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# A Note on Modified $k$-Pell Hybrid Numbers

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## Abstract

Modified $k$-Pell hybrid sequence is defined in this paper. The Binet formula and the generating functions for modified k-Pell hybrid numbers are also presented. By using the Binet formula, some properties involving this sequence, including Catalan's, Cassini's and d'Ocagne's identities, are obtained.

***Keywords:*** *Modified k-Pell numbers; Hybrid numbers; Recurrence relations; Generating functions; Binet's formula.*
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## 1. Introduction

A new non-commutative number system was introduced by Özdemir in [15] which can be considered as a generalization of three number systems: the complex, the hyperbolic and the dual. Let us consider the set $\mathbb{K}$ of hybrid numbers defined by

$$\mathbb{K} = \{a+bi+c\varepsilon+dh : a,\ b,\ c,\ d \in \mathbb{R}, i^2 = -1,\ \varepsilon^2 = 0,\ h^2 = 1,\ ih = -hi = \varepsilon + i\},$$

where $i$, $\varepsilon$ and $h$ are operators that satisfy the equalities previously stated. Using these equalities and the following multiplication table

**Table 1:** The multiplication table for $\mathbb{K}$

| $\cdot$ | $1$ | $i$ | $\varepsilon$ | $h$ |
|---|---|---|---|---|
| $1$ | $1$ | $i$ | $\varepsilon$ | $h$ |
| $i$ | $i$ | $-1$ | $1-h$ | $\varepsilon+i$ |
| $\varepsilon$ | $\varepsilon$ | $h+1$ | $0$ | $-\varepsilon$ |
| $h$ | $h$ | $-\varepsilon-i$ | $\varepsilon$ | $1$ |

the multiplication (not commutative, but associative) of two hybrid numbers $z_1 = a_1 + b_1 i + c_1\varepsilon + d_1 h$ and $z_2 = a_2 + b_2 i + c_2\varepsilon + d_2 h$ can be expressed as

$$\begin{aligned}&(a_1+b_1 i+c_1\varepsilon+d_1 h)(a_2+b_2 i+c_2\varepsilon+d_2 h)\\ &\quad= a_1a_2 - b_1b_2 + b_1c_2 + c_1b_2 + d_1d_2 + (a_1b_2 + b_1a_2 + b_1d_2 - d_1b_2)i + (a_1c_2 + b_1d_2 + c_1a_2 - c_1d_2 - d_1b_2 + d_1c_2)\varepsilon\\ &\qquad + (a_1d_2 - b_1c_2 + c_1b_2 + d_1a_2)h.\end{aligned}$$

About the addition of hybrid numbers, such algebraic operation is done component-wise and is commutative and associative.
The conjugate of a hybrid number $z = a+bi+c\varepsilon+dh$ is the hybrid number $\bar{z} = a - bi - c\varepsilon - dh$ and the real number $C(z) = z\cdot\bar{z} = \bar{z}\cdot z = a^2 + (b-c)^2 - c^2 - d^2$ is called the character of the hybrid number $z$. The norm of a hybrid number $z$ is denoted by $||z||$ and is given by $\sqrt{z\cdot\bar{z}}$. Note that a new expression for the character of a hybrid number $z$ is given by

$$C(z) = (a-b)^2 - 2b(c-a) - d^2. \tag{1.1}$$

The hybrid matrix corresponding to the hybrid number $z$ is the following $2\times 2$ matrix

$$A = \begin{bmatrix} a+c & b-c+d \\ c-b+d & a-c \end{bmatrix}. \tag{1.2}$$

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Several researchers have been devoted to the sequences of positive integers which are defined recursively. In this work we have as a base one of these sequences of integers - the Modified $k$-Pell sequence - which is also related to two other examples - that of the $k$-Pell sequence and the $k$-Pell-Lucas sequence. These sequences are generalizations of the sequences of Pell, Pell-Lucas and Modified Pell numbers. Such generalizations are studied in [2] as well as in [3] and in [4], which, for any positive real number $k$, they are defined by the recurrence relations of second order given by:

$$P_{k,n} = 2P_{k,n-1} + kP_{k,n-2},\ n \geq 2 \tag{1.3}$$

with initial terms $P_{k,0} = 0$ and $P_{k,1} = 1$, for $k$-Pell sequence;

