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Fuzzy Soft Ultrafilters and Convergence Properties of Fuzzy Soft Filters

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Abstract - In the present paper, we study the notion of a fuzzy soft filter by using fuzzy soft sets. We present the concepts of a fuzzy soft filter base and a fuzzy soft ultrafilter and obtain their related properties. Also, we show how a fuzzy soft topology is derived from a fuzzy soft filter. Moreover, we investigate convergence of fuzzy soft filters in a fuzzy soft topological space with related results.

Keywords - Fuzzy soft set, fuzzy soft topology, fuzzy soft filter, fuzzy soft ultrafilter, convergence.

1 Introduction

In 1999, Molodtsov [18] introduced the concept of a soft set theory as a new mathematical tool for dealing with uncertainties. This theory provides a very general framework with the involvement of parameters. The parameters can be expressed in the form of words, sentences, real numbers and so on. This implies that the problem of setting the membership function does not arise. Hence, soft set theory has attractive applications in other disciplines and real life problems, most of these applications was shown by Molodtsov [18]. Recently, researchers are contributing a lot regarding soft set theory

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and its applications [2, 14, 17, 22].

Maji et.al. [16] combined the concept of fuzzy set and soft set and introduced the new notion of the fuzzy soft set. Roy et al. [21] presented some results on an application of fuzzy soft sets in decision making problem. Then, Tanay and Kandemir [24] initiated the concept of a fuzzy soft topology and gave the some basic properties of it by following Chang [7]. Also, the fuzzy soft topology in Lowen's sense [13] was given by Varol and Aygün [25]. In recent years, there have been considerable advances in fuzzy soft sets and their applications [1, 3, 4, 9, 10, 12, 26].

Filters were introduced in 1937 by Cartan [6]. The study of filters is a very natural way to describe convergence in a topological space. Moreover, they play a fundamental role in the development of fuzzy spaces which have applications in computer science and engineering. Then, many authors have obtained the concept of a fuzzy filter structure in different approaches. By using fuzzy sets, Vicente and Aranguren [20] defined fuzzy filters. Burton et al. [5] introduced the different notion of a fuzzy filter. Later, as an extension of these definitions, Kim et al. [11] proposed a new definition of fuzzy filter on a set X as a map $\mathcal{F} : L^X \rightarrow M$ satisfying certain conditions, where L and M are a completely distributive lattice.

Extensions of filter structures to the soft sets and also fuzzy soft sets have been studied by some authors. More recently, Şahin et al. [23] defined soft filters and studied some of their properties. By using fuzzy soft sets, Çetkin et al. [8] introduced fuzzy soft filters on the base of definition suggested by Kim et al. [11].

In this work, we continue investigating the properties of fuzzy soft filters in Vicente and Aranguren's sense [20]. We define the notion of a fuzzy soft ultrafilter and obtain a few results analogous to the ones that hold for fuzzy ultrafilters. Also, we show that each fuzzy soft filter on X induces a fuzzy soft topology on the same set. Moreover, we investigate the convergence theory of fuzzy soft filters. Finally, we show that a fuzzy soft filter converge to at most one fuzzy soft point in a fuzzy soft Hausdorff space.

2 Preliminary

In this section, we recall some basic notions regarding fuzzy soft sets which will be used in the sequel. Throughout this work, let X be an initial universe, I^X be the set of all fuzzy subsets of X and E be the set of all parameters for X .

Definition 2.1. [16] *A fuzzy soft set f on the universe X with the set E of parameters is defined by the set of ordered pairs*

$$f = \{(e, f(e)) : e \in E, f(e) \in I^X\}$$

where f is a mapping given by $f : E \rightarrow I^X$.

Throughout this paper, the family of all fuzzy soft sets over X is denoted by $(I^X)^E$ [8].

Definition 2.2. [1, 16, 25] Let $f, g \in (I^X)^E$. Then,

(i) The fuzzy soft set f is called a null fuzzy soft set, denoted by $\tilde{\emptyset}$, if $f(e) = \bar{0}$ for every $e \in E$.

(ii) If $f(e) = \bar{1}$ for all $e \in E$, then f is called an absolute fuzzy soft set, denoted by \tilde{X} .

(iii) f is a fuzzy soft subset of g if $f(e) \leq g(e)$ for each $e \in E$. It is denoted by $f \sqsubseteq g$.

(iv) f and g are fuzzy soft equal if $f \sqsubseteq g$ and $g \sqsubseteq f$. It is denoted by $f = g$.

(v) The complement of f is denoted by f^c , where $f^c : E \rightarrow I^X$ is a mapping defined by $f^c(e) = \bar{1} - f(e)$ for all $e \in E$. Clearly, $(f^c)^c = f$.

(vi) The union of f and g is a fuzzy soft set h defined by $h(e) = f(e) \vee g(e)$ for all $e \in E$. h is denoted by $f \sqcup g$.

(vii) The intersection of f and g is a fuzzy soft set h defined by $h(e) = f(e) \wedge g(e)$ for all $e \in E$. h is denoted by $f \sqcap g$.

Definition 2.3. [9] Let f and g be two fuzzy soft sets. The difference of two fuzzy soft sets f and g over X , denoted by $f \setminus g$, is defined as $f \setminus g = f \sqcap g^c$.

Definition 2.4. [1] Let J be an arbitrary index set and let $\{f_i\}_{i \in J}$ be a family of fuzzy soft sets over X . Then,

(i) The union of these fuzzy soft sets is the fuzzy soft set h defined by $h(e) = \bigvee_{i \in J} f_i(e)$ for every $e \in E$ and this fuzzy soft set is denoted by $\bigsqcup_{i \in J} f_i$.

(ii) The intersection of these fuzzy soft sets is the fuzzy soft set h defined by $h(e) = \bigwedge_{i \in J} f_i(e)$ for every $e \in E$ and this fuzzy soft set is denoted by $\bigsqcap_{i \in J} f_i$.

Theorem 2.5. [25] Let J be an index set and $f, g, f_i, g_i \in (I^X)^E$, for all $i \in J$. Then, the following statements are satisfied.

$$(1) f \sqcap \left(\bigsqcup_{i \in J} g_i \right) = \bigsqcup_{i \in J} (f \sqcap g_i).$$

$$(2) f \sqcup \left(\bigsqcap_{i \in J} g_i \right) = \bigsqcap_{i \in J} (f \sqcup g_i).$$

$$(3) \left(\bigsqcap_{i \in J} f_i \right)^c = \bigsqcup_{i \in J} f_i^c.$$

$$(4) \left(\bigsqcup_{i \in J} f_i \right)^c = \bigsqcap_{i \in J} f_i^c.$$

$$(5) \text{ If } f \sqsubseteq g, \text{ then } g^c \sqsubseteq f^c.$$

Definition 2.6. [3, 26] A fuzzy soft set f over X is said to be a fuzzy soft point if there is an $e \in E$ such that $f(e)$ is a fuzzy point in X (i.e., there exists an $x \in X$ such that $f(e)(x) = \alpha \in (0, 1]$ and $f(e)(x') = 0$ for all $x' \in X - \{x\}$) and $f(e') = \bar{0}$ for every $e' \in E \setminus \{e\}$. It will be denoted by e_{x^α} .

