Abstract
The purpose of the present paper is to introduce a new subclass of harmonic univalent functions by using fractional calculus operator associated with $q$-calculus. Coefficient condition, extreme points, distortion bounds, convolution and convex combination are obtained for this class. Finally, we discuss a class preserving integral operator for this class.

Keywords: Harmonic functions; Fractional calculus; $q$-calculus.

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1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain $D$ if both $u$ and $v$ are real harmonic in $D$. In any simply connected domain we can write $f = h + \overline{g}$ where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $D$ is that $|h'(z)| > |g'(z)|$, $z \in D$. For detailed study one may refer to Clunie and Sheil-Small [3] and Duren [5], (see also [9]).

Let $S_H$ represent the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the open unit disk $U = \{ z : |z| < 1 \}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions $h$ and $g$ as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1.$$  

(1.1)

Note that the class $S_H$ reduces to the class $S$ of normalized analytic univalent functions if the co-analytic part of its member is zero. For this class the function $f(z)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$  

(1.2)
Further, we let $V_H^n$ be the subclass of $S_H$ consisting of functions of form $f = h + g^n$, where
\[
h(z) = z + \sum_{k=2}^{\infty} |a_k| z^k, \quad g_n(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k, \quad |b_1| < 1.
\] (1.3)

The following definitions of fractional derivatives and fractional integrals are due to Owa [11] and Srivastava and Owa [18].

**Definition 1.1.** The fractional integral of order $\lambda$ is defined for a function $f(z)$ of the form (1.2) by
\[
D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi,
\]
where $\lambda > 0$, $f(z)$ is an analytic functions in a simply connected region of the $z$-plane containing the origin and the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

**Definition 1.2.** The fractional derivative of order $\lambda$ is defined for a function $f(z)$ of the form (1.2) by
\[
D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{1+\lambda}} d\xi,
\]
where $0 \leq \lambda < 1$, $f(z)$ is an analytic functions in a simply connected region of the $z$-plane containing the origin and the multiplicity of $(z-\xi)^{-\lambda}$ is removed as in Definition 1.1 above.

**Definition 1.3.** Under the hypothesis of Definition 1.2 the fractional derivative of order $n + \lambda$ is defined for a function $f(z)$ by
\[
D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z),
\] (1.4)
where $0 \leq \lambda < 1$ and $n \in N_0 = \{0, 1, 2, \ldots\}$.

In 2011, Dixit and Porwal [4] introduce a new fractional derivative operator for function of the form (1.2) as follows
\[
\begin{align*}
\Omega^0 f(z) &= f(z) \\
\Omega^1 f(z) &= \Gamma(1-\lambda)z^{1+\lambda} D_z^{1+\lambda} f(z) \\
&\vdots \\
\Omega^n f(z) &= \Omega(\Omega^{n-1} f(z)).
\end{align*}
\]

Thus, we note that
\[
\Omega^n f(z) = z + \sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k z^k,
\] (1.5)
where
\[
\phi(k, \lambda) = \frac{\Gamma(k+1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)}.
\]

It is worthy to note that for $\lambda = 0$, $\Omega^n f(z)$ reduces to familiar Salagean operator introduced by Salagean in [16].

They define the above operator for function of the form $f = h + g$, where $h$ and $g$ are the form (1.1) as follows
\[
\Omega^n f(z) = \Omega^n (h(z)) + (-1)^n \Omega^n (g(z))
\]
where
\[
\Omega^n (h(z)) = z + \sum_{k=2}^{\infty} [\phi(k, \lambda)]^n a_k z^k
\]
and
\[
\Omega^n (g(z)) = \sum_{k=1}^{\infty} [\phi(k, \lambda)]^n b_k z^k.
\]
The applications of $q-$ calculus is a current and interesting topic of research in Geometric Function Theory. Very recently, Srivastava [17] gave definitions and properties of $q-$ calculus and fractional $q-$ calculus in detail and its applications in his survey-cum-expository review article. Several researchers e.g. see the work of Arif et al. [1], Ahuja et al. [2], Jahangiri [8], Najafzadeh and Makinde [10], Porwal and Gupta [12] and Ravindar et al. [14, 15] investigated various subclasses of univalent functions and obtain interesting results.

