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Weakly $\mathcal{I}_{\pi g}$-Closed Sets

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Abstract-   In this paper, another generalized class of τ^* called weakly $\mathcal{I}_{\pi g}$-open sets in ideal topological spaces is introduced and the notion of weakly $\mathcal{I}_{\pi g}$-closed sets in ideal topological spaces is studied. The relationships of weakly $\mathcal{I}_{\pi g}$-closed sets and various properties of weakly $\mathcal{I}_{\pi g}$-closed sets are investigated.

1   Introduction

In 1999, Dontchev et al. studied the notion of generalized closed sets in ideal topological spaces called $\mathcal{I}_g$-closed sets [2]. In 2008, Navaneethakrishnan and Paulraj Joseph have studied some characterizations of normal spaces via $\mathcal{I}_g$-open sets [10]. In 2009, Navaneethakrishnan et al. have introduced $\mathcal{I}_{rg}$-open sets to establish some new characterizations of mildly normal spaces [11]. In 2013, Ekici and Ozen [5] introduced weakly $\mathcal{I}_{rg}$-closed sets which is a generalized class of τ^*. The main aim of this paper is to introduce another generalized class of τ^* called weakly $\mathcal{I}_{\pi g}$-open sets in ideal topological spaces and to study the notion of weakly $\mathcal{I}_{\pi g}$-closed sets in ideal topological spaces. Moreover, this generalized class of τ^* generalize $\mathcal{I}_g$-open sets and $\mathcal{I}_{\pi g}$-open sets. The

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relationships of weakly $\mathcal{I}_{\pi g}$-closed sets and various properties of weakly $\mathcal{I}_{\pi g}$-closed sets are discussed.

2 Preliminaries

In this paper, (X, τ) represents topological space on which no separation axioms are assumed unless explicitly stated. The closure and the interior of a subset G of a space X will be denoted by cl(G) and int(G), respectively. A subset G of a topological space (X, τ) is said to be regular open [14] (resp. regular closed [14]) if G = int(cl(G)) (resp. G = cl(int(G))). A subset G of a topological space (X, τ) is said to be regular open if X\G is regular closed. A subset G of a topological space (X, τ) is said to be π-open [16] if the finite union of regular open sets. A subset G of a topological space (X, τ) is said to be π-closed [16] if X\G is π-open.

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

1. A $\in \mathcal{I}$ and B $\subseteq$ A imply B $\in \mathcal{I}$ and

2. A $\in \mathcal{I}$ and B $\in \mathcal{I}$ imply A$\cup$B $\in \mathcal{I}$ [8].

Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X if $\mathcal{P}$(X) is the set of all subsets of X, a set operator $(\bullet)^\star : \mathcal{P}(X) \to \mathcal{P}(X)$, called a local function [8] of A with respect to τ and $\mathcal{I}$ is defined as follows: for A $\subseteq$ X, $A^\star(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. A Kuratowski closure operator cl$^\star(\bullet)$ for a topology $\tau^\star(\mathcal{I}, \tau)$, called the $\star$-topology and finer than τ, is defined by cl*(A) = A $\cup$ $A^\star(\mathcal{I}, \tau)$ [15]. We will simply write $A^\star$ for $A^\star(\mathcal{I}, \tau)$ and $\tau^\star$ for $\tau^\star(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space. On the other hand, (A, τ_A, $\mathcal{I}_A$) where τ_A is the relative topology on A and $\mathcal{I}_A = \{A \cap J : J \in \mathcal{I}\}$ is an ideal topological space for an ideal topological space $(X, \tau, \mathcal{I})$ and A $\subseteq$ X [7]. For a subset A $\subseteq$ X, cl*(A) and int*(A) will, respectively, denote the closure and the interior of A in (X, τ*).

