E=\begin{bmatrix} L_k^{(0)} & L_k^{(1)} & L_k^{(2)} & \cdots & L_k^{(n-2)} & L_k^{(n-1)} \\ L_k^{(n-1)} & L_k^{(0)} & L_k^{(1)} & \cdots & L_k^{(n-3)} & L_k^{(n-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ L_k^{(2)} & L_k^{(3)} & L_k^{(4)} & \cdots & L_k^{(0)} & L_k^{(1)} \\ L_k^{(1)} & L_k^{(2)} & L_k^{(3)} & \cdots & L_k^{(n-1)} & L_k^{(0)} \end{bmatrix}.$$

That is, $L_r = B \circ C$. Then we obtain

$$r_1(D) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} \left|d_{ij}\right|^2} = \sqrt{n}$$

and

$$c_1(E) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} \left|e_{ij}\right|^2} = \sqrt{\sum_{s=0}^{n-1} \left(L_k^{(s)}\right)^2}.$$

(3.2) and Lemma 2.3 yield

$$\|L_r\|_2 \le r_1(B)\, c_1(E) \le \sqrt{n}\left(L_{k+1}^{(n-1)} - L_{k-1}\right).$$

Thus,

$$\frac{|r|}{\sqrt{n}}\left(L_{k+1}^{(n-1)} - L_{k-1}\right) \le \|L_r\|_2 \le \sqrt{n}\left(L_{k+1}^{(n-1)} - L_{k-1}\right).$$

This completes the proof. □

Example 3.14. By using Theorem 3.13 and the equations in (2.1), if $|r| \ge 1$, we have

$$2\sqrt{n} \le \|L_r\|_2 \le 4n^2|r|, \quad \text{if } k = 0,$$

$$\frac{1}{\sqrt{n}}\left(n^2+2n+1\right) \le \|L_r\|_2 \le |r|\left(n^2+2n+1\right)^2, \quad \text{if } k = 1,$$

and if $|r| < 1$*, we have*

$$2\sqrt{n}\,|r| \le \|L_r\|_2 \le 2n\sqrt{n}, \quad \text{if } k = 0,$$

$$\frac{|r|}{\sqrt{n}}\left(n^2+2n+1\right) \le \|L_r\|_2 \le \sqrt{n}\left(n^2+2n+1\right), \quad \text{if } k = 1.$$

Corollary 3.15. *The spectral norm of the Hadamard product of* $F_r = Circ - r(F_k^{(0)}, F_k^{(1)}, \ldots, F_k^{(n-1)})$ *and* $L_r = Circ - r(L_k^{(0)}, L_k^{(1)}, \ldots, L_k^{(n-1)})$ *holds*

i) If $|r| \ge 1$*, then*

$$\|F_r \circ L_r\|_2 \le |r|^2\left(F_{k+1}^{(n-1)} - F_{k-1}\right)^2\left(L_{k+1}^{(n-1)} - L_{k-1}\right)^2.$$

ii) If $|r| < 1$*, then*

$$\|F_r \circ L_r\|_2 \le n\left(F_{k+1}^{(n-1)} - F_{k-1}\right)\left(L_{k+1}^{(n-1)} - L_{k-1}\right).$$

Proof. The proof is trivial since $\|F_r \circ L_r\|_2 \le \|F_r\|_2\,\|L_r\|_2$. □

Corollary 3.16. *The spectral norm of the Kronecker product of* $F_r = Circ - r(F_k^{(0)}, F_k^{(1)}, \ldots, F_k^{(n-1)})$ *and* $L_r = Circ - r(L_k^{(0)}, L_k^{(1)}, \ldots, L_k^{(n-1)})$ *holds*

i) If $|r| \ge 1$*, then*

$$\frac{1}{n}\left(F_{k+1}^{(n-1)} - F_{k-1}\right)\left(L_{k+1}^{(n-1)} - L_{k-1}\right) \le \|F_r \otimes L_r\|_2 \le |r|^2\left(F_{k+1}^{(n-1)} - F_{k-1}\right)^2\left(L_{k+1}^{(n-1)} - L_{k-1}\right)^2.$$

ii) If $|r| < 1$*, then*

$$\frac{|r|^2}{n}\left(F_{k+1}^{(n-1)} - F_{k-1}\right)\left(L_{k+1}^{(n-1)} - L_{k-1}\right) \le \|F_r \otimes L_r\|_2 \le n\left(F_{n-1}^{(k+1)}\right)^2\left(L_{n-1}^{(k+1)}\right).$$

Proof. The proof is trivial since $\|F_r \otimes L_r\|_2 = \|F_r\|_2\,\|L_r\|_2$. □

4. Conclusion

In this study, we present some bounds for the spectral norms of a different form of $r-$circulant matrices with the hyper-Fibonacci and hyper-Lucas numbers by using some relations concerning the spectral norm, Euclidean norm, row norm, column norm. The importance of our results is that our results depend on hyper-Fibonacci and hyper-Lucas numbers.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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