Now, we recall the concept of $q$-calculus which was first introduced by Jackson [6, 7]. For $k \in \mathbb{N}$, the $q-$ number is defined as follows:

$$[k]_q = \frac{1 - q^k}{1 - q}, \quad 0 < q < 1.$$  \hfill (1.6)

Hence, $[k]_q$ can be expressed as a geometric series $\sum_{i=0}^{k-1} q^i$, when $k \to \infty$ the series converges to $\frac{1}{1-q}$. As $q \to 1$, $[k]_q \to k$ and this is the bookmark of a $q-$ analogue the limit as $q \to 1$ recovers the classical object.

The $q-$ derivative of a function $f$ is defined by

$$D_q(f(z)) = \frac{f(qz) - f(z)}{(q-1)z}, \quad q \neq 1, z \neq 0$$

and $D_q(f(0)) = f'(0)$ provided $f'(0)$ exists.

For a function $h(z) = z^k$ observe that

$$D_q(h(z)) = D_q(z^k) = \frac{1 - q^k}{1 - q} z^{k-1} = [k]_q z^{k-1}.$$  \hfill (1.7)

Then

$$\lim_{q \to 1} D_q(h(z)) = \lim_{q \to 1} [k]_q z^{k-1} = k z^{k-1} = h'(z)$$

where $h'$ is the ordinary derivative.

The $q-$ Jackson definite integral of the function $f$ is defined by

$$\int_0^z f(t) d_q t = (1 - q)z \sum_{n=0}^{\infty} f(zq^n)q^n, \quad z \in \mathbb{C}.$$  \hfill (1.8)

Now, we let $R_H(n, q, \beta, \lambda)$ denote the subclass $S_H$ consisting of functions $f = h + g$ of the form (1.1) that satisfy the condition

$$\Re \left\{ \Omega^n (D_q(h(z))) + (-1)^n \Omega^n (D_q(g(z))) \right\} < \beta, \quad z \in \mathbb{C}.$$  \hfill (1.9)

for some $\beta(1 < \beta \leq 2), 0 < q < 1, \lambda(0 \leq \lambda \leq 1), n \in \mathbb{N}$ and $z \in U$.

We further let $\overline{R}_H(n, q, \beta, \lambda)$ denote the subclass of $R_H(n, q, \beta, \lambda)$ consisting of functions $f = h + g \bar{n} \in S_H$ such that $h$ and $g\bar{n}$ are of the form (1.9).

If $f(z)$ is of the form (1.2) then the classes $R_H(n, q, \beta, \lambda)$ and $\overline{R}_H(n, q, \beta, \lambda)$ reduce to the classes $R(n, q, \beta, \lambda)$ and $\overline{R}(n, q, \beta, \lambda)$ By specializing the parameter we obtain the following known subclasses studied earlier by various researchers.

1. $R_H(n, 0, \beta, \lambda) \equiv R_H(n, \beta, \lambda)$ and $\overline{R}_H(n, 0, \beta, \lambda) \equiv \overline{R}_H(n, \beta, \lambda)$ studied by Porwal and Aouf [13].

2. $R(1, 0, \beta, 0) \equiv R(\beta)$ studied by Uralegaddi et al. [19].

In the present paper, we study the coefficient bounds, distortion bounds, extreme points, convolution condition, convex combinations and discuss a class preserving integral operator.

\section{Main Results}

First, we give a sufficient coefficient condition for functions in $R_H(n, q, \beta, \lambda)$.

\textbf{Theorem 2.1.} Let $f = h + g$ be such that $h$ and $g$ are given by (1.1). Furthermore, let

$$\sum_{k=2}^{\infty} [\phi(k, \lambda)]^n [k]_q |a_k| + \sum_{k=1}^{\infty} [\phi(k, \lambda)]^n [k]_q |b_k| \leq \beta - 1.$$  \hfill (2.1)

Then $f$ is sense-preserving, harmonic univalent in $U$ and $f \in R_H(n, q, \beta, \lambda)$. 

