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Balanced and Absorbing Soft Sets

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Abstract - The aim of this paper is to define the balanced soft set and absorbing soft set over a linear space and construct some useful theorem depending upon these concepts.

Keywords - *Soft set, balanced Soft set, absorbing Soft set.*

1 Introduction

Sometimes a few mathematical problems arise in economics, engineering and environment which can not be solved successfully by use classical methods because of various type of uncertainties are present in these problems. To solve these problems, a few concepts have been constructed in several times such as theory of probability, theory of fuzzy sets, and the interval mathematics etc. Soft set is the most recent notion of these concepts.

The concept of soft set was first introduced by D. Molodtsov [7] in 1999. In his work, he defined the operation on soft sets like union, intersection, cartesian product etc. Also he constructed some applications of soft sets in his first paper of soft set theory. Thereafter so many research works[1, 2, 4, 5, 6, 8, 9, 10] have been done on this concept in different disciplines of mathematics.

In functional analysis, certain types of sets viz. balanced set, absorbing set and convex set are found to play pivotal roles. In this paper, we also try to define the concept of balanced soft set, absorbing soft set over a linear space to study the functional analysis. Then we establish some theorems concerning the said notions.

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2 Preliminary

In this section, U refers to an initial universe, E is the set of parameters, $P(U)$ is the power set of U and $A \subseteq E$.

Definition 2.1. *[3] A soft set F_A on the universe U is defined by the set of ordered pairs $F_A = \{(e, F_A(e)) : e \in E, F_A(e) \in P(U)\}$ where $F_A : E \to P(U)$ such that $F_A(e) = \phi$ if e is not an element of A. The set of all soft sets over (U, E) is denoted by $S(U)$.*

Definition 2.2. *[3] Let $F_A \in S(U)$. If $F_A(e) = \phi$, for all $e \in E$, then F_A is called a empty soft set, denoted by Φ. $F_A(e) = \phi$ means that there is no element in U related to the parameter $e \in E$.*

Definition 2.3. *[3] Let $F_A, G_B \in S(U)$. We say that F_A is a soft subsets of G_B and we write $F_A \sqsubseteq G_B$ if and only if $F_A(e) \subseteq G_B(e)$ for all $e \in E$.*

Definition 2.4. *[3] Let $F_A, G_B \in S(U)$. Then F_A and G_B are said to be soft equal, denoted by $F_A = G_B$ if $F_A(e) = G_B(e)$ for all $e \in E$.*

Definition 2.5. *[3] Let $F_A, G_B \in S(U)$. Then the soft union of F_A and G_B is also a soft set $F_A \sqcup G_B = H_{A\cup B} \in S(U)$, defined by*
$H_{A\cup B}(e) = (F_A \sqcup G_B)(e) = F_A(e) \cup G_B(e)$ for all $e \in E$.

Definition 2.6. *[3] Let $F_A, G_B \in S(U)$. Then the soft intersection of F_A and G_B is also a soft set $F_A \sqcap G_B = H_{A\cap B} \in S(U)$, defined by*
$H_{A\cap B}(e) = (F_A \sqcap G_B)(e) = F_A(e) \cap G_B(e)$ for all $e \in E$.

Definition 2.7. *Let U be an initial universe and $f : X \to Y$ be a mapping, where X and Y are set of parameters. If F_A be a soft set over (U, X), then $f(F_A)$, a soft set over (U, Y), is defined by*

$$f(F_A)(y) = \begin{cases} \cup_{x\in f^{-1}(y)} F_A(x) & if \quad f^{-1}(y) \neq \Phi, \\ \Phi & otherwise. \end{cases}$$

Definition 2.8. *Let U be an initial universe and $f : X \to Y$ be a mapping, where X and Y are set of parameters. If G_B be a soft set over (U, Y), then $f^{-1}(G_B)$, a soft set over (U, X), is defined by*
$f^{-1}(G_B)(x) = G_B(f(x))$.

Definition 2.9. *[11] Let U be a universal set and E be a usual vector space over $\mathbb{R}$ or $\mathbb{C}$ and $F_{A_1}, F_{A_2}, \cdots, F_{A_n}$ be soft sets over (U, E) and $f : E^n \to E$ be a function defined by $f(e_1, e_2, \cdots, e_n) = e_1 + e_2 + \cdots + e_n$. Then the vector sum $F_{A_1} + F_{A_2} + \cdots + F_{A_n}$ is defined by*
$(F_{A_1} + F_{A_2} + \cdots + F_{A_n})(e)$
$= \bigcup_{(e_1,e_2,\cdots,e_n)\in f^{-1}(e)} \{F_{A_1}(e_1) \cap F_{A_2}(e_2) \cap \cdots \cap F_{A_n}(e_n)\}$