$$Q_{k,n} = 2Q_{k,n-1} + kQ_{k,n-2},\ n \geq 2 \tag{1.4}$$

with initial terms $Q_{k,0} = Q_{k,1} = 2$, for $k$-Pell-Lucas sequence; and

$$q_{k,n} = 2q_{k,n-1} + kq_{k,n-2},\ n \geq 2 \tag{1.5}$$

with initial terms $q_{k,0} = q_{k,1} = 1$, for Modified $k$-Pell sequence. Note that in the particular case where $k = 1$, (1.3)-(1.4)-(1.5) reduces to the case of Pell, Pell-Lucas and Modified Pell sequences.
The characteristic equation associated with the recurrence relations (1.3), (1.4) and (1.5) is $r^2 - 2r - k = 0$ whose roots are $r_1 = 1+\sqrt{1+k}$ and $r_2 = 1-\sqrt{1+k}$. Clearly, $r_1 + r_2 = 2,\ r_1 - r_2 = 2\sqrt{1+k},\ r_1 r_2 = -k$ and $\frac{r_1}{r_2} = \frac{(r_1)^2}{-k},\ \frac{r_2}{r_1} = \frac{(r_2)^2}{-k}$.
The Binet formulae for these sequences are given by

$$P_{k,n} = \frac{(r_1)^n - (r_2)^n}{r_1 - r_2},\ Q_{k,n} = (r_1)^n + (r_2)^n,\ q_{k,n} = \frac{(r_1)^n + (r_2)^n}{2}. \tag{1.6}$$

The following results involving Modified $k$-Pell and $k$-Pell numbers will be useful in the next section.

**Lemma 1.1.** *If $P_{k,j}$ and $q_{k,j}$ are respectively the jth k-Pell and Modified k-Pell numbers, the following identities are true:*

1. $q_{k,j} - q_{k,j+1} = -(1+k)P_{k,j}$;
2. $P_{k,2j+1} - (-k)^j = 2q_{k,j+1}P_{k,j}$;
3. $q_{k,j} - q_{k,j+2} = -(1+k)(2P_{k,j} + q_{k,j})$.

*Proof.* 1. Using the Binet formula of a Modified $k$-Pell number in (1.6), we easily obtain

$$\begin{aligned} q_{k,j} - q_{k,j+1} &= \frac{(r_1)^j + (r_2)^j - (r_1)^{j+1} - (r_2)^{j+1}}{2} \\ &= \frac{(r_1)^j\left(-\sqrt{1+k}\right) + (r_2)^j\left(\sqrt{1+k}\right)}{2} \\ &= \frac{\left(-\sqrt{1+k}\right)\left((r_1)^j - (r_2)^j\right)}{2} \end{aligned}$$

and by the use of the Binet formula of a $k$-Pell number in (1.6) and the fact that $r_1 - r_2 = 2\sqrt{1+k}$, the required result will follow.
2. Once more, the use the Binet formula of a Modified $k$-Pell and a $k$-Pell number stated in (1.6) is the main key for the proof. Hence we obtain

$$\begin{aligned} 2q_{k,j+1}P_{k,j} &= 2\left(\frac{(r_1)^{j+1} + (r_2)^{j+1}}{2}\right)\left(\frac{(r_1)^j - (r_2)^j}{r_1 - r_2}\right) \\ &= \frac{\left((r_1)^{2j+1} - (r_2)^{2j+1}\right) - (r_1 r_2)^j (r_1 - r_2)}{r_1 - r_2} \end{aligned}$$

and using the fact that $r_1 r_2 = -k$, the result follows.
3. The proof of this identity is similar to the proof of item 1. □

More detail can be found in the extensive literature dedicated to these sequences. Even so, some of their properties can be found in [1], [2], [3], [4], [5], [7], [8], [9], [10], [11], [12], [14], [20] and [21], among others.
Also different types of Hybrid numbers have been introduced and studied. For instance, this is the case of Jacobsthal and Jacobsthal-Lucas Hybrid numbers in [17], the case of $k$-Pell hybrid numbers in [6], the case of Pell and Pell-Lucas Hybrid numbers in [18], the case of Horadam Hybrid Numbers in [19] and the case of a new generalization of Fibonacci hybrid and Lucas hybrid numbers in [13].
In this paper, it is our aim to introduce a new sequence with Hybrid numbers using the Modified $k$-Pell numbers referred before. Motivated essentially by the recent work of Özdemir in [15], we introduce the Modified $k$-Pell Hybrid sequences and we give some properties, including the Binet formula and the generating function for these sequences. Some identities involving these sequences are also provided.