Proof. If \( z_1 \neq z_2 \), then

\[
\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| \geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| \geq 1 - \left( \sum_{k=1}^{\infty} b_k (z_1^k - z_2^k) \right) \left( \frac{1}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)} \right)
\]

\[
\geq 1 - \sum_{k=1}^{\infty} k|b_k|
\]

\[
\geq 0,
\]

which proves univalence.

Note that \( f \) is sense-preserving in \( U \). This is because

\[
|h'(z)| \geq 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1}
\]

\[
> 1 - \sum_{k=2}^{\infty} k|a_k|
\]

\[
\geq 1 - \sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n [k]^q |b_k|}{\beta - 1}
\]

\[
\geq \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]^q |b_k|}{\beta - 1}
\]

\[
\geq \sum_{k=1}^{\infty} k|b_k|
\]

\[
> \sum_{k=1}^{\infty} k|b_k||z|^{k-1}
\]

\[
\geq |g'(z)|.
\]

Now, we show that \( f \in R_H(n,q,\beta,\lambda) \). Using the fact that \( \Re \omega < \beta \), if and only if, \( |\omega - 1| < |\omega + 1 - 2\beta| \), it suffices to show that

\[
\left| \frac{\Omega^n (D_q(h(z))) + (-1)^n \Omega^n (D_q(g(z)))}{\Omega^n (D_q(h(z))) + (-1)^n \Omega^n (D_q(g(z))) - (2\beta - 1)} - 1 \right| < 1, \ z \in U.
\]
We have

\[
\frac{z + \sum_{k=2}^{\infty} [\phi (k, \lambda)]^n [k]_q a_k z^k + (-1)^n \sum_{k=1}^{\infty} [\phi (k, \lambda)]^n [k]_q b_k z^k}{z + \sum_{k=2}^{\infty} [\phi (k, \lambda)]^n [k]_q a_k z^k + (-1)^n \sum_{k=1}^{\infty} [\phi (k, \lambda)]^n [k]_q b_k z^k} = 1
\]

which is the required condition.

The harmonic univalent functions of the form

\[
\sum_{n=1}^{\infty} [\phi (k, \lambda)]^n [k]_q a_k z^k \leq 1
\]

is equivalent to

\[
\sum_{k=1}^{\infty} [\phi (k, \lambda)]^n [k]_q |a_k| \leq 1
\]

which is bounded above by 1 by using (2.1) and so the proof is complete.

The harmonic univalent functions of the form

\[
f(z) = z + \sum_{k=2}^{\infty} [\phi (k, \lambda)]^n [k]_q a_k z^k + \sum_{k=1}^{\infty} [\phi (k, \lambda)]^n [k]_q b_k z^k
\]

where \(1 < \beta \leq 2, 0 < q < 1, 0 \leq \lambda \leq 1, n \in \mathbb{N}\) and \(\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1\), show that the coefficient bound given by (2.1) is sharp. It is worthy to note that the function of the form (2.2) belongs to the class \(R_H(n, \beta, \lambda)\) for all \(\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| \leq 1\) because coefficient inequality (2.1) holds.

**Theorem 2.2.** Let \(f_n\) be given by (1.3). Then \(f_n \in R_H(n, q, \beta, \lambda)\) if and only if

\[
\sum_{k=2}^{\infty} [\phi (k, \lambda)]^n [k]_q |a_k| + \sum_{k=1}^{\infty} [\phi (k, \lambda)]^n [k]_q |b_k| \leq \beta - 1.
\]