The fuzzy soft point e_{x^α} is said to belong to a fuzzy soft set f , denoted by $e_{x^\alpha} \tilde{\in} f$, if $\alpha \leq f(e)(x)$.

Let $\mathcal{P}(X, E)$ be the family of all fuzzy soft points on X .

Definition 2.7. [10] Let $(I^X)^E$ and $(I^Y)^K$ be the families of all fuzzy soft sets over X and Y , respectively. Let $\varphi : X \rightarrow Y$ and $\psi : E \rightarrow K$ be two mappings. Then, the mapping φ_ψ is called a fuzzy soft mapping from X to Y , denoted by $\varphi_\psi : (I^X)^E \rightarrow (I^Y)^K$.

(1) Let $f \in (I^X)^E$. Then $\varphi_\psi(f)$ is the fuzzy soft set over Y defined as follows:

$$\varphi_\psi(f)(k)(y) = \begin{cases} \bigvee_{x \in \varphi^{-1}(y)} \left(\bigvee_{e \in \psi^{-1}(k)} f(e) \right)(x), & \text{if } \psi^{-1}(k) \neq \emptyset, \varphi^{-1}(y) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

for all $k \in K$ and all $y \in Y$.

(2) Let $g \in (I^Y)^K$. Then $\varphi_\psi^{-1}(g)$ is the soft set over X defined as follows:

$$\varphi_\psi^{-1}(g)(e)(x) = g(\psi(e))(\varphi(x))$$

for all $e \in E$ and all $x \in X$.

Theorem 2.8. [3, 10, 25] Let $f, f_i \in (I^X)^E$ and $g, g_i \in (I^Y)^K$ for all $i \in J$ where J is an index set. Then, for a fuzzy soft mapping $\varphi_\psi : (I^X)^E \rightarrow (I^Y)^K$, the following conditions are satisfied.

- (1) If $f_1 \sqsubseteq f_2$, then $\varphi_\psi(f_1) \sqsubseteq \varphi_\psi(f_2)$.
- (2) If $g_1 \sqsubseteq g_2$, then $\varphi_\psi^{-1}(g_1) \sqsubseteq \varphi_\psi^{-1}(g_2)$.
- (3) $f \sqsubseteq \varphi_\psi^{-1}(\varphi_\psi(f))$, $\varphi_\psi(\varphi_\psi^{-1}(g)) \sqsubseteq g$.
- (4) $\varphi_\psi\left(\bigsqcup_{i \in J} f_i\right) = \bigsqcup_{i \in J} \varphi_\psi(f_i)$.
- (5) $\varphi_\psi\left(\prod_{i \in J} f_i\right) \sqsubseteq \prod_{i \in J} \varphi_\psi(f_i)$.
- (6) $\varphi_\psi^{-1}\left(\bigsqcup_{i \in J} g_i\right) = \bigsqcup_{i \in J} \varphi_\psi^{-1}(g_i)$, $\varphi_\psi^{-1}\left(\prod_{i \in J} g_i\right) = \prod_{i \in J} \varphi_\psi^{-1}(g_i)$.
- (7) $\varphi_\psi^{-1}(\tilde{Y}) = \tilde{X}$, $\varphi_\psi^{-1}(\tilde{\emptyset}) = \tilde{\emptyset}$
- (8) $\varphi_\psi(\tilde{\emptyset}) = \tilde{\emptyset}$.

Proposition 2.9. [26] Let $\varphi_\psi : (I^X)^E \rightarrow (I^Y)^K$ be a fuzzy soft mapping and $e_{x^\alpha} \in \mathcal{P}(X, E)$. Then $\varphi_\psi(e_{x^\alpha}) = \psi(e)_{\varphi(x)^\alpha} \in \mathcal{P}(Y, K)$.

Definition 2.10. [25] Let $f \in (I^X)^E$ and $g \in (I^Y)^K$. The fuzzy soft product $f \times g$ is defined by the fuzzy soft set h where $h : E \times K \rightarrow I^{X \times Y}$ and $h(e, k) = f(e) \times g(k)$ for all $(e, k) \in E \times K$.

Definition 2.11. [25] Let $f \in (I^X)^E, g \in (I^Y)^K$ and let $p_X : X \times Y \rightarrow X, q_E : E \times K \rightarrow E$ and $p_Y : X \times Y \rightarrow Y, q_K : E \times K \rightarrow K$ be the projection mappings in classical meaning. The fuzzy soft mappings $(p_X)_{q_E}$ and $(p_Y)_{q_K}$ are called fuzzy soft projection mappings from $X \times Y$ to X and $X \times Y$ to Y , respectively, where $(p_X)_{q_E}(f \times g) = f$ and $(p_Y)_{q_K}(f \times g) = g$.

Definition 2.12. [24] Let τ be the collection of fuzzy soft sets over X , then τ is said to be a fuzzy soft topology on X if

(FST1) $\tilde{\emptyset}, \tilde{X}$ belong to τ .

(FST2) the union of any number of fuzzy soft sets in τ belongs to τ .

(FST3) the intersection of any two fuzzy soft sets in τ belongs to τ

(X, τ) is called a fuzzy soft topological space. The members of τ are called fuzzy soft open sets in X . A fuzzy soft set f over X is called a fuzzy soft closed in X if $f^c \in \tau$.

Definition 2.13. [24] Let (X, τ) be a fuzzy soft topological space and $f \in (I^X)^E$. The fuzzy soft interior of f is the fuzzy soft set $f^\circ = \bigsqcup \{g : g \text{ is a fuzzy soft open set and } g \sqsubseteq f\}$.

By property (FST2) for fuzzy soft open sets, f° is fuzzy soft open. It is the largest fuzzy soft open set contained in f .

Definition 2.14. [19, 25] Let (X, τ) be a fuzzy soft topological space and $f \in (I^X)^E$. The fuzzy soft closure of f is the fuzzy soft set $\bar{f} = \bigsqcap \{g : g \text{ is a fuzzy soft closed set and } f \sqsubseteq g\}$.

Clearly \bar{f} is the smallest fuzzy soft closed set over X which contains f .

Definition 2.15. [24] A fuzzy soft set f in a fuzzy soft topological space (X, τ) is called a fuzzy soft neighborhood of the fuzzy soft point e_{x^α} if there exists a fuzzy soft open set g such that $e_{x^\alpha} \tilde{\in} g \sqsubseteq f$.

The fuzzy soft neighborhood system of a fuzzy soft point e_{x^α} , denoted by $\mathcal{N}(e_{x^\alpha})$, is the family of all its fuzzy soft neighborhoods.

Theorem 2.16. [15, 26] A fuzzy soft set f over X is fuzzy soft open iff f is a fuzzy soft neighborhood of each of its fuzzy soft points.