Definition 2.10. *[11] If U be a universal set and E be a usual vector space over $\mathbb{R}$ or $\mathbb{C}$ and t be a scalar and $g : E \to E$ be a mapping defined by $g(e) = te$, then the scalar multiplication tF_A of a soft set F_A is defined by $tF_A = g(F_A)$.*

Proposition 2.11. *[11] If F_A is a soft set over the universal set U and the parameter set E, where E is an usual vector space over $K(\mathbb{R}$ or $\mathbb{C})$ and $t \in K$, then*

$$tF_A(e) = \begin{cases} F_A(t^{-1}e) & if \quad t \neq 0, \\ \Phi & if \quad t = 0 \quad and \quad e \neq 0, \\ \bigcup_{p \in E} F_A(p) & if \quad t = 0 \quad and \quad e = 0. \end{cases}$$

2.1 Balanced Soft set

Throughout this work, we denote E as a vector space over the field $K(\mathbb{R}$ or $\mathbb{C})$ and U as an initial universe. Also we denote 0 and 1 as the zero element and unity of the field respectively. Also the zero vector of the linear space is denoted by 0 which can be easily separated from the zero element of the field.

Definition 2.12. *A soft set F_A over (U, E) is said to be balanced soft set if $tF_A \sqsubseteq F_A$ for all $t \in K$ with $|t| \leq 1$.*

Example 2.13. *Let the universal set $U=$ the set of all real numbers and E be a real vector space and $0 \in A \subseteq E$. Let F_A be a soft set defined by*

$$F_A(e) = \begin{cases} (|e|, \infty) & if \quad e \neq 0, \\ U & if \quad e = 0, \quad where \quad e \in A. \end{cases}$$

Then obviously, F_A is a balanced soft set.

Theorem 2.14. *If F_A is a soft set over (U, E), then $\sqcup_{|\lambda| \leq 1} \lambda F_A$ is a balanced soft set.*

Proof: Let $|\alpha| \leq 1$ and $e \in E$.
Case 1. $0 < |\alpha| \leq 1$.
$\alpha(\sqcup_{|\lambda| \leq 1} \lambda F_A)(e)$
$= (\sqcup_{|\lambda| \leq 1} \lambda F_A)(\frac{1}{\alpha} e)$
$= \cup_{|\lambda| \leq 1} \lambda F_A(\frac{1}{\alpha} e)$
$= (\sqcup_{|\lambda| \leq 1} \alpha \lambda F_A)(e)$
$\subseteq (\sqcup_{|\lambda| \leq 1} \lambda F_A)(e)$, since $|\alpha| \leq 1$ and $|\lambda| \leq 1$, $|\alpha\lambda| \leq 1$
Case 2. $\alpha = 0$.
Subcase 1. If $e \neq 0$, then obviously, $0(\sqcup_{|\lambda| \leq 1} \lambda F_A)(e) = \Phi \subseteq (\sqcup_{|\lambda| \leq 1} \lambda F_A)(e)$.
Subcase 2. If $e = 0$, then
$0(\sqcup_{|\lambda| \leq 1} \lambda F_A)(0)$
$= \cup_{x \in E}\{(\sqcup_{|\lambda| \leq 1} \lambda F_A)(x)\}$
$= \cup_{x \in E}\{\cup_{|\lambda| \leq 1} \lambda F_A(x)\}$
$= \{0F_A(0)\} \cup \{\cup_{x(\neq 0) \in E} 0F_A(x)\} \cup \{\cup_{x \in E}\{\cup_{0<|\lambda| \leq 1} \lambda F_A(x)\}\}$
$= \{0F_A(0)\} \cup \{\cup_{x \in E}\{\cup_{0<|\lambda| \leq 1} F_A(\frac{1}{\lambda} x)\}$, as $\cup_{x(\neq 0) \in E} 0F_A(x) = \Phi$
$= 0F_A(0)$.
Again, $(\sqcup_{|\lambda| \leq 1} \lambda F_A)(0)$
$= \cup_{|\lambda| \leq 1} \lambda F_A(0)$
$= \{\cup_{0<|\lambda| \leq 1} \lambda F_A(0)\} \cup \{0F_A(0)\}$
$= \{\cup_{0<|\lambda| \leq 1} F_A(0)\} \cup \{0F_A(0)\}$

$= 0F_A(0)$.
Thus, $0(\sqcup_{|\lambda|\leq 1}\lambda F_A)(0) = (\sqcup_{|\lambda|\leq 1}\lambda F_A)(0)$.
Hence, $\sqcup_{|\lambda|\leq 1}\lambda F_A$ is a balanced soft set.