**Proof.** Since \(R_H(n, q, \beta, \lambda) \subset R_H(n, q, \beta, \lambda)\), we only need to prove the "only if" part of the theorem. To this end, for functions \(f_n\) of the form (1.3), we notice that the condition

\[
\Re \left\{ \frac{\Omega^n (D_q(h(z))) + (-1)^n \Omega^n (D_q(g(z)))}{z} \right\} < \beta
\]

is equivalent to

\[
\Re \left\{ 1 + \sum_{k=2}^{\infty} [\phi (k, \lambda)]^n [k]_q a_k z^{k-1} + (-1)^n \sum_{k=1}^{\infty} [\phi (k, \lambda)]^n [k]_q b_k z^{k-1} \right\}
\]

\[
\leq 1 + \sum_{k=2}^{\infty} [\phi (k, \lambda)]^n [k]_q |a_k| z^{k-1} + \sum_{k=1}^{\infty} [\phi (k, \lambda)]^n [k]_q |b_k| z^{k-1} < \beta, \ z \in U.
\]

The above condition must hold for all values of \(z, |z| = r < 1\). Upon choosing the values of \(z\) to be real and let \(z \to 1^-\), we obtain

\[
\sum_{k=2}^{\infty} [\phi (k, \lambda)]^n [k]_q |a_k| + \sum_{k=1}^{\infty} [\phi (k, \lambda)]^n [k]_q |b_k| \leq \beta - 1,
\]

which is the required condition.

The harmonic univalent functions of the form

\[
f_n(z) = z + \sum_{k=2}^{\infty} \frac{\beta - 1}{\phi (k, \lambda)^n [k]_q} x_k z^k + (-1)^n \sum_{k=1}^{\infty} \frac{\beta - 1}{\phi (k, \lambda)^n [k]_q} y_k z^k,
\]

where \(1 < \beta \leq 2, 0 < q < 1, 0 \leq \lambda \leq 1, n \in \mathbb{N}, x_k \geq 0, y_k \geq 0\) and \(\sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k \leq 1\) belongs to the class \(R_H(n, q, \beta, \lambda)\).
Theorem 2.3. If \( f \in H(n, q, \beta, \lambda) \), then

\[
|f(z)| \leq (1 + |b_1|)r + \left( \frac{1 - \lambda}{2} \right)^n \frac{1}{(1 + q)^n} (\beta - 1 - |b_1|)r^2, \quad |z| = r < 1
\]

and

\[
|f(z)| \geq (1 - |b_1|)r - \left( \frac{1 - \lambda}{2} \right)^n \frac{1}{(1 + q)^n} (\beta - 1 - |b_1|)r^2, \quad |z| = r < 1.
\]

Proof. Let \( f \in H(n, q, \beta, \lambda) \). Taking the absolute value of \( f \), we have

\[
|f(z)| \leq (1 + |b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|)r^k
\]

\[
\leq (1 + |b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|)r^2
\]

\[
\leq (1 + |b_1|)r + \left( \frac{1 - \lambda}{2} \right)^n \frac{1}{(1 + q)^n} \sum_{k=2}^{\infty} \left( \frac{2}{1 - \lambda} \right)^n (1 + q)(|a_k| + |b_k|)r^2
\]

\[
\leq (1 + |b_1|)r + \left( \frac{1 - \lambda}{2} \right)^n \frac{1}{(1 + q)^n} \sum_{k=2}^{\infty} \phi(k, \lambda)^n[k]_q (|a_k| + |b_k|)r^2
\]

\[
\leq (1 + |b_1|)r + \left( \frac{1 - \lambda}{2} \right)^n \frac{1}{(1 + q)^n} (\beta - 1 - |b_1|)r^2
\]

and

\[
|f(z)| \geq (1 - |b_1|)r - \sum_{k=2}^{\infty} (|a_k| + |b_k|)r^k
\]

\[
\geq (1 - |b_1|)r - \sum_{k=2}^{\infty} (|a_k| + |b_k|)r^2
\]

\[
\geq (1 - |b_1|)r - \left( \frac{1 - \lambda}{2} \right)^n \frac{1}{(1 + q)^n} \sum_{k=2}^{\infty} \left( \frac{2}{1 - \lambda} \right)^n (1 + q)(|a_k| + |b_k|)r^2
\]

\[
\geq (1 - |b_1|)r - \left( \frac{1 - \lambda}{2} \right)^n \frac{1}{(1 + q)^n} \sum_{k=2}^{\infty} \phi(k, \lambda)^n[k]_q (|a_k| + |b_k|)r^2
\]

\[
\geq (1 - |b_1|)r - \left( \frac{1 - \lambda}{2} \right)^n \frac{1}{(1 + q)^n} (\beta - 1 - |b_1|)r^2.
\]