## 2. The Modified $k$-Pell Hybrid Sequence, the Binet Formula and the Generating Functions

The principal goals of this section will be to define the Modified $k$-Pell hybrid sequence and to present some elementary results involving it. First of all, we define the Modified $k$-Pell sequence, denoted by $\{Hq_{k,n}\}$

$$Hq_{k,n} = q_{k,n} + q_{k,n+1}i + q_{k,n+2}\varepsilon + q_{k,n+3}h. \tag{2.1}$$

Note that the sequence $\{Hq_{k,n}\}$ of Modified $k$-Pell hybrid numbers satisfies the following second order recursive relation:

$$Hq_{k,n+1} = 2Hq_{k,n} + kHq_{k,n-1} \tag{2.2}$$

with initial conditions $Hq_{k,0} = 1 + i + (2+k)\varepsilon + (4+3k)h$ and $Hq_{k,1} = 1 + (2+k)i + (4+3k)\varepsilon + \left(8+8k+k^2\right)h$.
Taking into account (1.1), the norm and character of a Modified $k$-Pell hybrid number is given as

$$||Hq_{k,n}|| = \sqrt{C(Hq_{k,n})}$$

where

$$C(Hq_{k,n}) = \left(q_{k,n} - q_{k,n+1}\right)^2 - 2q_{k,n+1}\left(q_{k,n+2} - q_{k,n}\right) - q_{k,n+3}^2. \tag{2.3}$$

Using the matrix in (1.2), the corresponding hybrid matrix of $Hq_{k,n}$ is the following $2\times 2$ matrix

$$A_{Hq_{k,n}} = \begin{bmatrix} q_{k,n} + q_{k,n+2} & q_{k,n+1} - q_{k,n+2} + q_{k,n+3} \\ q_{k,n+2} - q_{k,n+1} + q_{k,n+3} & q_{k,n} - q_{k,n+2} \end{bmatrix}. \tag{2.4}$$

Using the definition of norm, the character of a hybrid number and the hybrid matrix in (2.4), we easily obtain the identities stated in the following result:

**Proposition 2.1.** *For a natural number n and for any positive real number k, the norm of the nth Modified k-Pell hybrid numbers* $||Hq_{k,n}||^2$ *and the determinant of the corresponding hybrid matrix satisfy respectively the following identities:*

$$||Hq_{k,n}||^2 = (1+k)\left((1+k)\left(P_{k,n}\right)^2 - P_{k,2n+1} - P_{k,2n+2} + (-k)^n - \left(P_{k,n+3}\right)^2\right) - (-k)^{n+3}$$

*and*

$$det(A_{Hq_{k,n}}) = C(Hq_{k,n}).$$

*Proof.* Since $||Hq_{k,n}||^2 = C(Hq_{k,n})$ and taking into account (2.3), the identities 1. and 3. of Lemma 1.1, we easily obtain

$$\begin{aligned} ||Hq_{k,n}||^2 &= \left(q_{k,n} - q_{k,n+1}\right)^2 - 2q_{k,n+1}\left(q_{k,n+2} - q_{k,n}\right) - q_{k,n+3}^2 \\ &= (1+k)^2\left(P_{k,n}\right)^2 - 2(1+k)q_{k,n+1}\left(2P_{k,n} + q_{k,n}\right) - \left(q_{k,n+3}\right)^2. \end{aligned}$$

Now by the use of the identity 2. of Lemma 1.1, items (ii), (vi) of Proposition 2.1 of [21] and doing some calculations we get the required result.
For the second identity, it is enough to use the Theorem 3.3 of [15] and then the result follows. □

In order to find the generating function for the Modified $k$-Pell hybrid sequence, we shall write the sequence as a power series where each term of the sequence correspond to coefficients of the series. For background about generating functions, see, for example, [16]. As a consequence of the definition of generating function of a sequence, the generating function associated to $\{Hq_{k,n}\}_{n=0}^{\infty}$, denoted by $g_{Hq_{k,n}}(t)$, is defined by

$$g_{Hq_{k,n}}(t) = \sum_{n=0}^{\infty} Hq_{k,n}t^n. \tag{2.5}$$

Consequently, we obtain the following result:

**Theorem 2.2.** *The generating function for the Modified k-Pell hybrid numbers is given by*

$$g_{Hq_{k,n}}(t) = \frac{Hq_{k,0} + \left(Hq_{k,1} - 2Hq_{k,0}\right)t}{1 - 2t - kt^2}.$$

*Proof.* Using (2.5), we have

$$g_{Hq_{k,n}}(t) = Hq_{k,0} + Hq_{k,1}t + Hq_{k,2}t^2 + \cdots + Hq_{k,n}t^n + \cdots \tag{2.6}$$

Multiplying both sides of (2.6) by $-2t$ we obtain

$$-2tg_{Hq_{k,n}}(t) = -2Hq_{k,0}t - 2Hq_{k,1}t^2 - 2Hq_{k,2}t^3 - \cdots - 2Hq_{k,n}t^{n+1} - \cdots \tag{2.7}$$

Now multiplying both sides of (2.6) by $-kt^2$ we get

$$-kg_{Hq_{k,n}}(t)t^2 = -kHq_{k,0}t^2 - kHq_{k,1}t^3 - kHq_{k,2}t^4 - \cdots - kHq_{k,n}t^{n+2} - \cdots \tag{2.8}$$

From (2.2), (2.5), (2.7) and (2.8), we have

$$(1 - 2t - kt^2)g_{Hq_{k,n}}(t) = Hq_{k,0} + (Hq_{k,1} - 2Hq_{k,0})t$$

and the result follows. □

The following result gives the Binet formula for $Hq_{k,n}$.