Theorem 2.17. [15, 26] Let (X, τ) be a fuzzy soft topological space and $\mathcal{N}(e_{x^\alpha})$ be the fuzzy soft neighborhood system of fuzzy soft point e_{x^α} . Then,

(FSN1) $\mathcal{N}(e_{x^\alpha}) \neq \emptyset$,

(FSN2) If $f \in \mathcal{N}(e_{x^\alpha})$, then $e_{x^\alpha} \tilde{\in} f$,

(FSN3) If $f \in \mathcal{N}(e_{x^\alpha})$ and $f \sqsubseteq g$, then $g \in \mathcal{N}(e_{x^\alpha})$,

(FSN4) If $f, g \in \mathcal{N}(e_{x^\alpha})$, then $f \sqcap g \in \mathcal{N}(e_{x^\alpha})$,

(FSN5) If $f \in \mathcal{N}(e_{x^\alpha})$, then there is a $g \in \mathcal{N}(e_{x^\alpha})$ such that $g \sqsubseteq f$ and $g \in \mathcal{N}(e'_{y^\lambda})$, for each $e'_{y^\lambda} \tilde{\in} g$.

Theorem 2.18. Let there be assigned to each fuzzy soft point $e_{x^\alpha} \in \mathcal{P}(X, E)$ a collection $\mathcal{N}(e_{x^\alpha})$ of subsets of $(I^X)^E$ satisfying the above axioms (FSN1)-(FSN5). Then, there exists a fuzzy soft topology τ on X such that, for each $e_{x^\alpha} \in \mathcal{P}(X, E)$, $\mathcal{N}(e_{x^\alpha})$ is the τ -fuzzy soft neighborhood system of e_{x^α} .

Proof: Let $\tau = \{f \in (I^X)^E : f \in \mathcal{N}(e_{x^\alpha}) \text{ for each } e_{x^\alpha} \tilde{\in} f\}$. It is clear that $\tilde{\emptyset}, \tilde{X} \in \tau$.

Let $f, g \in \tau$ and $e_{x^\alpha} \tilde{\sqsubseteq} f \sqcap g$. Then, $e_{x^\alpha} \tilde{\sqsubseteq} f$ and $e_{x^\alpha} \tilde{\sqsubseteq} g$. Therefore, $f, g \in \mathcal{N}(e_{x^\alpha})$ and by (FSN4), we have $f \sqcap g \in \mathcal{N}(e_{x^\alpha})$. Thus, $f \sqcap g \in \tau$.

Let $\{f_i\}_{i \in J} \subseteq \tau$ and $e_{x^\alpha} \tilde{\sqsubseteq} \bigsqcup_{i \in J} f_i$. Then, there exists an $i_0 \in J$ such that $e_{x^\alpha} \tilde{\sqsubseteq} f_{i_0}$. By the definition of τ , $f_{i_0} \in \mathcal{N}(e_{x^\alpha})$. Hence, from (FSN3), we have $\bigsqcup_{i \in J} f_i \in \mathcal{N}(e_{x^\alpha})$.

Now, let us show that $\mathcal{N}(e_{x^\alpha})$ is the fuzzy soft neighborhood system of e_{x^α} . Let f be a fuzzy soft neighborhood of e_{x^α} . Then there is a $g \in \tau$ such that $e_{x^\alpha} \tilde{\sqsubseteq} g \sqsubseteq f$. By the definition of τ , we have $g \in \mathcal{N}(e_{x^\alpha})$. Thus, since $g \sqsubseteq f$, we obtain $f \in \mathcal{N}(e_{x^\alpha})$. Conversely, let $f \in \mathcal{N}(e_{x^\alpha})$. Then, from (FSN5), there exists a $g \in \mathcal{N}(e_{x^\alpha})$ such that $g \sqsubseteq f$ and $g \in \mathcal{N}(e'_{y^\lambda})$, for each $e'_{y^\lambda} \tilde{\sqsubseteq} g$. By the definition of τ , $g \in \tau$. Also, from $g \in \mathcal{N}(e_{x^\alpha})$ it follows that $e_{x^\alpha} \tilde{\sqsubseteq} g$. Hence f is a fuzzy soft neighborhood of e_{x^α} .

Definition 2.19. [3, 26] Let (X, τ_1) and (Y, τ_2) be two fuzzy soft topological spaces. A fuzzy soft mapping $\varphi_\psi : (X, \tau_1) \rightarrow (Y, \tau_2)$ is called fuzzy soft continuous if for each fuzzy soft point e_{x^α} in X and each fuzzy soft neighborhood h of $\varphi_\psi(e_{x^\alpha})$, there is a fuzzy soft neighborhood g of e_{x^α} such that $\varphi_\psi(g) \sqsubseteq h$.

Theorem 2.20. [25] Let $\{(X_i, \tau_i)\}_{i \in J}$ be a family of fuzzy soft topological spaces and let τ be the product fuzzy soft topology on $X (= \prod_{i \in J} X_i)$. τ has as a base the set of finite intersections of fuzzy soft sets of the form $(p_{X_i})_{(q_{E_i})}^{-1}(f_i)$, where $f_i \in \tau_i$, $i \in J$.

3 Fuzzy Soft Filter

In this section, we study some elementary properties of fuzzy soft filter structures in Vicente and Aranguren's sense. We introduce the concepts of a fuzzy soft filter base and a fuzzy soft ultrafilter and give several related properties. Also, we investigate the convergence theory of the fuzzy soft filter in a fuzzy soft topological space.

Definition 3.1. [8] A fuzzy soft filter \mathcal{F} on X is a nonempty collection of subsets of $(I^X)^E$ with the following properties:

- (FSF1) $\tilde{\emptyset} \notin \mathcal{F}$,
- (FSF2) If $f, g \in \mathcal{F}$, then $f \sqcap g \in \mathcal{F}$,
- (FSF3) If $f \in \mathcal{F}$ and $f \sqsubseteq g$, then $g \in \mathcal{F}$.

If \mathcal{F}_1 and \mathcal{F}_2 are two fuzzy soft filters on X , we say that \mathcal{F}_1 is finer than \mathcal{F}_2 (or \mathcal{F}_2 is coarser than \mathcal{F}_1) iff $\mathcal{F}_1 \supseteq \mathcal{F}_2$.

Example 3.2. For each $\alpha \in (0, 1]$,

$$\mathcal{F}_\alpha = \{f \in (I^X)^E : f(e)(x) \geq \alpha, \forall e \in E, \forall x \in X\}$$

is a fuzzy soft filter on X .

Definition 3.3. Let $f \in (I^X)^E$. Then,

(i) f is said to be finite if there exists a finite parameter subset $A \subseteq E$ such that $f(e) \neq \bar{0}$ for every $e \in A$ and $f(e') = \bar{0}$ for every $e' \in E \setminus A$.

(ii) f is said to be countable if there exists a countable parameter subset $A \subseteq E$ such that $f(e) \neq \bar{0}$ for every $e \in A$ and $f(e') = \bar{0}$ for every $e' \in E \setminus A$.