Theorem 2.15. *Let F_A be a balanced soft set over (U, E). Then*
i) $\alpha, \beta \in K$ and $|\alpha| \leq |\beta| \Rightarrow \alpha F_A \sqsubseteq \beta F_A$
ii) $\alpha \in K$ and $|\alpha| = 1 \Rightarrow \alpha F_A = F_A$

Proof: *i)* **Case 1.** $\alpha = 0$.
Subcase 1. If $\beta = 0$, then clearly, $\alpha F_A = \beta F_A$.
Subcase 2. If $\beta \neq 0$, then for any non-zero $e \in E$, $\alpha F_A(e) = \Phi \subseteq \beta F_A(e)$ and for zero element of E, we have $\alpha F_A(0) = 0F_A(0) \subseteq F_A(0)$ as F_A is a balanced soft set. That is, $\alpha F_A(0) \subseteq F_A(\frac{1}{\beta}0) = \beta F_A(0)$. Therefore in this case $\alpha F_A \sqsubseteq \beta F_A$.
Case 2. $\alpha \neq 0$. Since $|\frac{\alpha}{\beta}| \leq 1$, we have $\frac{\alpha}{\beta}F_A \sqsubseteq F_A$.
Let $e \in E$ and $\frac{1}{\beta}e = e_1$. Now since $\frac{\alpha}{\beta}F_A \sqsubseteq F_A$, then
$\frac{\alpha}{\beta}F_A(e_1) \subseteq F_A(e_1)$
or, $F_A(\frac{\beta}{\alpha}e_1) \subseteq F_A(e_1)$
or, $F_A(\frac{\beta}{\alpha}\frac{1}{\beta}e) \subseteq F_A(\frac{1}{\beta}e)$
or, $F_A(\frac{1}{\alpha}e) \subseteq F_A(\frac{1}{\beta}e)$
or, $\alpha F_A(e) \subseteq \beta F_A(e)$
that is, $\alpha F_A \sqsubseteq \beta F_A$.

ii) Let $|\alpha| \leq 1$. Then $\alpha F_A \sqsubseteq F_A$. Again let $|\alpha| \geq 1$. Then $|\frac{1}{\alpha}| \leq 1$ which implies that $\frac{1}{\alpha}F_A \sqsubseteq F_A$. So as the procedure of case 2 for proof (i), we get $F_A \sqsubseteq \alpha F_A$. Thus, $\alpha F_A = F_A$ if $|\alpha| = 1$.

Theorem 2.16. *If $\{F_{A_\alpha} : \alpha \in \Lambda\}$ is a collection of balanced soft sets over (U, E), then $\sqcap_{\alpha\in\Lambda}F_{A_\alpha}$ is balanced.*

Proof: Let $\{F_{A_\alpha} : \alpha \in \Lambda\}$ be a collection of balanced soft sets over (U, E) and $F_A = \sqcap_{\alpha\in\Lambda}F_{A_\alpha}$. Also let $|t| \leq 1$ and $e \in E$.
Case 1. $t = 0$.
Subcase 1. If $e \neq 0$, then $tF_A(e) = \Phi$. So $tF_A \sqsubseteq F_A$.
Subcase 2. If $e = 0$, then

$$\begin{aligned} tF_A(e) = 0F_A(0) &= \cup_{p\in E}F_A(p) \\ &= \cup_{p\in E}(\sqcap_{\alpha\in\Lambda}F_{A_\alpha})(p) \\ &= \cup_{p\in E}(\cap_{\alpha\in\Lambda}F_{A_\alpha}(p)) \\ &\subseteq \cap_{\alpha\in\Lambda}(\cup_{p\in E}F_{A_\alpha}(p)) \\ &= \cap_{\alpha\in\Lambda}0F_{A_\alpha}(0) \\ &\subseteq \cap_{\alpha\in\Lambda}F_{A_\alpha}(0) \qquad \text{[as each } F_{A_\alpha} \text{ is balanced, } 0F_{A_\alpha}(0) \subseteq F_{A_\alpha}(0)] \\ &= F_A(0). \end{aligned}$$

Case 2. $t \neq 0$.

$$\begin{aligned} tF_A(e) &= F_A(t^{-1}e) \\ &= (\sqcap_{\alpha\in\Lambda}F_{A_\alpha})(t^{-1}e) \end{aligned}$$

$= \cap_{\alpha\in\Lambda} F_{A_\alpha}(t^{-1}e)$
$= \cap_{\alpha\in\Lambda} tF_{A_\alpha}(e)$
$\subseteq \cap_{\alpha\in\Lambda} F_{A_\alpha}(e)$ [as each F_{A_α} is balanced]
$= F_A(e)$.
Hence F_A is also a balanced soft set.