Definition 2.1. *A subset G of an ideal topological space (X, τ, $\mathcal{I}$) is said to be*

1. *$\mathcal{I}_g$-closed [2] if G^* $\subseteq$ H whenever G $\subseteq$ H and H is open in (X, τ, $\mathcal{I}$).*

2. *$\mathcal{I}_g$-open [2] if X\G is an $\mathcal{I}_g$-closed.*

3. *pre*$^*_\mathcal{I}$-open [4] if G $\subseteq$ int*(cl(G)).*

4. *pre*$^*_\mathcal{I}$-closed [4] if X\G is pre*$^*_\mathcal{I}$-open.*

5. *$\mathcal{I}_{rg}$-closed [11] if G^* $\subseteq$ H whenever G $\subseteq$ H and H is a regular open set in (X, τ, $\mathcal{I}$).*

6. *$\mathcal{I}_{rg}$-open [11] if X\G is an $\mathcal{I}_{rg}$-closed set.*

7. *$\mathcal{I}_{\pi g}$-closed [13] if G^* $\subseteq$ H whenever G $\subseteq$ H and H is a π-open set in (X, τ, $\mathcal{I}$).*

8. *$\mathcal{I}_{\pi g}$-open [13] if X\G is an $\mathcal{I}_{\pi g}$-closed set.*

9. *$\mathcal{I}$-R closed [1] if $G = cl^*(int(G))$.*

10. *$*$-closed [7] if $G = cl^*(G)$ or $G^* \subseteq G$.*

11. *weakly $\mathcal{I}_{rg}$-closed set [5] if $(int(G))^* \subseteq H$ whenever $G \subseteq H$ and H is a regular open set in X.*

Remark 2.2. *[3] In any topological space (X, τ), the following properties hold:*

1. *Every regular open set is π-open but not conversely.*
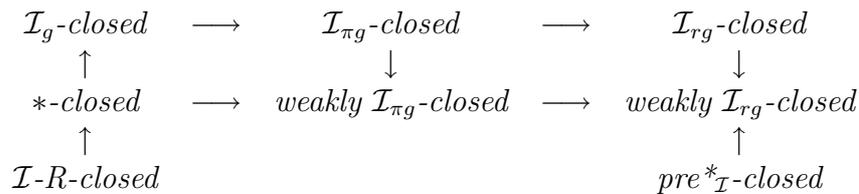
2. *Every π-open set is open but not conversely.*

3   Another Generalized Class of τ^*

Definition 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset G of $(X, \tau, \mathcal{I})$ is said to be a weakly $\mathcal{I}_{\pi g}$-closed set if $(int(G))^* \subseteq H$ whenever $G \subseteq H$ and H is a π-open set in X.*

Example 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space such that $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $\{a\}$ is weakly $\mathcal{I}_{\pi g}$-closed set and $\{b\}$ is not a weakly $\mathcal{I}_{\pi g}$-closed set.*

Definition 3.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. Then G is said to be a weakly $\mathcal{I}_{\pi g}$-open set if $X \backslash G$ is a weakly $\mathcal{I}_{\pi g}$-closed set.*

Remark 3.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The following diagram holds for a subset $G \subseteq X$:*

$$
\begin{array}{ccccc}
\mathcal{I}_g\text{-closed} & \longrightarrow & \mathcal{I}_{\pi g}\text{-closed} & \longrightarrow & \mathcal{I}_{rg}\text{-closed} \\
\uparrow & & \downarrow & & \downarrow \\
*\text{-closed} & \longrightarrow & weakly\ \mathcal{I}_{\pi g}\text{-closed} & \longrightarrow & weakly\ \mathcal{I}_{rg}\text{-closed} \\
\uparrow & & & & \uparrow \\
\mathcal{I}\text{-R-closed} & & & & pre^*{}_{\mathcal{I}}\text{-closed}
\end{array}
$$

These implications are not reversible as shown in the following examples and in [1, 2, 4, 5, 10, 11, 13].