Theorem 2.4. Let \( f \in \text{clco} H(n, q, \beta, \lambda) \), if and only if

\[
f(z) = \sum_{k=1}^{\infty} (\lambda_k h_k(z) + \gamma_k g_k(z)), \quad (2.4)
\]

where \( h_1(z) = z \)

\[
h_k(z) = z + \frac{\beta - 1}{\phi(k, \lambda)^n[k]_q} z^k, \quad (k = 2, 3, ...)
\]

\[
g_k(z) = z + (-1)^n \frac{\beta - 1}{\phi(k, \lambda)^n[k]_q} z^k, \quad (k = 1, 2, 3, ...)
\]

and \( \sum_{k=1}^{\infty} (\lambda_k + \gamma_k) = 1, \quad \lambda_k \geq 0 \) and \( \gamma_k \geq 0. \)

In particular the extreme points of \( H(n, q, \beta, \lambda) \) are \( \{h_k\} \) and \( \{g_k\} \).
Theorem 2.5. For functions \( f \) of the form (2.4) we may write

\[
f(z) = \sum_{k=1}^{\infty} \left( \lambda_k h_k(z) + \gamma_k g_k(z) \right)
\]

\[
= z + \sum_{k=2}^{\infty} \left( \frac{\beta - 1}{\beta - 1} k \right) \lambda_k z^k + (-1)^n \sum_{k=1}^{\infty} \left( \frac{\beta - 1}{\beta - 1} k \right) \gamma_k z^k.
\]

Then

\[
\sum_{k=2}^{\infty} \frac{\phi(k, \lambda)^n [k]_q}{\beta - 1} \lambda_k + \sum_{k=1}^{\infty} \frac{\phi(k, \lambda)^n [k]_q}{\beta - 1} \gamma_k = \sum_{k=2}^{\infty} \lambda_k + \sum_{k=1}^{\infty} \gamma_k = 1 - \lambda_1 \leq 1,
\]

and so \( f \in \text{clco} \overline{\mathcal{H}}(n, q, \beta, \lambda) \).

Conversely, suppose that \( f \in \text{clco} \overline{\mathcal{H}}(n, q, \beta, \lambda) \).

Set

\[
\lambda_k = \frac{\phi(k, \lambda)^n [k]_q}{\beta - 1} |a_k|, \quad (k = 2, 3, 4, \ldots)
\]

and

\[
\gamma_k = \frac{\phi(k, \lambda)^n [k]_q}{\beta - 1} |b_k|, \quad (k = 1, 2, 3, \ldots).
\]

Then note that by Theorem 2.2,

\[
0 \leq \lambda_k \leq 1, \quad (k = 2, 3, 4, \ldots)
\]

and

\[
0 \leq \gamma_k \leq 1, \quad (k = 1, 2, 3, \ldots).
\]

We define \( \lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k - \sum_{k=1}^{\infty} \gamma_k \) and note that by Theorem 2.2, \( \lambda_1 \geq 0 \). Consequently, we obtain \( f(z) = \sum_{k=1}^{\infty} \left( \lambda_k h_k(z) + \gamma_k g_k(z) \right) \) as required.

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic function of the form

\[
f_n(z) = z + \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| z^k
\]

and

\[
F_n(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |B_k| z^k
\]

we define their convolution

\[
(f_n * F_n)(z) = f_n(z) * F_n(z) = z + \sum_{k=2}^{\infty} |a_k A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k B_k| z^k,
\]

using this definition, we show that the class \( \overline{\mathcal{H}}(n, q, \beta, \lambda) \) is closed under convolution.