It is readily shown that the union of any finite number of finite fuzzy soft sets is a finite fuzzy soft set. Again, the union of any countable number of countable fuzzy soft sets is a countable fuzzy soft set.

Example 3.4. (i) Let $X = \{x_1, x_2\}$ and $E = \{e_1, e_2, e_3, \dots\}$. Then,

$$f = \{(e_1, \{x_1 \setminus 0.2, x_2 \setminus 0\}), (e_2, \{x_1 \setminus 0.5, x_2 \setminus 0.1\}), (e_3, \bar{0}), (e_4, \bar{0}), \dots\}$$

(that is, $f(e) = \bar{0}$ for each $e \in E \setminus \{e_1, e_2\}$) is a finite fuzzy soft set.

(ii) Let $X = \{x_1, x_2\}$ and $E = \{e_i : i \in \mathbb{R}\}$. If $f(e) \neq \bar{0}$ for every $e \in A = \{e_i : i \in \mathbb{N}\}$ and $f(e') = \bar{0}$ for every $e' \in E \setminus A$, then f is a countable fuzzy soft set.

Example 3.5. (i) Let X be an arbitrary set and E be an infinite set. Then,

$$\mathcal{F} = \{f \in (I^X)^E : f^c \text{ is finite}\}$$

is a fuzzy soft filter on X .

(ii) Let X be an arbitrary set and E be an uncountable set. Then,

$$\mathcal{F} = \{f \in (I^X)^E : f^c \text{ is countable}\}$$

is a fuzzy soft filter on X .

Definition 3.6. A collection \mathcal{B} of subsets of $(I^X)^E$ is called a base for a fuzzy soft filter on X if the following two conditions are satisfied :

(B1) $\mathcal{B} \neq \emptyset$ and $\tilde{\emptyset} \notin \mathcal{B}$,

(B2) If $f, g \in \mathcal{B}$, then there is an $h \in (I^X)^E$ such that $h \sqsubseteq f \sqcap g$.

One readily sees that if \mathcal{B} is a base for a fuzzy soft filter on X , the collection

$$\mathcal{F}_{\mathcal{B}} = \{f \in (I^X)^E : \text{there exists an } h \in \mathcal{B} \text{ such that } h \sqsubseteq f\}$$

is a fuzzy soft filter on X . We say that the fuzzy soft filter $\mathcal{F}_{\mathcal{B}}$ is generated by \mathcal{B}

Example 3.7. Let f be a nonempty fuzzy soft set. Then, $\mathcal{B} = \{f\}$ is a base for a fuzzy soft filter on X .

Theorem 3.8. Let $\varphi_\psi : (I^X)^E \rightarrow (I^Y)^K$ be a fuzzy soft mapping and let \mathcal{F} be a fuzzy soft filter on X . Then, $\{\varphi_\psi(f) : f \in \mathcal{F}\}$ is a base for a fuzzy soft filter $\varphi_\psi(\mathcal{F})$ on Y .

Proof: We need to verify axioms (B1) – (B2). It is clear from the definition of fuzzy soft filter.

Definition 3.9. A fuzzy soft filter \mathcal{F} on X is called a fuzzy soft ultrafilter if there is no finer fuzzy soft filter than \mathcal{F} (i.e., it is maximal for the inclusion relation among fuzzy soft filters).

Theorem 3.10. Every fuzzy soft filter \mathcal{F} on X is contained in some fuzzy soft ultrafilter on X .

Proof: Let Φ be the collection of all fuzzy soft filters on X finer than \mathcal{F} , partially ordered by $\mathcal{F}_1 \lesssim \mathcal{F}_2$ if and only if $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Now, let us take a chain $\{\mathcal{F}_\alpha : \alpha \in \Lambda\} \subseteq \Phi$. From the fact that if f_1 and f_2 belong to $\bigcup_{\alpha \in \Lambda} \mathcal{F}_\alpha$, then they both belong to some \mathcal{F}_α by linearity of the inclusion order on $\{\mathcal{F}_\alpha : \alpha \in \Lambda\}$ it follows that $\bigcup_{\alpha \in \Lambda} \mathcal{F}_\alpha$ is a fuzzy soft filter on X . Hence, $\bigcup_{\alpha \in \Lambda} \mathcal{F}_\alpha$ is an upper bound of $\{\mathcal{F}_\alpha : \alpha \in \Lambda\}$. Thus, by Zorn's Lemma, Φ has a maximal element \mathcal{G} and therefore, clearly, \mathcal{G} is a fuzzy soft ultrafilter containing \mathcal{F} .

Lemma 3.11. If \mathcal{A} is a collection of fuzzy soft sets with the finite intersection property, then there is a fuzzy soft filter \mathcal{F} on X such that $\mathcal{A} \subseteq \mathcal{F}$.

Proof: Let \mathcal{F} be the collection of all $f \in (I^X)^E$ such that there is a finite $\{f_1, \dots, f_n\} \subseteq \mathcal{A}$ with $f_1 \sqcap \dots \sqcap f_n \subseteq f$. Then \mathcal{F} is a fuzzy soft filter containing \mathcal{A} .

Theorem 3.12. Let \mathcal{F} be a fuzzy soft filter on X . Then,

(i) \mathcal{F} is a fuzzy soft ultrafilter on X if and only if each $g \in (I^X)^E$ such that $f \sqcap g \neq \tilde{\emptyset}$ for all $f \in \mathcal{F}$ belongs to \mathcal{F} .

(ii) If \mathcal{F} is a fuzzy soft ultrafilter on X and $f_1 \sqcup f_2 \in \mathcal{F}$, then we have either $f_1 \in \mathcal{F}$ or $f_2 \in \mathcal{F}$.

(iii) If \mathcal{F} is a fuzzy soft ultrafilter on X , then for every $f \in (I^X)^E$ we have either $f \in \mathcal{F}$ or $f^c \in \mathcal{F}$.

Proof: (i) Let \mathcal{F} be a fuzzy soft ultrafilter on X and let $g \in (I^X)^E$ be such that $f \sqcap g \neq \tilde{\emptyset}$ for every $f \in \mathcal{F}$. Put $\mathcal{G} = \mathcal{F} \cup \{g\}$. Since \mathcal{G} has the finite intersection property, by Lemma 3.11, there exists a fuzzy soft filter \mathcal{H} on X such that $\mathcal{G} \subseteq \mathcal{H}$. By the maximality of \mathcal{F} , we have $\mathcal{F} = \mathcal{H}$ and thus $g \in \mathcal{F}$.

Conversely, suppose that \mathcal{F} contains every $g \in (I^X)^E$ such that $f \sqcap g \neq \tilde{\emptyset}$ for all $f \in \mathcal{F}$. Let us take a fuzzy soft filter \mathcal{G} such that $\mathcal{F} \subseteq \mathcal{G}$. Then, there is an $h \in (I^X)^E$ such that $h \in \mathcal{G}$ and $h \notin \mathcal{F}$. If $f \in \mathcal{F}$, then $f, h \in \mathcal{G}$ and therefore $f \sqcap h \neq \tilde{\emptyset}$. Thus, by hypothesis, we have $h \in \mathcal{F}$. This is a contradiction.