Example 3.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space such that $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$ and $\mathcal{I} = \{\emptyset\}$. Then $\{c\}$ is a weakly $\mathcal{I}_{\pi g}$-closed set but not an $\mathcal{I}_{\pi g}$-closed set.*

Example 3.6. *Let X, τ and $\mathcal{I}$ be as in Example 3.5. Then $\{c\}$ is a weakly $\mathcal{I}_{\pi g}$-closed set but not $*$-closed.*

Proposition 3.7. *Every weakly $\mathcal{I}_{\pi g}$-closed set is a weakly $\mathcal{I}_{rg}$-closed but not conversely.*

Example 3.8. *Let X, τ and $\mathcal{I}$ be as in Example 3.2. Then $\{a, b\}$ is a weakly $\mathcal{I}_{rg}$-closed set but not a weakly $\mathcal{I}_{\pi g}$-closed set.*

Theorem 3.9. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and $G \subseteq X$. The following properties are equivalent:*

1. *G is a weakly $\mathcal{I}_{\pi g}$-closed set,*

2. *cl*(int(G)) $\subseteq$ H whenever $G \subseteq H$ and H is a π-open set in X.*

Proof. (1) $\Rightarrow$ (2) : Let G be a weakly $\mathcal{I}_{\pi g}$-closed set in (X, τ, $\mathcal{I}$). Suppose that $G \subseteq$ H and H is a π-open set in X. We have $(int(G))^* \subseteq$ H. Since $int(G) \subseteq G \subseteq$ H, then $(int(G))^* \cup int(G) \subseteq$ H. This implies that cl*(int(G)) $\subseteq$ H.

(2) $\Rightarrow$ (1) : Let cl*(int(G)) $\subseteq$ H whenever $G \subseteq$ H and H is π-open in X. Since $(int(G))^* \cup int(G) \subseteq$ H, then $(int(G))^* \subseteq$ H whenever $G \subseteq$ H and H is a π-open set in X. Thus G is a weakly $\mathcal{I}_{\pi g}$-closed set.

Theorem 3.10. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and $G \subseteq X$. If G is π-open and weakly $\mathcal{I}_{\pi g}$-closed, then G is $*$-closed.*

Proof. Let G be a π-open and weakly $\mathcal{I}_{\pi g}$-closed set in (X, τ, $\mathcal{I}$). Since G is π-open and weakly $\mathcal{I}_{\pi g}$-closed, cl*(G) = cl*(int(G)) $\subseteq$ G. Thus, G is a $*$-closed set in (X, τ, $\mathcal{I}$).

Theorem 3.11. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and $G \subseteq X$. If G is a weakly $\mathcal{I}_{\pi g}$-closed set, then $(int(G))^* \setminus G$ contains no any nonempty π-closed set.*

Proof. Let G be a weakly $\mathcal{I}_{\pi g}$-closed set in (X, τ, $\mathcal{I}$). Suppose that H is a π-closed set such that $H \subseteq (int(G))^* \setminus G$. Since G is a weakly $\mathcal{I}_{\pi g}$-closed set, $X \setminus H$ is π-open and $G \subseteq X \setminus H$, then $(int(G))^* \subseteq X \setminus H$. We have $H \subseteq X \setminus (int(G))^*$. Hence, $H \subseteq (Int(G))^* \cap (X \setminus (int(G))^*) = \emptyset$. Thus, $(int(G))^* \setminus G$ contains no any nonempty π-closed set.

Theorem 3.12. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and $G \subseteq X$. If G is a weakly $\mathcal{I}_{\pi g}$-closed set, then cl*(int(G))\G contains no any nonempty π-closed set.*

Proof. Suppose that H is a π-closed set such that $H \subseteq$ cl*(int(G))\G. By Theorem 3.11, it follows from the fact that cl*(int(G))\G = $((int(G))^* \cup int(G))$\G.