Theorem 2.17. *If $\{F_{A_\alpha} : \alpha \in \Lambda\}$ is a collection of balanced soft sets over (U, E), then $\sqcup_{\alpha\in\Lambda} F_{A_\alpha}$ is balanced.*

Proof: Let $\{F_{A_\alpha} : \alpha \in \Lambda\}$ be a collection of balanced soft sets over (U, E) and $F_A = \sqcup_{\alpha\in\Lambda} F_{A_\alpha}$. Also let $|t| \leq 1$ and $e \in E$.
Case 1. $t = 0$.
Subcase 1. If $e \neq 0$, then $tF_A(e) = \Phi$. So $tF_A \sqsubseteq F_A$.
Subcase 2. If $e = 0$, then
$tF_A(e) = 0F_A(0) = \cup_{p\in E} F_A(p)$
$= \cup_{p\in E}(\sqcup_{\alpha\in\Lambda} F_{A_\alpha})(p)$
$= \cup_{p\in E}(\cup_{\alpha\in\Lambda} F_{A_\alpha}(p))$
$= \cup_{\alpha\in\Lambda}(\cup_{p\in E} F_{A_\alpha}(p))$
$= \cup_{\alpha\in\Lambda} 0F_{A_\alpha}(0)$
$\subseteq \cup_{\alpha\in\Lambda} F_{A_\alpha}(0)$ [as each F_{A_α} is balanced, $0F_{A_\alpha}(0) \subseteq F_{A_\alpha}(0)$]
$= F_A(0)$.
Case 2. $t \neq 0$.
$tF_A(e) = F_A(t^{-1}e)$
$= (\sqcup_{\alpha\in\Lambda} F_{A_\alpha})(t^{-1}e)$
$= \cup_{\alpha\in\Lambda} F_{A_\alpha}(t^{-1}e)$
$= \cup_{\alpha\in\Lambda} tF_{A_\alpha}(e)$
$\subseteq \cup_{\alpha\in\Lambda} F_{A_\alpha}(e)$ [as each F_{A_α} is balanced]
$= F_A(e)$.
Hence, F_A is also a balanced soft set.

Theorem 2.18. *If $\lambda \in K$ and F_A is balanced soft set over (U, E), then λF_A is balanced.*

Proof: Let $|t| \leq 1$. If $\lambda = 0$, there is nothing to prove. So we take $\lambda \neq 0$.
Case 1. $t = 0$.
Subcase 1. If $e \neq 0$, then $t\lambda F_A(e) = 0F_A(e) = \Phi \subseteq \lambda F_A(e)$.
Subcase 2. If $e = 0$, then $0\lambda F_A(0) = 0F_A(0) \subseteq F_A(0) = F_A(\lambda^{-1}0) = \lambda F_A(0)$.
Case 2. $t \neq 0$.
$t\lambda F_A(e) = F_A((t\lambda)^{-1}e) = F_A(\lambda^{-1}t^{-1}e) = F_A(t^{-1}\lambda^{-1}e) = tF_A(\lambda^{-1}e) \subseteq F_A(\lambda^{-1}e) = \lambda F_A(e)$.
Hence, λF_A is balanced soft set.

Theorem 2.19. *If F_A and G_B are balanced soft sets over (U, E) and $\lambda, \mu \in K$ then $\lambda F_A + \mu G_B$ is also a balanced soft set.*

Proof: Since F_A and G_B are balanced soft sets, λF_A and μG_B are balanced soft sets. Let $|t| \leq 1$.
Case 1. $t = 0$.
Subcase 1. If $e \neq 0$, then $t(\lambda F_A + \mu G_B)(e) = \Phi \subseteq (\lambda F_A + \mu G_B)(e)$.
Subcase 2. If $e = 0$, then
$0(\lambda F_A + \mu G_B)(0)$
$= \cup_{e \in E}(\lambda F_A + \mu G_B)(e)$
$= \cup_{e \in E}\{\cup_{e=e_1+e_2}(\lambda F_A(e_1) \cap \mu G_B(e_2))\}$
$= \cup_{e_1, e_2 \in E}(\lambda F_A(e_1) \cap \mu G_B(e_2))$
$\subseteq \{\cup_{e_1 \in E}\lambda F_A(e_1)\} \cap \{\cup_{e_2 \in E}\mu G_B(e_2)\}$
$= 0\lambda F_A(0) \cap 0\mu G_B(0)$
$\subseteq \lambda F_A(0) \cap \mu G_B(0)$ [as both λF_A and μG_B are balanced soft sets.]
$\subseteq \cup_{e \in E}(\lambda F_A(e) \cap \mu G_B(-e))$
$= \cup_{0=e+(-e)}(\lambda F_A(e) \cap \mu G_B(-e))$
$= (\lambda F_A \cap \mu G_B)(0)$.
Case 2. $t \neq 0$.
$t(\lambda F_A + \mu G_B)(e)$
$= (\lambda F_A + \mu G_B)(t^{-1}e)$
$= \cup_{t^{-1}e=e_1+e_2}(\lambda F_A(e_1) \cap \mu G_B(e_2))$
$= \cup_{e=te_1+te_2}(\lambda F_A(e_1) \cap \mu G_B(e_2))$
$= \cup_{e=x_1+x_2, x_1=te_1, x_2=te_2}(\lambda F_A(t^{-1}x_1) \cap \mu G_B(t^{-1}x_2))$
$= \cup_{e=x_1+x_2}(t\lambda F_A(x_1) \cap t\mu G_B(x_2))$
$\subseteq \cup_{e=x_1+x_2}(\lambda F_A(x_1) \cap \mu G_B(x_2))$ [as both λF_A and μG_B are balanced soft sets.]
$= (\lambda F_A + \mu G_B)(e)$.
This completes the proof of the theorem.