(ii) Suppose that $f_1, f_2 \notin \mathcal{F}$ and let $f = f_1 \sqcup f_2 \in \mathcal{F}$. By (i), there exist $g_1, g_2 \in \mathcal{F}$ with $f_1 \sqcap g_1 = f_2 \sqcap g_2 = \tilde{\emptyset}$. If $g = g_1 \sqcap g_2$, then we have $g \in \mathcal{F}$ and $f \sqcap g = \tilde{\emptyset}$. This implies that $f \notin \mathcal{F}$, which is a contradiction.

(iii) Assume that neither f nor f^c belongs to \mathcal{F} . From (i) it follows that there is a $g \in \mathcal{F}$ such that $g \sqcap (f \sqcup f^c) = \tilde{\emptyset}$. Since $g \neq \tilde{\emptyset}$, there exist an $e \in E$ and an $x \in X$ with $g(e)(x) \neq 0$. For these $e \in E$ and $x \in X$ we also have $f(e)(x) \vee f^c(e)(x) \neq 0$. Thus we obtain $g(e)(x) \wedge (f(e)(x) \vee f^c(e)(x)) \neq 0$, a contradiction.

Theorem 3.13. Let $\varphi_\psi : (I^X)^E \rightarrow (I^Y)^K$ be a fuzzy soft mapping and let \mathcal{B} be a base for a fuzzy soft ultrafilter on X . Then, $\mathcal{B}^* = \{\varphi_\psi(f) : f \in \mathcal{B}\}$ is a base for a fuzzy soft ultrafilter on Y .

Proof: We first show that \mathcal{B}^* is a base for a fuzzy soft filter on Y .

(B1) is obvious.

(B2) Let $\varphi_\psi(f_1), \varphi_\psi(f_2) \in \mathcal{B}^*$. Then, there exists an $f_3 \in \mathcal{B}$ such that $f_3 \sqsubseteq f_1 \sqcap f_2$. Therefore, we have $\varphi_\psi(f_3) \sqsubseteq \varphi_\psi(f_1) \sqcap \varphi_\psi(f_2)$. Thus, \mathcal{B}^* is a base for a fuzzy soft filter on Y .

Let \mathcal{F}^* be the fuzzy soft filter on Y generated by \mathcal{B}^* and suppose that \mathcal{G} is another fuzzy soft filter on Y with $\mathcal{F}^* \subset \mathcal{G}$. Then, there is a $g \in (I^Y)^K$ such that $g \in \mathcal{G}$ and $g \notin \mathcal{F}^*$. Let \mathcal{F} be the fuzzy soft ultrafilter on X generated by \mathcal{B} . If $f \in \mathcal{F}$, then there is an $h \in \mathcal{B}$ such that $h \sqsubseteq f$. Because $\varphi_\psi(h)$ and g belong both to the fuzzy soft filter \mathcal{G} , we have $\varphi_\psi(h) \sqcap g \neq \tilde{\emptyset}$. Therefore, there exist a $k \in K, y \in Y$ and an $\alpha > 0$ such that $\varphi_\psi(h)(k)(y) > \alpha$ and $g(k)(y) > \alpha$. By the definition of $\varphi_\psi(h)$, there exist an $e \in E$ and an $x \in X$ such that $\psi(e) = k, \varphi(x) = y$ and $h(e)(x) > \alpha$. Then,

$$\begin{aligned} \alpha &< \min\{h(e)(x), g(\psi(e))(\varphi(x))\} = \min\{h(e)(x), \varphi_\psi^{-1}(g)(e)(x)\} \\ &= (h \sqcap \varphi_\psi^{-1}(g))(e)(x) \\ &\leq (f \sqcap \varphi_\psi^{-1}(g))(e)(x). \end{aligned}$$

This says that $f \sqcap \varphi_\psi^{-1}(g) \neq \tilde{\emptyset}$ for every $f \in \mathcal{F}$. By Theorem 3.12 (i) we have $\varphi_\psi^{-1}(g) \in \mathcal{F}$ and therefore there exists a $g_1 \in \mathcal{B}$ with $g_1 \sqsubseteq \varphi_\psi^{-1}(g)$. From the fact that $\varphi_\psi(g_1) \sqsubseteq g$ and $\varphi_\psi(g_1) \in \mathcal{B}^*$ it follows that $g \in \mathcal{F}^*$. This contradiction completes the proof.

Definition 3.14. A fuzzy soft filter \mathcal{F} on X is called a fuzzy soft free if $\bigcap\{f : f \in \mathcal{F}\} = \tilde{\emptyset}$.

Theorem 3.15. Every fuzzy soft ultrafilter \mathcal{F} is fuzzy soft free.

Proof: Assume that $\bigcap\{f : f \in \mathcal{F}\} \neq \tilde{\emptyset}$. Then, there exists a fuzzy soft point $e_{x^\alpha} \tilde{\in} \bigcap\{f : f \in \mathcal{F}\}$. Now, take a fuzzy soft point e_{x^λ} such that $\lambda < \alpha$. For each $f \in \mathcal{F}$, $e_{x^\lambda} \sqcap f \neq \tilde{\emptyset}$. Therefore, by Theorem 3.12 (i), we have $e_{x^\lambda} \in \mathcal{F}$. Thus, $e_{x^\alpha} \tilde{\in} e_{x^\lambda}$, that is, $\alpha \leq \lambda$. This is a contradiction.

Definition 3.16. [19] Let X be a set and let A be a subset of X . Then, $\tilde{\chi}_A$ is a fuzzy soft set on A defined as the following:

$$\tilde{\chi}_A(e) = \chi_A \text{ for every } e \in E$$

where χ_A is a characteristic function of A .

Theorem 3.17. Let \mathcal{F} be a fuzzy soft ultrafilter on X , $A \subseteq X$ and suppose that for every $e \in E$ and for every $x \in X \setminus A$, there is an $f^* \in \mathcal{F}$ such that $f^*(e)(x) = 0$. If we set $f_A = f \sqcap \tilde{\chi}_A$ for each $f \in \mathcal{F}$, then

$$\mathcal{F}_A = \{f_A : f \in \mathcal{F}\}$$

is a fuzzy soft ultrafilter on A .

Proof: We first prove that \mathcal{F}_A is a fuzzy soft filter on A .

(FSF1) Assume that $\tilde{\emptyset} \in \mathcal{F}_A$. Then, there exists an $f \in \mathcal{F}$ such that $f \sqcap \tilde{\chi}_A = \tilde{\emptyset}$. Thus, we have $f \sqcap f^* = \tilde{\emptyset} \in \mathcal{F}$, which is a contradiction.