**Theorem 2.3.** *[The Binet formula] For $n \geq 0$, we have*

$$Hq_{k,n} = \frac{\widehat{r_1}(r_1)^n + \widehat{r_2}(r_2)^n}{2}$$

*where $\widehat{r_1}$ and $\widehat{r_2}$ are hybrid numbers defined by $\widehat{r_1} = 1 + r_1 i + (k+2r_1)\varepsilon + (2k + r_1(k+4))h$ and $\widehat{r_2} = 1 + r_2 i + (k+2r_2)\varepsilon + (2k + r_2(k+4))h$, respectively.*

*Proof.* 1. Using (2.1) and the Binet formula for the Modified $k$-Pell numbers considered in (1.6), we have

$$\begin{aligned}
Hq_{k,n} &= q_{k,n} + q_{k,n+1}i + q_{k,n+2}\varepsilon + q_{k,n+3}h \\
&= q_{k,n} + q_{k,n+1}i + q_{k,n+2}\varepsilon + \left(2q_{k,n+2} + kq_{k,n+1}\right)h \\
&= q_{k,n} + (i + kh)\, q_{k,n+1} + (\varepsilon + 2h)\, q_{k,n+2} \\
&= q_{k,n} + (i + kh)\, q_{k,n+1} + 2q_{k,n+1}\varepsilon + 4q_{k,n+1}h + kq_{k,n}\varepsilon + 2kq_{k,n}h \\
&= q_{k,n}(1 + k\varepsilon + 2kh) + q_{k,n+1}(i + kh + 2\varepsilon + 4h) \\
&= \left(\frac{(r_1)^n + (r_2)^n}{2}\right)(1 + k\varepsilon + 2kh) + \left(\frac{(r_1)^{n+1} + (r_2)^{n+1}}{2}\right)(i + kh + 2\varepsilon + 4h).
\end{aligned}$$

The last equation proves the theorem. □

## 3. Some Identities Involving This Type of Sequence

In this section, we state some identities related to these type of hybrid sequences. As a consequence of the Binet formula of Theorem 2.3, we get for these sequences the following interesting identities. The following identities obtained by the multiplication of hybrid numbers are very useful for later use:

**Lemma 3.1.**

$$\widehat{r_1}\widehat{r_2} = U + V,$$
$$\widehat{r_2}\widehat{r_1} = U - V$$

*and*

$$\widehat{r_1}\widehat{r_2} + \widehat{r_2}\widehat{r_1} = 2U \tag{3.1}$$

*where $U = 2Hq_{k,0} - (k^3 + k + 1)$ and $V = 2k\sqrt{1+k}\Big(2i + (2+k)\varepsilon - h\Big)$.*

We begin with Catalan's identity.

**Proposition 3.2.** *[Catalan's identity] For natural numbers $n$, $s$ and $n \geq s$, if $Hq_{k,n}$ is the nth Modified k-Pell hybrid number, then the following identity is true:*

$$Hq_{k,n+s}Hq_{k,n-s} - (Hq_{k,n})^2 = (-k)^{n-s}(1+k)\left[k(2i + (2+k)\varepsilon - h)P_{k,2s} + UP_{k,s}^2\right]$$

*Proof.* Using the Binet Formula of Theorem 2.3, the identities which involve sum, subtraction, product and quotient of $r_1$, $r_2$ and the fact that the multiplication of two Modified $k$-Pell hybrid numbers is not commutative, we have