Definition 2.20. *Let F_A be a soft set over (U, E). Then*
i) the intersection of all balanced soft sets over (U, E), each containing F_A, is called the balanced hull of F_A.
ii) the union of all balanced soft sets over (U, E), each contained in F_A, is called the balanced core of F_A.

Theorem 2.21. *Let H and C respectively denotes the balanced hull and the balanced core of a soft set F_A over (U, E). Then*
i) $H = \sqcup\{\lambda F_A : |\lambda| \leq 1\}$,
ii) $C = \sqcap\{\lambda F_A : |\lambda| \geq 1\}$ if $F_A(0) = 0F_A(0)$.

Proof: *i)* Let $B = \sqcup\{\lambda F_A : |\lambda| \leq 1\}$. Then B is a balanced soft set by theorem 2.14. Also clearly, $F_A \sqsubseteq B$. Then by definition of H, $H \sqsubseteq B$. Again $F_A \sqsubseteq H$. Then $\lambda F_A \sqsubseteq \lambda H \sqsubseteq H$ for $|\lambda| \leq 1$, as H is balanced. So $B = \sqcup\{\lambda F_A : |\lambda| \leq 1\} \sqsubseteq H$. Hence $H = B$.

ii) Let G_D be a soft set such that $G_D \sqsubseteq C$. Then we have $\alpha G_D \sqsubseteq \alpha C \sqsubseteq C$ for all $|\alpha| \leq 1$, as C is balanced

$\Rightarrow \alpha G_D \sqsubseteq F_A$ for all $|\alpha| \leq 1$

$\Rightarrow G_D \sqsubseteq \frac{1}{\alpha}F_A$ for all $0 < |\alpha| \leq 1$

$\Rightarrow G_D \sqsubseteq \lambda F_A$ for all $|\lambda| \geq 1$

$\Rightarrow G_D \sqsubseteq \sqcap_{|\lambda|\geq 1} \lambda F_A$.

Therefore for any soft subset G_D of C we have $G_D \sqsubseteq \sqcap_{|\lambda|\geq 1} \lambda F_A$. Since C is also a soft subset of C, $C \sqsubseteq \sqcap_{|\lambda|\geq 1} \lambda F_A$.

Again let, $G_D \sqsubseteq \sqcap_{|\lambda|\geq 1} \lambda F_A$

$\Rightarrow G_D \sqsubseteq \lambda F_A$ for all $|\lambda| \geq 1$

$\Rightarrow \frac{1}{\lambda} G_D \sqsubseteq F_A$ for all $|\lambda| \geq 1$

$\Rightarrow \alpha G_D \sqsubseteq F_A$ for all $0 < |\alpha| \leq 1$

$\Rightarrow \sqcup_{0<|\alpha|\leq 1} \alpha G_D \sqsubseteq F_A$

$\Rightarrow \sqcup_{0\leq|\alpha|\leq 1} \alpha G_D \sqsubseteq F_A$, as $0G_D(0) \sqsubseteq 0F_A(0) = F_A(0)$.

Again $\sqcup_{0\leq|\alpha|\leq 1} \alpha G_D$ is balanced by theorem 2.14, So $\sqcup_{0\leq|\alpha|\leq 1} \alpha G_D \sqsubseteq C$. Then $G_D \sqsubseteq C$. Therefore for any soft subset G_D of $\sqcap_{|\lambda|\geq 1} \lambda F_A$ we have $G_D \sqsubseteq C$. So, $\sqcap_{|\lambda|\geq 1} \lambda F_A \sqsubseteq C$. Hence $C = \sqcap_{|\lambda|\geq 1} \lambda F_A$.

Example 2.22. *Let $E =$ the real vector space $\mathbb{R}$, $U =$ the set of all natural numbers including zero and $A = E$. Let F_A be a soft set defined by*

$F_A(x) = \{[|x|],\ [|x|] + 1,\ [|x|] + 2,\ \cdots\}$ *for* $x \in A$ *and* $x \neq 0$,

$= \{2,\ 3,\ 4,\ \cdots\}$ *for* $x = 0$,

where $[|x|]$ is the greatest integer less than equal to $|x|$.

Let $D = [2,\ \infty) \cup (-\infty,\ -2]$ and G_A be a soft set defined by

$G_A(x) = F_A(x)$ *for all* $x \in D$.