Theorem 3.13. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. The following properties are equivalent:*

1. *G is pre*$_\mathcal{I}$-closed for each weakly $\mathcal{I}_{\pi g}$-closed set G in (X, τ, $\mathcal{I}$),*

2. *Each singleton $\{x\}$ of X is a π-closed set or $\{x\}$ is pre*$_\mathcal{I}$-open.*

Proof. (1) $\Rightarrow$ (2) : Let G be pre*$_\mathcal{I}$-closed for each weakly $\mathcal{I}_{\pi g}$-closed set G in (X, τ, $\mathcal{I}$) and $x \in X$. We have cl*(int(G)) $\subseteq$ G for each weakly $\mathcal{I}_{\pi g}$-closed set G in (X, τ, $\mathcal{I}$). Assume that $\{x\}$ is not a π-closed set. It follows that X is the only π-open set containing $X \setminus \{x\}$. Then, $X \setminus \{x\}$ is a weakly $\mathcal{I}_{\pi g}$-closed set in (X, τ, $\mathcal{I}$). Thus, cl*(int($X \setminus \{x\}$)) $\subseteq X \setminus \{x\}$ and hence $\{x\} \subseteq$ int*(cl($\{x\}$)). Consequently, $\{x\}$ is pre*$_\mathcal{I}$-open.

(2) $\Rightarrow$ (1) : Let G be a weakly $\mathcal{I}_{\pi g}$-closed set in (X, τ, $\mathcal{I}$). Let $x \in$ cl*(int(G)).

Suppose that $\{x\}$ is pre*$_\mathcal{I}$-open. We have $\{x\} \subseteq$ int*(cl($\{x\}$)). Since $x \in$ cl*(int(G)), then int*(cl($\{x\}$)) $\cap$ int(G) $\neq \emptyset$. It follows that cl($\{x\}$) $\cap$ int(G) $\neq \emptyset$. We have cl($\{x\}$ $\cap$ int(G)) $\neq \emptyset$ and then $\{x\} \cap$ int(G) $\neq \emptyset$. Hence, $x \in$ int(G). Thus, we have $x \in$ G.

Suppose that $\{x\}$ is a π-closed set. By Theorem 3.12, cl*(int(G))\G does not contain $\{x\}$. Since $x \in$ cl*(int(G)), then we have $x \in$ G. Consequently, we have $x \in$ G.

Thus, cl*(int(G)) $\subseteq$ G and hence G is pre*$_\mathcal{I}$-closed.

Theorem 3.14. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. If cl*(int(G))\G contains no any nonempty $*$-closed set, then G is a weakly $\mathcal{I}_{\pi g}$-closed set.*

Proof. Suppose that cl*(int(G))\G contains no any nonempty $*$-closed set in $(X, \tau, \mathcal{I})$. Let $G \subseteq H$ and H be a π-open set. Assume that cl*(int(G)) is not contained in H. It follows that cl*(int(G)) $\cap$ (X\H) is a nonempty $*$-closed subset of cl*(int(G))\G. This is a contradiction. Hence, G is a weakly $\mathcal{I}_{\pi g}$-closed set.

Theorem 3.15. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. If G is a weakly $\mathcal{I}_{\pi g}$-closed set, then int(G) = H\K where H is $\mathcal{I}$-R closed and K contains no any nonempty π-closed set.*

Proof. Let G be a weakly $\mathcal{I}_{\pi g}$-closed set in $(X, \tau, \mathcal{I})$. Take K = (int(G))*\G. Then, by Theorem 3.11, K contains no any nonempty π-closed set. Take H = cl*(int(G)). Then H = cl*(int(H)). Moreover, we have H\K = ((int(G))* $\cup$ int(G))\((int(G))*\G) = ((int(G))* $\cup$ int(G)) $\cap$ (X\(int(G))* $\cup$ G) = int(G).

Theorem 3.16. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. Assume that G is a weakly $\mathcal{I}_{\pi g}$-closed set. The following properties are equivalent:*

1. *G is pre*$_{\mathcal{I}}$-closed,*

2. *cl*(int(G))\ G is a π-closed set,*

3. *(int(G))* \ G is a π-closed set.*

Proof. (1) $\Rightarrow$ (2) : Let G be pre*$_{\mathcal{I}}$-closed. We have cl*(int(G)) $\subseteq$ G. Then, cl*(int(G))\G = $\emptyset$. Thus, cl*(int(G))\G is a π-closed set.