Theorem 2.5. For \( 1 < \beta \leq \alpha \leq 2 \), let \( f_n \in \overline{\mathcal{H}}(n, q, \beta, \lambda) \) and \( F_n \in \overline{\mathcal{H}}(n, q, \alpha, \lambda) \).

Then \( (f_n * F_n)(z) \in \overline{\mathcal{H}}(n, q, \beta, \lambda) \).

Proof. Let \( f_n(z) = z + \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| z^k \) be in \( \overline{\mathcal{H}}(n, q, \beta, \lambda) \) and \( F_n(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |B_k| z^k \) be in \( \overline{\mathcal{H}}(n, q, \alpha, \lambda) \). Then the convolution \( (f_n * F_n)(z) \) is given by (2.5). We wish to show that the coefficients of
Theorem 3.1. Let $f_n * f_n$ satisfy the required condition given in Theorem 2.2. For $F_n(z) \in \overline{R}_H(n, q, \alpha, \lambda)$, we note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Now, for the convolution function $(f_n * f_n)(z)$ we have
\[
\sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n [k]_q}{\beta - 1} |a_k A_k| + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n [k]_q}{\beta - 1} |b_k B_k| \leq \sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n [k]_q}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n [k]_q}{\beta - 1} |b_k|
\]
\[
\leq 1, \quad \text{(since } f \in \overline{R}_H(n, q, \beta, \lambda)).
\]

Therefore $(f_n * f_n)(z) \in \overline{R}_H(n, q, \beta, \lambda) \subseteq \overline{R}_H(n, q, \alpha, \lambda)$.

Theorem 2.6. The class $\overline{R}_H(n, q, \beta, \lambda)$ is closed under convex combination.

Proof. For $i = 1, 2, 3...$ let $f_{n_i}(z) \in \overline{R}_H(n, q, \beta, \lambda)$ where $f_i(z)$ is given by
\[
f_{n_i}(z) = z + \sum_{k=2}^{\infty} |a_{k_i}| z^k + (-1)^n \sum_{k=1}^{\infty} |b_{k_i}| z^k.
\]

Then by Theorem 2.2, we have
\[
\sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n [k]_q}{\beta - 1} |a_{k_i}| + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n [k]_q}{\beta - 1} |b_{k_i}| \leq 1.
\]

For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of $f_{n_i}$ may be written as
\[
\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{k=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_{k_i}| \right) z^k + (-1)^n \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_{k_i}| \right) z^k.
\]

Then by Theorem 2.2, we have
\[
\sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n [k]_q}{\beta - 1} \left( \sum_{i=1}^{\infty} t_i |a_{k_i}| \right) + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n [k]_q}{\beta - 1} \left( \sum_{i=1}^{\infty} t_i |b_{k_i}| \right)
\]
\[
= \sum_{i=1}^{\infty} t_i \left( \sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n [k]_q}{\beta - 1} |a_{k_i}| + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n [k]_q}{\beta - 1} |b_{k_i}| \right)
\]
\[
\leq \sum_{i=1}^{\infty} t_i = 1.
\]

Therefore
\[
\sum_{i=1}^{\infty} t_i f_{n_i}(z) \in \overline{R}_H(n, q, \beta, \lambda).
\]

3. A Family of Class Preserving Integral Operator

Let $f(z) = h(z) + g(z) \in S_H$ be given by (1.1) then $F(z)$ defined by relation
\[
F(z) = \frac{c + 1}{z^c} \int_0^z t^{c-1} h(t) dt + \frac{c + 1}{z^c} \int_0^z t^{c-1} g(t) dt, \quad (c > -1).
\]

Theorem 3.1. Let $f(z) = h(z) + g(z) \in S_H$ be given by (1.3) and $f(z) \in \overline{R}_H(n, q, \beta, \lambda)$ then $F(z)$ be defined by (3.1) also belong to $\overline{R}_H(n, q, \beta, \lambda)$.
Proof. Let
\[ f(z) = z + \sum_{k=2}^{\infty} |a_k|z^k + (-1)^n \sum_{k=1}^{\infty} |b_k|z^k \]
be in \( \overline{R_\mathbb{H}}(n, q, \beta, \lambda) \) then by Theorem 2.2, we have
\[ \sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n[k]_q}{\beta - 1} a_k + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n[k]_q}{\beta - 1} b_k \leq 1. \]  
(3.2)