$= \{2,\ 3,\ 4,\ \cdots\}$ *for* $x \in A \backslash D = (-2,\ 2)$

It is easy to see that $G_A(y) \subseteq G_A(x)$ if $x,\ y \in A$ with $|x| \leq |y|$

We now show that G_A is a balanced soft set. Let $x \in A$.

If $0 < |\lambda| \leq 1$, then $\lambda G_A(x) = G_A(\frac{1}{\lambda}x) \subseteq G_A(x)$, as $|x| \leq |\frac{1}{\lambda}x|$.

If $\lambda = 0$, then

Case 1. *for $x \neq 0$, we have $0G_A(x) = \Phi \subseteq G_A(x)$.*

Case 2. *for $x = 0$, we have*

$0G_A(0) = \cup_{e\in E} G_A(e)$

$= \{\cup_{e\in D} G_A(e)\} \cup \{\cup_{e\in A\backslash D} G_A(e)\}$

$= \{\cup_{e\in D} F_A(e)\} \cup \{2,\ 3,\ 4,\ \cdots\}$

$= \{2,\ 3,\ 4,\ \cdots\}$

$= G_A(0)$.

Thus, G_A is a balanced soft set. We now show that G_A is the greatest balanced soft set contained in F_A. Let G_P be any balanced soft set contained in F_A. Then obviously, $P \subseteq A$. Let $x \in P$. Then either $x \in D$ or $x \in A\backslash D$. If $x \in D$, then $G_P(x) \subseteq F_A(x) = G_A(x)$. Again $\cup_{e\in P} G_P(e) = 0G_P(0) \subseteq G_P(0) \subseteq F_A(0) = \{2, 3, 4, \cdots\}$. So, if $x \in A\backslash D$, then $G_P(x) \subseteq \{2,\ 3,\ 4,\ \cdots\} = G_A(x)$. Thus, $G_P(x) \subseteq G_A(x)$ for all $x \in P$ and so, G_A is the greatest balanced soft set contained in F_A. Therefore the balanced core of F_A is G_A.

Note 2.23. *In the above example the balanced core $G_A \neq \sqcap\{\lambda F_A : |\lambda| \geq 1\}$. Infact, if $x = 1$, then $G_A(1) = \{2, 3, 4, \cdots\}$ and $(\sqcap_{|\lambda|\geq 1} \lambda F_A)(1) = \cap_{|\lambda|\geq 1} F_A(\frac{1}{\lambda}) = \{0, 1, 2, 3, 4, \cdots\}$.*

Again $0F_A(0) = \cup_{e\in E} F_A(e) = \{0, 1, 2, 3, 4, \cdots\} \neq F_A(0)$. *So, if* $0F_A(0) \neq F_A(0)$, *then the balanced core of* F_A *may not be equal to* $\sqcap\{\lambda F_A : |\lambda| \geq 1\}$.

Theorem 2.24. *Let* X *and* Y *be linear spaces over* $K(\mathbb{R}$ *or* $\mathbb{C})$ *and* $f : X \to Y$ *be a linear map. Then* F_A *is balanced soft set over* (U, X) *imply that* $f(F_A)$ *is also a balanced soft set over* (U, Y).

Proof: Let $\lambda \in K$ such that $|\lambda| \leq 1$. We now verify that $\lambda f(F_A) \sqsubseteq f(F_A)$.
Let $y \in Y$.
Case 1. $\lambda = 0$.
Subcase 1. If $y = 0$, then

$$\begin{aligned} 0f(F_A)(0) &= \cup_{y'\in Y} f(F_A)(y') \\ &= \cup_{y'\in Y} \cup_{x\in f^{-1}(y')} F_A(x) \\ &= \cup_{x\in X} F_A(x) \\ &= 0F_A(0) \\ &\subseteq F_A(0) \qquad [\text{ as } F_A \text{ is balanced soft set.}] \\ &\subseteq \cup_{e\in f^{-1}(0)} F_A(e) \qquad [\text{ as } 0 \in f^{-1}(0)\,] \\ &= f(F_A)(0). \end{aligned}$$

Subcase 2. If $y \neq 0$, then $0f(F_A)(y) = \Phi \subseteq f(F_A)(y)$.
Case 2. $\lambda \neq 0$.

$$\begin{aligned} \lambda f(F_A)(y) &= f(F_A)(\lambda^{-1}y) \\ &= \cup_{x\in f^{-1}(\lambda^{-1}y)} F_A(x) \\ &= \cup_{\lambda x\in f^{-1}(y)} F_A(x) \\ &= \cup_{x'\in f^{-1}(y)} F_A(\lambda^{-1}x'), \text{ where } \lambda x = x', \text{ i.e., } x = \lambda^{-1}x' \\ &= \cup_{x'\in f^{-1}(y)} \lambda F_A(x') \\ &\subseteq \cup_{x'\in f^{-1}(y)} F_A(x') \ [\text{ as } F_A \text{ is a balanced soft set and } |\lambda| \leq 1\,] \\ &= f(F_A)(y) \end{aligned}$$

Thus, from case 1 and case 2, we have $\lambda f(F_A)(e) \subseteq f(F_A)(e)$ for every $|\lambda| \leq 1$ and $e \in Y$. Hence, $\lambda f(F_A) \sqsubseteq f(F_A)$ for all $|\lambda| \leq 1$. This completes the proof.