(2) $\Rightarrow$ (1) : Let cl*(int(G))\G be a π-closed set. Since G is a weakly $\mathcal{I}_{\pi g}$-closed set in $(X, \tau, \mathcal{I})$, then by Theorem 3.12, cl*(int(G))\G = $\emptyset$. Hence, we have cl*(int(G)) $\subseteq$ G. Thus, G is pre*$_{\mathcal{I}}$-closed.

(2) $\Leftrightarrow$ (3) : It follows easily from that cl*(int(G))\G = (int(G))*\G.

Theorem 3.17. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$ be a weakly $\mathcal{I}_{\pi g}$-closed set. Then $G \cup (X \backslash (int(G))^*)$ is a weakly $\mathcal{I}_{\pi g}$-closed set in $(X, \tau, \mathcal{I})$.*

Proof. Let G be a weakly $\mathcal{I}_{\pi g}$-closed set in $(X, \tau, \mathcal{I})$. Suppose that H is a π-open set such that G $\cup$ (X\(int(G))*) $\subseteq$ H. We have X\H $\subseteq$ X\(G $\cup$ (X\(int(G))*)) = (X\G) $\cap$ (int(G))* = (int(G))*\G. Since X\H is a π-closed set and G is a weakly $\mathcal{I}_{\pi g}$-closed set, it follows from Theorem 3.11 that X\H = $\emptyset$. Hence, X = H. Thus, X is the only π-open set containing G $\cup$ (X\(int(G))*). Consequently, G $\cup$ (X\(int(G))*) is a weakly $\mathcal{I}_{\pi g}$-closed set in $(X, \tau, \mathcal{I})$.

Corollary 3.18. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$ be a weakly $\mathcal{I}_{\pi g}$-closed set. Then $(int(G))^* \backslash G$ is a weakly $\mathcal{I}_{\pi g}$-open set in $(X, \tau, \mathcal{I})$.*

Proof. Since X\((int(G))*\G) = G $\cup$ (X\(int(G))*), it follows from Theorem 3.17 that (int(G))*\G is a weakly $\mathcal{I}_{\pi g}$-open set in $(X, \tau, \mathcal{I})$.

Theorem 3.19. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. The following properties are equivalent:*

1. *G is a $*$-closed and π-open set,*

2. *G is $\mathcal{I}$-R closed and a π-open set,*

3. *G is a weakly $\mathcal{I}_{\pi g}$-closed and π-open set.*

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$: Obvious.
  $(3) \Rightarrow (1)$: It follows from Theorem 3.10.

Proposition 3.20. *Every pre**$_{\mathcal{I}}$-closed set is weakly $\mathcal{I}_{\pi g}$-closed but not conversely.*

Proof. Let H $\subseteq$ G and G is a π-open set in X. Since H is pre*$_{\mathcal{I}}$-closed, cl*(int(H)) $\subseteq$ H $\subseteq$ G. Hence H is weakly $\mathcal{I}_{\pi g}$-closed.

Example 3.21. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space such that $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then $\{b, c\}$ is weakly $\mathcal{I}_{\pi g}$-closed set but not pre**$_{\mathcal{I}}$-closed.*

4 Further Properties

Theorem 4.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The following properties are equivalent:*

1. *Each subset of $(X, \tau, \mathcal{I})$ is a weakly $\mathcal{I}_{\pi g}$-closed set,*