(FSF2) Let $f_A, g_A \in \mathcal{F}_A$. Then, we have $f, g \in \mathcal{F}$ and therefore $f \sqcap g \in \mathcal{F}$. Since $(f \sqcap g)_A = f_A \sqcap g_A$, we get $f_A \sqcap g_A \in \mathcal{F}_A$.

(FSF3) Let $f_A \in \mathcal{F}_A$ and let us take a $g \in (I^A)^E$ satisfying $f_A \sqsubseteq g$. We define an $h \in (I^X)^E$ by

$$\text{For each } e \in E, \quad h(e)(x) = \begin{cases} g(e)(x), & \text{if } x \in A; \\ 1, & \text{if } x \notin A. \end{cases}$$

Then, since $f \sqsubseteq h$, we have $h \in \mathcal{F}$. Thus $g = h_A \in \mathcal{F}_A$.

Now, we shall show that \mathcal{F}_A is a fuzzy soft ultrafilter on A . Let $g \in (I^A)^E$ such that $f_A \sqcap g \neq \tilde{\emptyset}$ for each $f_A \in \mathcal{F}_A$. Let us define an $h \in (I^X)^E$ by

$$\text{For each } e \in E, \quad h(e)(x) = \begin{cases} g(e)(x), & \text{if } x \in A; \\ 1, & \text{if } x \notin A. \end{cases}$$

Since $f \sqcap h \neq \tilde{\emptyset}$ for each $f \in \mathcal{F}$, by Theorem 3.12 (i), we have $h \in \mathcal{F}$. Therefore, $g = h_A \in \mathcal{F}_A$. Thus, from Theorem 3.12 (i) it follows that \mathcal{F}_A is a fuzzy soft filter on A .

Theorem 3.18. Let there be assigned to each fuzzy soft point $e_{x^\alpha} \in \mathcal{P}(X, E)$ a fuzzy soft filter $\mathcal{F}(e_{x^\alpha})$ satisfying the following properties:

- (a) If $f \in \mathcal{F}(e_{x^\alpha})$, then $e_{x^\alpha} \tilde{\in} f$,
- (b) If $f \in \mathcal{F}(e_{x^\alpha})$, then there is a $g \in \mathcal{F}(e_{x^\alpha})$ such that $g \sqsubseteq f$ and $g \in \mathcal{N}(e'_{y^\lambda})$, for each $e'_{y^\lambda} \tilde{\in} g$.

Then, there exists a fuzzy soft topology τ on X such that, for each $e_{x^\alpha} \in \mathcal{P}(X, E)$, $\mathcal{F}(e_{x^\alpha})$ is the τ -fuzzy soft neighborhood system of e_{x^α} .

Proof: It is clear by Theorem 2.18, since $\mathcal{F}(e_{x^\alpha})$ has the properties (FSN1) – (FSN5).

Definition 3.19. Let (X, τ) be a fuzzy soft topological space. A fuzzy soft point $e_{x^\alpha} \tilde{\in} \tilde{X}$ is said to be in the adherence of a fuzzy soft set f on X (e.g., is adherent to f) if for each $g \in \mathcal{N}((e_{x^\alpha})^c)$, $g \not\sqsubseteq f^c$, where $(e_{x^\alpha})^c = e_{x^{1-\alpha}}$.

Example 3.20. Let $X = \{x_1, x_2\}$ and $E = \{e^1, e^2\}$. Let us consider the following fuzzy soft sets on X with the set E of parameters:

$$f = \{(e^1, \{x_1 \setminus 0.1, x_2 \setminus 0.7\}), (e^2, \{x_1 \setminus 0.5, x_2 \setminus 0.3\})\},$$

$$g = \{(e^1, \{x_1 \setminus 0.3, x_2 \setminus 0.6\}), (e^2, \{x_1 \setminus 0.2, x_2 \setminus 0.3\})\}.$$

Then, $\tau = \{\tilde{\emptyset}, \tilde{X}, f, g, f \sqcup g, f \sqcap g\}$ is a fuzzy soft topology on X . Also, since $h \not\sqsubseteq f^c$ for every $h \in \mathcal{N}(e_{x_2^{0.6}}^1)$, we see that $e_{x_2^{0.4}}^1$ is in the adherence of the fuzzy soft set f .

Theorem 3.21. Let $e_{x^\alpha} \in \mathcal{P}(X, E)$ and $f \in (I^X)^E$. Then $e_{x^\alpha} \tilde{\in} \bar{f}$ if and only if e_{x^α} is in the adherence of f .

Proof: Let $e_{x^\alpha} \tilde{\in} \bar{f}$. For every fuzzy soft closed set k which contains f , $\alpha \leq k(e)(x)$. By taking complement, this fact can be stated as follows: for every fuzzy soft open set $h \sqsubseteq f^c$, $h(e)(x) \leq (1 - \alpha)$. In other words, for every fuzzy soft open set h satisfying $(1 - \alpha) < h(e)(x)$, $h \not\sqsubseteq f^c$. Thus, e_{x^α} is in the adherence of f .

Conversely, let e_{x^α} be in the adherence of f . Then, for every fuzzy soft open set h satisfying $e_{x^{1-\alpha}} \tilde{\in} h$, $h \not\sqsubseteq f^c$. Therefore, for every fuzzy soft open set h such that $h \sqsubseteq f^c$, we have $(1 - \alpha) > h(e)(x)$. In other words, for every fuzzy soft closed set k such that $f \sqsubseteq k$, we obtain $\alpha \leq k(e)(x)$. Thus, $e_{x^\alpha} \tilde{\in} \bar{f}$.

Definition 3.22. A fuzzy soft filter \mathcal{F} on a fuzzy soft topological space (X, τ) is said to converge to the fuzzy soft point e_{x^α} , denoted by $\mathcal{F} \rightarrow e_{x^\alpha}$, if $\mathcal{N}(e_{x^\alpha}) \subseteq \mathcal{F}$.

Definition 3.23. Let (X, τ) be a fuzzy soft topological space, \mathcal{F} be a fuzzy soft filter on X and $e_{x^\alpha} \in \mathcal{P}(X, E)$. e_{x^α} is called a fuzzy soft cluster point of \mathcal{F} , denoted by $\mathcal{F} \infty e_{x^\alpha}$, if every fuzzy soft neighborhood of e_{x^α} intersects all members of \mathcal{F} .

Remark 3.24. It is clear that if $\mathcal{F} \rightarrow e_{x^\alpha}$, then $\mathcal{F} \infty e_{x^\alpha}$. But the converse is not always true. For example, consider the fuzzy soft topological space (X, τ) as defined in Example 3.20. The fuzzy soft filter $\mathcal{F} = \{h \in (I^X)^E : f \sqsubseteq h\}$ on X has a fuzzy soft cluster point $e_{x_1^{0.3}}^1$; but does not converge.

Theorem 3.25. Let (X, τ) be a fuzzy soft topological space and let \mathcal{F} be a fuzzy soft filter on X . Then, $\mathcal{F} \infty e_{x^\alpha}$ if and only if there exists a fuzzy soft filter \mathcal{G} such that $\mathcal{G} \supseteq \mathcal{F}$ and $\mathcal{G} \rightarrow e_{x^\alpha}$.