Theorem 2.25. *Let* X *and* Y *be linear spaces over* $K(\mathbb{R}$ *or* $\mathbb{C})$ *and* $f : X \to Y$ *be a linear map. Then* G_B *is balanced soft set over* (U, Y) *imply that* $f^{-1}(G_B)$ *is also a balanced soft set over* (U, X).

Proof: Let $\lambda \in K$ such that $|\lambda| \leq 1$. We now verify that $\lambda f^{-1}(G_B) \sqsubseteq f^{-1}(G_B)$.
Let $x \in X$.
Case 1. $\lambda = 0$.
Subcase 1. If $x = 0$, then

$$\begin{aligned} 0f^{-1}(G_B)(0) &= \cup_{x'\in X} f^{-1}(G_B)(x') \\ &= \cup_{x'\in X} G_B(f(x')) \\ &\subseteq \cup_{y\in Y} G_B(y) \\ &= 0G_B(0) \\ &\subseteq G_B(0) \qquad [\text{ as } G_B \text{ is balanced soft set.}] \end{aligned}$$

$= G_B(f(0))$
$= f^{-1}(G_B)(0).$

Subcase 2. If $x \neq 0$, then $0f^{-1}(G_B)(x) = \Phi \subseteq f^{-1}(G_B)(x)$.

Case 2. $\lambda \neq 0$.

$$\begin{aligned}\lambda f^{-1}(G_B)(x) &= f^{-1}(G_B)(\lambda^{-1}x)\\ &= G_B(f(\lambda^{-1}x))\\ &= G_B(\lambda^{-1}f(x))\\ &= \lambda G_B(f(x))\\ &\subseteq G_B(f(x)) \qquad [\text{ as } G_B \text{ is a balanced soft set and } |\lambda| \leq 1.]\\ &= f^{-1}(G_B)(x)\end{aligned}$$

Thus, from case 1 and case 2, we get $\lambda f^{-1}(G_B)(x) \subseteq f^{-1}(G_B)(x)$ for every $|\lambda| \leq 1$ and $x \in X$. Hence, $\lambda f^{-1}(G_B) \sqsubseteq f^{-1}(G_B)$ for all $|\lambda| \leq 1$. This completes the proof.

2.2 Absorbing Soft set

Definition 2.26. *A soft set F_A over (U, E) is called an absorbing soft set if $\sqcup_{\lambda>0}\lambda F_A = \overline{1}$, where $\overline{1}(e) = U$ for all $e \in E$.*

Example 2.27. *Let E = real vector space $\mathbb{R}$ and the universal set U = the set of all natural numbers. Let $0 \in A \subseteq E$ such that $A \cap (-1, 1) \setminus \{0\} \neq \Phi$ and for $e \in A$, the soft set F_A is defined by*
$F_A(e) = \{[|e|] + 1, [|e|] + 2, [|e|] + 3, \cdots\}$, where $[|e|]$ is the greatest integer less than equal to $|e|$. Then obviously $\sqcup_{\lambda>0}\lambda F_A = \overline{1}$ and so F_A is an absorbing soft set.

Example 2.28. *Let E = real vector space $\mathbb{R}$ and the universal set U_1 = the set of all natural numbers including zero. Let B be an unbounded (both below and above) subset of E containing 0 and G_B be a soft set defined by $G_B(e) = \{0, 1, 2, \cdots, [|e|]\}$ for $e(\neq 0) \in B$ and $G_B(0) = U_1$, where $[|e|]$ is the greatest integer less than equal to $|e|$. Then obviously $\sqcup_{\lambda>0}\lambda G_B = \overline{1}$ and so G_B is an absorbing soft set.*

Note 2.29. *Consider the absorbing soft set F_A of the example 2.27 and the absorbing soft set G_B of the example 2.28. If $A \cap B = \{0\}$, then obviously $(F_A \sqcap G_B) = U$. Again if $\{0\} \subsetneqq A \cap B$, let $e(\neq 0) \in A \cap B$. Then $(F_A \sqcap G_B)(e) = F_A(e) \cap G_B(e) = \Phi$. Hence, $(F_A \sqcap G_B)$ is not an absorbing soft set. Thus, intersection of two absorbing soft sets may or may not be an absorbing soft set.*