2. *G is pre**$_{\mathcal{I}}$-closed for each π-open set G in X.*

Proof. $(1) \Rightarrow (2)$: Suppose that each subset of $(X, \tau, \mathcal{I})$ is a weakly $\mathcal{I}_{\pi g}$-closed set. Let G be a π-open set in X. Since G is weakly $\mathcal{I}_{\pi g}$-closed, then we have cl*(int(G)) $\subseteq$ G. Thus, G is pre*$_{\mathcal{I}}$-closed.
  $(2) \Rightarrow (1)$: Let G be a subset of $(X, \tau, \mathcal{I})$ and H be a π-open set in X such that G $\subseteq$ H. By (2), we have cl*(int(G)) $\subseteq$ cl*(int(H)) $\subseteq$ H. Thus, G is a weakly $\mathcal{I}_{\pi g}$-closed set in $(X, \tau, \mathcal{I})$.

Theorem 4.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If G is a weakly $\mathcal{I}_{\pi g}$-closed set and $G \subseteq H \subseteq cl*(int(G))$, then H is a weakly $\mathcal{I}_{\pi g}$-closed set.*

Proof. Let H $\subseteq$ K and K be a π-open set in X. Since G $\subseteq$ K and G is a weakly $\mathcal{I}_{\pi g}$-closed set, then cl*(int(G)) $\subseteq$ K. Since H $\subseteq$ cl*(int(G)), then cl*(int(H)) $\subseteq$ cl*(int(G)) $\subseteq$ K. Thus, cl*(int(H)) $\subseteq$ K and hence, H is a weakly $\mathcal{I}_{\pi g}$-closed set.

Corollary 4.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If G is a weakly $\mathcal{I}_{\pi g}$-closed and open set, then cl*(G) is a weakly $\mathcal{I}_{\pi g}$-closed set.*

Proof. Let G be a weakly $\mathcal{I}_{\pi g}$-closed and open set in $(X, \tau, \mathcal{I})$. We have G $\subseteq$ cl*(G) $\subseteq$ cl*(G) = cl*(int(G)). Hence, by Theorem 4.2, cl*(G) is a weakly $\mathcal{I}_{\pi g}$-closed set in $(X, \tau, \mathcal{I})$.

Theorem 4.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. If G is a nowhere dense set, then G is a weakly $\mathcal{I}_{\pi g}$-closed set.*

Proof. Let G be a nowhere dense set in X. Since int(G) $\subseteq$ int(cl(G)), then int(G) = $\emptyset$. Hence, cl*(int(G)) = $\emptyset$. Thus, G is a weakly $\mathcal{I}_{\pi g}$-closed set in $(X, \tau, \mathcal{I})$.

Remark 4.5. *The reverse of Theorem 4.4 is not true in general as shown in the following example.*

Example 4.6. *Let X, τ and $\mathcal{I}$ be as in Example 3.2. Then $G = \{a\}$ is a weakly $\mathcal{I}_{\pi g}$-closed set but not a nowhere dense set.*

Remark 4.7. *The union of two weakly $\mathcal{I}_{\pi g}$-closed sets in an ideal topological space need not be a weakly $\mathcal{I}_{\pi g}$-closed set.*

Example 4.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space such that $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{b\}, \{c, d\}, \{b, c, d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Clearly $A = \{c\}$ and $B = \{d\}$ are weakly $\mathcal{I}_{\pi g}$-closed sets. But their union $A \cup B = \{c, d\}$ is not a weakly $\mathcal{I}_{\pi g}$-closed set.*

Lemma 4.9. *[12] Let G be an open subset of a topological space (X, τ).*
 If $K (\subseteq G)$ is π-open in (G, τ_G), then there exists a π-open set L in (X, τ) such that $K = L \cap G$.

Theorem 4.10. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $H \subseteq G \subseteq X$. If G is an open set in X and H is a weakly $\mathcal{I}_{\pi g}$-closed in G, then H is a weakly $\mathcal{I}_{\pi g}$-closed set in X.*

Proof. Let K be a π-open set in X and $H \subseteq K$. We have $H \subseteq K \cap G$. By Lemma 4.9, K $\cap$ G is a π-open set in G. Since H is a weakly $\mathcal{I}_{\pi g}$-closed set in G, then cl*$_G$(int$_G$(H)) $\subseteq K \cap G$. Also, we have cl*(int(H)) $\subseteq$ cl*$_G$(int(H)) = cl*$_G$(int$_G$(H)) $\subseteq K \cap G \subseteq K$. Hence, cl*(int(H)) $\subseteq$ K. Thus, H is a weakly $\mathcal{I}_{\pi g}$-closed in $(X, \tau, \mathcal{I})$.