Proof: If $\mathcal{F} \infty e_{x^\alpha}$, then the collection $\mathcal{B} = \{f \sqcap g : f \in \mathcal{N}(e_{x^\alpha}), g \in \mathcal{F}\}$ is a base for a fuzzy soft filter \mathcal{G} which is finer than \mathcal{F} and converges to e_{x^α} .

On the other hand, if $\mathcal{G} \supseteq \mathcal{F}$ and $\mathcal{G} \rightarrow e_{x^\alpha}$, then every $f \in \mathcal{N}(e_{x^\alpha})$ and every $g \in \mathcal{F}$ belong to \mathcal{G} . Since \mathcal{G} is fuzzy soft filter, $f \sqcap g \neq \tilde{\emptyset}$. Thus, $\mathcal{F} \infty e_{x^\alpha}$.

Theorem 3.26. *Let (X, τ) be a fuzzy soft topological space, $f \in (I^X)^E$ and $e_{x^\alpha} \in \mathcal{P}(X, E)$. Then, the following conditions are satisfied.*

(i) *$f \in \tau$ if and only if for each fuzzy soft filter \mathcal{F} on X converging to $e_{x^\alpha} \tilde{\cong} f$, we have $f \in \mathcal{F}$.*

(ii) *e_{x^α} is adherent to f if and only if there exists a fuzzy soft filter \mathcal{F} on X such that $f^c \notin \mathcal{F}$ and $\mathcal{F} \rightarrow e_{x^{1-\alpha}}$.*

(iii) *$f \in \tau^c$ if and only if whenever \mathcal{F} is a fuzzy soft filter on X such that $f^c \notin \mathcal{F}$ and $\mathcal{F} \rightarrow e_{x^{1-\alpha}}$, then $e_{x^\alpha} \tilde{\cong} f$.*

Proof: (i) The necessity is clear from the definition of a fuzzy soft open set.

To prove sufficiency, by Theorem 2.16 it is enough to show that f is a fuzzy soft neighborhood of each of its fuzzy soft points. Let $e_{x^\alpha} \tilde{\cong} f$ and take $\mathcal{N}(e_{x^\alpha}) = \mathcal{F}$. Then, by hypothesis, we have $f \in \mathcal{N}(e_{x^\alpha})$.

(ii) If e_{x^α} is adherent to f , then for each $g \in \mathcal{N}(e_{x^{1-\alpha}})$, $g \not\sqsubseteq f^c$. Letting $\mathcal{F} = \mathcal{N}(e_{x^{1-\alpha}})$, we obtain $\mathcal{F} \rightarrow e_{x^{1-\alpha}}$ and $f^c \notin \mathcal{N}(e_{x^{1-\alpha}})$.

For the converse, let \mathcal{F} be a fuzzy soft filter on X such that $\mathcal{F} \rightarrow e_{x^{1-\alpha}}$ and $f^c \notin \mathcal{F}$. Then, for each $g \in \mathcal{N}(e_{x^{1-\alpha}})$ we have $g \not\sqsubseteq f^c$, because otherwise we would have $f^c \in \mathcal{F}$ which is impossible. Thus, e_{x^α} is adherent to f .

(iii) Necessity follows from (ii) and Theorem 3.21.

To prove sufficiency, we shall show that $\bar{f} \sqsubseteq f$. Let $e_{x^\alpha} \tilde{\cong} \bar{f}$. By Theorem 3.21, for each $g \in \mathcal{N}(e_{x^{1-\alpha}})$, $g \not\sqsubseteq f^c$. Considering $\mathcal{F} = \mathcal{N}(e_{x^{1-\alpha}})$, we have $\mathcal{F} \rightarrow e_{x^{1-\alpha}}$ and $f^c \notin \mathcal{N}(e_{x^{1-\alpha}})$. Thus, from hypothesis it follows that $e_{x^\alpha} \tilde{\cong} f$.

Theorem 3.27. *Let (X, τ_1) and (Y, τ_2) be two fuzzy soft topological spaces and $e_{x^\alpha} \in \mathcal{P}(X, E)$. A fuzzy soft mapping $\varphi_\psi : (X, \tau_1) \rightarrow (Y, \tau_2)$ is fuzzy soft continuous if and only if whenever \mathcal{F} converges to e_{x^α} , $\varphi_\psi(\mathcal{F})$ converges to $\varphi_\psi(e_{x^\alpha})$.*

Proof: Suppose φ_ψ is fuzzy soft continuous at e_{x^α} and $\mathcal{F} \rightarrow e_{x^\alpha}$. Let f be any fuzzy soft neighborhood of $\varphi_\psi(e_{x^\alpha})$ in Y . Then, for some fuzzy soft neighborhood g of e_{x^α} , $\varphi_\psi(g) \sqsubseteq f$. Thus, from the fact that $g \in \mathcal{F}$ it follows that $f \in \varphi_\psi(\mathcal{F})$.

Conversely, let $\mathcal{F} = \mathcal{N}(e_{x^\alpha})$. Since $\mathcal{F} \rightarrow e_{x^\alpha}$, by hypothesis, we have $\varphi_\psi(\mathcal{F}) \rightarrow \varphi_\psi(e_{x^\alpha})$. Then every fuzzy soft neighborhood f of $\varphi_\psi(e_{x^\alpha})$ belongs to $\varphi_\psi(\mathcal{F})$. Therefore, there is a fuzzy soft neighborhood g of e_{x^α} such that $\varphi_\psi(g) \sqsubseteq f$. Thus, φ_ψ is fuzzy soft continuous at e_{x^α} .

Definition 3.28. *Let X_i be a set for each $i \in J$ and let $e_{x_i}^{\alpha_i} \in \mathcal{P}(X_i, E_i)$. Then, the fuzzy soft product $\prod_{i \in J} e_{x_i}^{\alpha_i}$ is a fuzzy soft point in $\prod_{i \in J} X_i$, denoted by $(e^i)_{(x_i)^\alpha}$ where $\alpha = \inf\{\alpha_i : i \in J\}$.*

Theorem 3.29. Let $\{(X_i, \tau_i)\}_{i \in J}$ be a family of fuzzy soft topological spaces and let τ be the product fuzzy soft topology on $X (= \prod_{i \in J} X_i)$. Then,

(i) A fuzzy soft filter \mathcal{F} on X converges to $(e^i)_{(x_i)^\alpha} \in \mathcal{P}(X, E)$ if and only if $(p_{X_i})_{(q_{E_i})}(\mathcal{F}) \rightarrow (p_{X_i})_{(q_{E_i})}((e^i)_{(x_i)^\alpha}) = e^i_{x_i^{\alpha_i}} \in \mathcal{P}(X_i, E_i)$ for each $i \in J$.