By definition of \( F(z) \) we have
\[ F(z) = z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} |a_k|z^k + (-1)^n \sum_{k=1}^{\infty} \frac{c+1}{c+k} |b_k|z^k. \]

Now
\[ \sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n[k]_q}{\beta - 1} \left( \frac{c+1}{c+k} |a_k| \right) + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n[k]_q}{\beta - 1} \left( \frac{c+1}{c+k} |b_k| \right) \leq \sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n[k]_q}{\beta - 1} a_k + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n[k]_q}{\beta - 1} b_k \]
\[ \leq 1. \]

Thus \( F(z) \in \overline{R_\mathbb{H}}(n, q, \beta, \lambda) \).

Definition 3.1. Let \( f = h + g \) be defined by (1.1). Then, the \( q \)-Jackson integral operator \( F_q \) is defined by the relation
\[ F_q(z) = \frac{[c]_q}{z^{c+1}} \int_0^z t^c h(t) dt + \frac{[c]_q}{z^{c+1}} \int_0^z t^c g(t) dt, \]
(3.3)
where \([c]_q\) is the \( q \)-number defined by (1.6).

Theorem 3.2. Let \( f(z) = h(z) + g(z) \) be given by (1.3) and \( f(z) \in \overline{R_\mathbb{H}}(n, q, \beta, \lambda) \) where \( 1 < \beta \leq 2, \ 0 < q < 1, 0 \leq \lambda < 1 \). Then \( F_q \) defined by (3.3) is also in the class \( \overline{R_\mathbb{H}}(n, q, \beta, \lambda) \).

Proof. Let
\[ f(z) = z + \sum_{k=2}^{\infty} |a_k|z^k + (-1)^n \sum_{k=1}^{\infty} |b_k|z^k \]
be in \( \overline{R_\mathbb{H}}(n, q, \beta, \lambda) \). Then by Theorem 2.2, the condition (3.2) is satisfied.

From the representation (3.3) of \( F_q \), it follows that,
\[ F_q(z) = z + \sum_{k=2}^{\infty} \frac{[c]_q}{[k+c+1]_q} |a_k|z^k + (-1)^n \sum_{k=1}^{\infty} \frac{[c]_q}{[k+c+1]_q} |b_k|z^k. \]

Since
\[ [k+c+1]_q - [c]_q = \sum_{i=0}^{k+c} q^i - \sum_{i=0}^{c-1} q^i = \sum_{i=c}^{k+c} q^i > 0 \]
\[ [k+c+1]_q > [c]_q \quad \text{or} \quad \frac{[c]_q}{[k+c+1]_q} < 1. \]

Now
\[ \sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n[k]_q}{\beta - 1} \frac{[c]_q}{[k+c+1]_q} |a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n[k]_q}{\beta - 1} \frac{[c]_q}{[k+c+1]_q} |b_k| \leq \sum_{k=2}^{\infty} \frac{[\phi(k, \lambda)]^n[k]_q}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n[k]_q}{\beta - 1} |b_k| \leq 1. \]

Thus the proof of Theorem 3.2 is established. \( \square \)
4. Conclusion

This paper deals with a new class of harmonic univalent functions defined by using $q-$ calculus. Coefficient condition, extreme points, distortion bounds, convolution and convex combination are determined for this class. We also study a class preserving integral operator for this class.

Motivated by a recently-published survey-cum-expository review article by Srivastava [17], the interested reader’s attention is drawn toward the possibility of investigating the basic (or $q-$ ) extensions of the results which are presented in this paper. However, as already pointed out by Srivastava [17], their further extensions using the so-called $(p,q)-$ calculus will be rather trivial and inconsequential variations of the suggested extensions which are based upon the classical $q-$ calculus, the additional parameter $p$ being redundant or superfluous (see, for details, [[17], p. 340]).

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References


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