$$\begin{aligned}
Hq_{k,n+s}Hq_{k,n-s} - (Hq_{k,n})^2 &= \left(\frac{\widehat{r_1}r_1^{n+s} + \widehat{r_2}r_2^{n+s}}{2}\right)\left(\frac{\widehat{r_1}r_1^{n-s} + \widehat{r_2}r_2^{n-s}}{2}\right) - \left(\frac{\widehat{r_1}r_1^{n} + \widehat{r_2}r_2^{n}}{2}\right)^2 \\
&= \frac{1}{4}\left[\widehat{r_1}\widehat{r_2}r_1^{n+s}r_2^{n-s} + \widehat{r_2}\widehat{r_1}r_1^{n-s}r_2^{n+s} - (r_1r_2)^n(\widehat{r_1}\widehat{r_2} + \widehat{r_1}\widehat{r_2})\right] \\
&= \frac{(r_1r_2)^{n-s}}{4}\left[\widehat{r_1}\widehat{r_2}r_1^{2s} + \widehat{r_2}\widehat{r_1}r_2^{2s} - 2(-k)^s U\right] \\
&= \frac{(r_1r_2)^{n-s}}{4}\left[U(r1^{2s} + r_2^{2s}) + V(r1^{2s} - r_2^{2s}) - 2(-k)^s U\right] \\
&= \frac{(r_1r_2)^{n-s}}{4}\left[2k\sqrt{1+k}(2i + (2+k)\varepsilon - h)(r1^{2s} - r_2^{2s}) + 2U(q_{k,2s} - (-k)^s)\right] \\
&= \frac{(r_1r_2)^{n-s}}{4}\left[4k(1+k)(2i + (2+k)\varepsilon - h)P_{k,2s} + 2U(q_{k,2s} - (-k)^s)\right]
\end{aligned}$$

by using the identity $q_{k,2s} - (-k)^s = 2(1+k)P_{k,s}^2$, we get the required result. □

Note that for $s = 1$ in Catalan's identity obtained, we get the Cassini identity for the Modified $k$-Pell hybrid sequence. In fact, for $s = 1$, the identity stated in Proposition 3.2, yields:

**Proposition 3.3.** *[Cassini's identity] For a natural number n, if $Hq_{k,n}$ is the nth Modified k-Pell hybrid number, then the following identity is true:*

$$Hq_{k,n+1}Hq_{k,n-1} - (Hq_{k,n})^2 = (-k)^{n-1}(1+k)\Big[2k(2i+(2+k)\varepsilon - h) + U\Big].$$

□

The d'Ocagne identity can also be obtained using the Binet formula and in this case we obtain:

**Proposition 3.4.** *[d'Ocagne's identity] For natural numbers $m,\ n$, if $m > n$ and $Hq_{k,n}$ is the nth Modified k-Pell hybrid number, then the following identity is true:*

$$Hq_{k,m}Hq_{k,n+1} - Hq_{k,m+1}Hq_{k,n} = (-k)^m(1+k)\Big[(2Hq_{k,0} - k^3 - k - 1)P_{k,n-m} - 2k(2i+2\varepsilon+k\varepsilon - h)q_{k,n-m}\Big].$$

*Proof.* Once more, using the Binet Formula of Theorem 2.3 and the fact that $r_1r_2 = -k,\ r_1 - r_2 = 2\sqrt{1+k}$,

$$\begin{aligned} Hq_{k,m}Hq_{k,n+1} - Hq_{k,m+1}Hq_{k,n} &= \frac{1}{4}\left(\widehat{r_1}r_1^m + \widehat{r_2}r_2^m\right)\left(\widehat{r_1}r_1^{n+1} + \widehat{r_2}r_2^{n+1}\right) - \frac{1}{4}\left(\widehat{r_1}r_1^{m+1} + \widehat{r_2}r_2^{m+1}\right)\left(\widehat{r_1}r_1^n + \widehat{r_2}r_2^n\right) \\ &= \frac{1}{4}\left[\widehat{r_1}\widehat{r_2}(r_1^m r_2^{n+1} - r_1^{m+1}r_2^n) + \widehat{r_2}\widehat{r_1}(r_1^{n+1}r_2^m - r_1^n r_2^{m+1})\right] \\ &= \frac{\sqrt{1+k}}{2}(-k)^m\left(\widehat{r_2}\widehat{r_1}r_1^{n-m} - \widehat{r_1}\widehat{r_2}r_2^{n-m}\right) \\ &= (-k)^m\left[(1+k)UP_{k,n-m} - \sqrt{1+k}Vq_{k,n-m}\right] \end{aligned}$$

after substituting $U$ and $V$ into the last equation, the result follows. □

It should be noted that $P_{k,-n} = -(-k)^n P_{k,n}$ and $q_{k,-n} = (-k)^n q_{k,n}$.

## 4. Conclusion

In this paper, the sequence of Modified $k$-Pell hybrid numbers defined by a recurrence relation of second order was introduced. Some properties involving these sequences, including the Binet-style formulae and the generating functions were presented. Note that the results stated in this paper for the particular case of $k = 1$, give us the correspondent results for the sequences of the Modified Pell hybrid numbers.

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