Theorem 4.11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $H \subseteq G \subseteq X$. If G is an open set in X and H is a weakly $\mathcal{I}_{\pi g}$-closed set in X, then H is a weakly $\mathcal{I}_{\pi g}$-closed set in G.*

Proof. Let $H \subseteq K$ and K be a π-open set in G. By Lemma 4.9, there exists a π-open set L in X such that $K = L \cap G$. Since H is a weakly $\mathcal{I}_{\pi g}$-closed set in X, then cl*(int(H)) $\subseteq$ K. Also, we have cl*$_G$(int$_G$(H)) = cl*$_G$(int(H)) = cl*(int(H)) $\cap G \subseteq K \cap G = K$. Thus, cl*$_G$(int$_G$(H)) $\subseteq$ K. Hence, H is a weakly $\mathcal{I}_{\pi g}$-closed set in G.

Theorem 4.12. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. Then G is a weakly $\mathcal{I}_{\pi g}$-open set if and only if $H \subseteq int^*(cl(G))$ whenever $H \subseteq G$ and H is a π-closed set.*

Proof. Let H be a π-closed set in X and $H \subseteq G$. It follows that $X \backslash H$ is a π-open set and $X \backslash G \subseteq X \backslash H$. Since $X \backslash G$ is a weakly $\mathcal{I}_{\pi g}$-closed set, then cl*(int($X \backslash G$)) $\subseteq X \backslash H$. We have $X \backslash int^*(cl(G)) \subseteq X \backslash H$. Thus, $H \subseteq int^*(cl(G))$.
 Conversely, let K be a π-open set in X and $X \backslash G \subseteq K$. Since $X \backslash K$ is a π-closed set such that $X \backslash K \subseteq G$, then $X \backslash K \subseteq int^*(cl(G))$. We have $X \backslash int^*(cl(G)) = cl^*(int(X \backslash G)) \subseteq K$. Thus, $X \backslash G$ is a weakly $\mathcal{I}_{\pi g}$-closed set. Hence, G is a weakly $\mathcal{I}_{\pi g}$-open set in $(X, \tau, \mathcal{I})$.

Theorem 4.13. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. If G is a weakly $\mathcal{I}_{\pi g}$-closed set, then $cl^*(int(G)) \backslash G$ is a weakly $\mathcal{I}_{\pi g}$-open set in $(X, \tau, \mathcal{I})$.*

Proof. Let G be a weakly $\mathcal{I}_{\pi g}$-closed set in (X, τ, $\mathcal{I}$). Suppose that H is a π-closed set such that H $\subseteq$ cl*(int(G))\G. Since G is a weakly $\mathcal{I}_{\pi g}$-closed set, it follows from Theorem 3.12 that H = $\emptyset$. Thus, we have H $\subseteq$ int*(cl(cl*(int(G))\G)). It follows from Theorem 4.12 that cl*(int(G))\G is a weakly $\mathcal{I}_{\pi g}$-open set in (X, τ, $\mathcal{I}$).

Theorem 4.14. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. If G is a weakly $\mathcal{I}_{\pi g}$-open set, then H = X whenever H is a π-open set and int*(cl(G)) $\cup$ (X\G) $\subseteq$ H.*

Proof. Let H be a π-open set in X and int*(cl(G)) $\cup$ (X\G) $\subseteq$ H. We have X\H $\subseteq$ (X\int*(cl(G))) $\cap$ G = cl*(int(X\G))\(X\G). Since X\H is a π-closed set and X\G is a weakly $\mathcal{I}_{\pi g}$-closed set, it follows from Theorem 3.12 that X\H = $\emptyset$. Thus, we have H = X.