(ii) If $\mathcal{F} \infty (e^i)_{(x_i)^\alpha}$, then $(p_{X_i})_{(q_{E_i})}(\mathcal{F}) \infty e^i_{x_i^{\alpha_i}}$ for each $i \in J$.

Proof: (i) Because $(p_{X_i})_{(q_{E_i})}$ is fuzzy soft continuous for each $i \in J$, necessity follows immediately from Theorem 3.27.

On the other hand, suppose $(p_{X_i})_{(q_{E_i})}(\mathcal{F}) \rightarrow e^i_{x_i^{\alpha_i}}$ for each $i \in J$. By Theorem 2.20, we know that the collection

$$\mathcal{B} = \left\{ \prod_{i \in \Lambda} (p_{X_i})_{(q_{E_i})}^{-1}(f_i) : f_i \in \tau_i, \forall i \in \Lambda \text{ and } \Lambda \subset J \text{ is finite} \right\}$$

is a fuzzy soft base for τ . We shall show that for each $h \in \mathcal{B}$ such that $(e^i)_{(x_i)^\alpha} \tilde{\in} h$, $h \in \mathcal{F}$, which will complete the proof. Let us take $h = \prod_{j=1}^n (p_{X_{i_j}})_{(q_{E_{i_j}})}^{-1}(f_{i_j}) \in \mathcal{B}$ such that $(e^i)_{(x_i)^\alpha} \tilde{\in} h$. By hypothesis, we have $(p_{X_{i_j}})_{(q_{E_{i_j}})}(\mathcal{F}) \rightarrow e^{i_j}_{x_{i_j}^{\alpha_{i_j}}}$ for each $j \in \{1, 2, \dots, n\}$ and therefore $f_{i_j} \in (p_{X_{i_j}})_{(q_{E_{i_j}})}(\mathcal{F})$. From Theorem 3.8 it follows that there exists a $g_j \in \mathcal{F}$ such that $(p_{X_{i_j}})_{(q_{E_{i_j}})}(g_j) \sqsubseteq f_{i_j}$ and so that $g_j \sqsubseteq (p_{X_{i_j}})_{(q_{E_{i_j}})}^{-1}(f_{i_j})$ for each $j \in \{1, 2, \dots, n\}$. Thus, since $\prod_{j=1}^n g_j \in \mathcal{F}$ and $\prod_{j=1}^n g_j \sqsubseteq \prod_{j=1}^n (p_{X_{i_j}})_{(q_{E_{i_j}})}^{-1}(f_{i_j}) = h$, we obtain $h \in \mathcal{F}$.

(ii) It is clear from (i) and Theorem 3.25.

Definition 3.30. Two fuzzy soft points $e^1_{x_1^{\alpha_1}}, e^2_{x_2^{\alpha_2}}$ are said to be equal if $e^1 = e^2, x_1 = x_2$ and $\alpha_1 = \alpha_2$. Thus, $e^1_{x_1^{\alpha_1}} \neq e^2_{x_2^{\alpha_2}} \Leftrightarrow x_1 \neq x_2$ or $e^1 \neq e^2$ or $\alpha_1 \neq \alpha_2$.

Definition 3.31. [15] A fuzzy soft topological space (X, τ) is called fuzzy soft Hausdorff space if for any two distinct fuzzy soft points $e^1_{x_1^{\alpha_1}}, e^2_{x_2^{\alpha_2}} \in \mathcal{P}(X, E)$ there exist fuzzy soft open sets f and g such that $e^1_{x_1^{\alpha_1}} \tilde{\in} f, e^2_{x_2^{\alpha_2}} \tilde{\in} g$ and $f \cap g = \tilde{\emptyset}$.

Theorem 3.32. A fuzzy soft topological space (X, τ) is a fuzzy soft Hausdorff space if and only if every fuzzy soft filter on (X, τ) converges to at most one fuzzy soft point.

Proof: Let (X, τ) be a fuzzy soft Hausdorff space and \mathcal{F} be a fuzzy soft filter on (X, τ) . Suppose that \mathcal{F} converges to two distinct fuzzy soft points $e^1_{x_1^{\alpha_1}}$ and $e^2_{x_2^{\alpha_2}}$. By the fuzzy soft Hausdorff property, there exist fuzzy soft open sets f and g such that $e^1_{x_1^{\alpha_1}} \tilde{\in} f, e^2_{x_2^{\alpha_2}} \tilde{\in} g$ and $f \cap g = \tilde{\emptyset}$. Since \mathcal{F} converges to $e^1_{x_1^{\alpha_1}}$ and $e^2_{x_2^{\alpha_2}}$, then $f, g \in \mathcal{F}$. Therefore, $f \cap g = \tilde{\emptyset} \in \mathcal{F}$, contradicting the definition of fuzzy soft filter.

Conversely, assume that every fuzzy soft filter on (X, τ) converges to at most one

fuzzy soft point, but suppose that (X, τ) is not a fuzzy soft Hausdorff space. Then, there are two distinct fuzzy soft points $e_{x_1}^1, e_{x_2}^2 \in \mathcal{P}(X, E)$ such that every pair of fuzzy soft neighborhoods f of $e_{x_1}^1$ and g of $e_{x_2}^2$ intersect. Thus, $\mathcal{F} = \{f \sqcap g : f \in \mathcal{N}(e_{x_1}^1), g \in \mathcal{N}(e_{x_2}^2)\}$ is a fuzzy soft filter on X . Since every fuzzy soft neighborhood of $e_{x_1}^1$ and every fuzzy soft neighborhood of $e_{x_2}^2$ belongs to \mathcal{F} , we get $\mathcal{F} \rightarrow e_{x_1}^1$ and $\mathcal{F} \rightarrow e_{x_2}^2$. This is a contradiction.

The following example shows that a fuzzy soft filter can converge to more than one fuzzy soft point in a fuzzy soft topological space which is not fuzzy soft Hausdorff space.

Example 3.33. *Let (X, τ) be a fuzzy soft topological space which is defined in Example 3.20. Then, (X, τ) is not a fuzzy soft Hausdorff space. Also, from the fact that $\mathcal{F} = \mathcal{N}(e_{x_0.7}^1) = \mathcal{N}(e_{x_1}^2)$ it follows that $\mathcal{F} \rightarrow e_{x_0.7}^1$ and $\mathcal{F} \rightarrow e_{x_1}^2$.*

4 Conclusion

The study of filters is a very natural way to talk about convergence in an arbitrary topological space. Also, they are an important tool used by researchers describing non-topological convergence notions in functional analysis. Hence, the concept of filter have been studied by many authors in both the fuzzy setting and the soft setting. In the present work, we mainly establish some properties of fuzzy soft filters in Vicente and Aranguren's sense. We present the concept of a fuzzy soft ultrafilter and study some of their properties. Also, we investigate convergence of fuzzy soft filters in a fuzzy soft topological space with related results. Moreover, we show that a fuzzy soft filter converge to at most one fuzzy soft point in fuzzy soft Hausdorff space. We believe that the results of this work will contribute to advance and promote the further study on fuzzy soft topology.

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