Theorem 2.30. *Union of two absorbing soft sets is an absorbing soft set.*

Example 2.31. *Consider the soft set F_A as defined in example 2.27 and we take $A = E$. Here it is obvious that $0F_A(0) = F_A(0)$. Therefore the balanced core of F_A is $\sqcap_{|\lambda|\geq 1}\lambda F_A$. Let $e \in E$. Now*

$$\begin{aligned}\sqcap_{|\lambda|\geq 1}\lambda F_A(e) &= \cap_{|\lambda|\geq 1}(\lambda F_A)(e)\\ &= \cap_{|\lambda|\geq 1}F_A(\lambda^{-1}e)\\ &= \cap_{|\lambda|\geq 1}\{[|\lambda^{-1}e|] + 1, [|\lambda^{-1}e|] + 2, [|\lambda^{-1}e|] + 3, \cdots\}\\ &= \{[|e|] + 1, [|e|] + 2, [|e|] + 3, \cdots\}\end{aligned}$$

$= F_A(e)$

Since F_A is absorbing soft set, $\sqcap_{|\lambda|\geq 1}\lambda F_A$ is absorbing soft set.

Example 2.32. *Consider the soft set G_B as defined in example 2.28. Here it is obvious that $0G_B(0) = G_B(0)$. Therefore the balanced core of G_B is $\sqcap_{|\lambda|\geq 1}\lambda G_B$. Let $e \in E$. If $e \neq 0$, then*

$$\begin{aligned}(\sqcap_{|\lambda|\geq 1}\lambda G_B)(e) &= \cap_{|\lambda|\geq 1}(\lambda G_B)(e)\\ &= \cap_{|\lambda|\geq 1}(G_B)(\lambda^{-1}e)\\ &= \cap_{|\lambda|\geq 1}\{0,\,1,\,2,\,\cdots,\,[|\lambda^{-1}e|]\}\\ &= \{0\}.\end{aligned}$$

If $e = 0$, then $(\sqcap_{|\lambda|\geq 1}\lambda G_B)(0) = \cap_{|\lambda|\geq 1}\lambda G_B(0) = \cap_{|\lambda|\geq 1}G_B(0) = U_1$.
So, it is clear that balanced core of G_B is not an absorbing soft set.

Note 2.33. *Balanced core of an absorbing soft set may or may not be an absorbing soft set.*

Theorem 2.34. *Let X and Y be linear spaces over $K(\mathbb{R}$ or $\mathbb{C})$ and $f : X \to Y$ be a linear map. Then F_A is an absorbing soft set over (U, X) imply that $f(F_A)$ is also an absorbing soft set over(U, Y).*

Proof: Let $y \in Y$. Now

$(\sqcup_{\lambda>0}\lambda f(F_A))(y)$
$= \cup_{\lambda>0}(\lambda f(F_A))(y)$
$= \cup_{\lambda>0}f(F_A)(\lambda^{-1}y)$
$= \cup_{\lambda>0}\cup_{x\in f^{-1}(\lambda^{-1}y)} F_A(x)$
$= \cup_{\lambda>0}\cup_{\lambda x\in f^{-1}(y)} F_A(x)$
$= \cup_{\lambda>0}\cup_{z\in f^{-1}(y)} F_A(\lambda^{-1}z)$, where $\lambda x = z$
$\supseteq \cup_{\lambda>0}F_A(\lambda^{-1}z)$ for some $z \in f^{-1}(y)$
$= \cup_{\lambda>0}\lambda F_A(z)$
$= U$, as F_A is an absorbing soft set.
Therefore, $\sqcup_{\lambda>0}\lambda f(F_A) = \overline{1}$. Hence, $f(F_A)$ is an absorbing soft set.

Theorem 2.35. *Let X and Y be linear spaces over $K(\mathbb{R}$ or $\mathbb{C})$ and $f : X \to Y$ be a linear map. Then G_B is an absorbing soft set over (U, Y) imply that $f^{-1}(G_B)$ is also an absorbing soft set over (U, X).*

Proof: Let $x \in X$. Now

$(\sqcup_{\lambda>0}\lambda f^{-1}(G_B))(x)$
$= \cup_{\lambda>0}(\lambda f^{-1}(G_B))(x)$
$= \cup_{\lambda>0}f^{-1}(G_B)(\lambda^{-1}x)$
$= \cup_{\lambda>0}G_B(f(\lambda^{-1}x))$
$= \cup_{\lambda>0}G_B(\lambda^{-1}f(x))$
$= \cup_{\lambda>0}\lambda G_B(f(x))$
$= U$, as G_B is an absorbing soft set.
Thus, $\sqcup_{\lambda>0}\lambda f^{-1}(G_B) = \overline{1}$. Hence, $f^{-1}(G_B)$ is an absorbing soft set.

3 Conclusion

To study the functional analysis on soft sets, balanced and absorbing soft sets are being defined over a linear space in this paper. Then we have established some theorems which will be needed in future for construction of a convex soft set, absolutely convex soft set etc.

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