Theorem 4.15. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If G is a weakly $\mathcal{I}_{\pi g}$-open set and int*(cl(G)) $\subseteq$ H $\subseteq$ G, then H is a weakly $\mathcal{I}_{\pi g}$-open set.*

Proof. Let G be a weakly $\mathcal{I}_{\pi g}$-open set and int*(cl(G)) $\subseteq$ H $\subseteq$ G. Since int*(cl(G)) $\subseteq$ H $\subseteq$ G, then int*(cl(G)) = int*(cl(H)). Let K be a π-closed set and K $\subseteq$ H. We have K $\subseteq$ G. Since G is a weakly $\mathcal{I}_{\pi g}$-open set, it follows from Theorem 4.12 that K $\subseteq$ int*(cl(G)) = int*(cl(H)). Hence, by Theorem 4.12, H is a weakly $\mathcal{I}_{\pi g}$-open set in (X, τ, $\mathcal{I}$).

Corollary 4.16. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. If G is a weakly $\mathcal{I}_{\pi g}$-open and closed set, then int*(G) is a weakly $\mathcal{I}_{\pi g}$-open set.*

Proof. Let G be a weakly $\mathcal{I}_{\pi g}$-open and closed set in (X, τ, $\mathcal{I}$). Then int*(cl(G)) = int*(G) $\subseteq$ int*(G) $\subseteq$ G. Thus, by Theorem 4.15, int*(G) is a weakly $\mathcal{I}_{\pi g}$-open set in (X, τ, $\mathcal{I}$).

Definition 4.17. *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called E-set if A = M$\cup$N where M is π-closed and N is pre*$_{\mathcal{I}}$-open.*

Remark 4.18. *Every pre*$_{\mathcal{I}}$-open (resp. π-closed) set is E-set but not conversely.*

Example 4.19. *Let (X, τ, $\mathcal{I}$) be an ideal topological space such that X = {a, b, c, d}, τ = {$\emptyset$, {a}, {b}, {a, b}, {b, c}, {a, b, c}, X} and $\mathcal{I}$ = {$\emptyset$, {d}}. Then {d} is a E-set but not a pre*$_{\mathcal{I}}$-open set. Also {a} is a E-set but not a π-closed set.*

Theorem 4.20. *For a subset H of (X, τ, $\mathcal{I}$), the following are equivalent.*

1. *H is pre*$_{\mathcal{I}}$-open.*

2. *H is a E-set and weakly $\mathcal{I}_{\pi g}$-open.*

Proof. (1) $\Rightarrow$ (2): By Remark 4.18, H is a E-set and H is a weakly $\mathcal{I}_{\pi g}$-open by Proposition 3.20.

(2) $\Rightarrow$ (1): Let H be a E-set and weakly $\mathcal{I}_{\pi g}$-open. Then there exist a π-closed set M and a pre*$_{\mathcal{I}}$-open set N such that H =M$\cup$N. Since M $\subseteq$ H and H is weakly $\mathcal{I}_{\pi g}$-open, by Theorem 4.12, M $\subseteq$ int*(cl(H)). Also, we have N $\subseteq$ int*(cl(N)). Since N $\subseteq$ H, N $\subseteq$ int*(cl(N)) $\subseteq$ int*(cl(H)). Then H = M$\cup$N $\subseteq$ int*(cl(H)). So H is pre*$_{\mathcal{I}}$-open.

The following Example shows that the concepts of weakly $\mathcal{I}_{\pi g}$-open set and E-set are independent.

Example 4.21. *Let X, τ and $\mathcal{I}$ be as in Example 4.19. Then $\{d\}$ is a E-set but not a weakly $\mathcal{I}_{\pi\}}$-open set. Also $\{a, c\}$ is a weakly $\mathcal{I}_{\pi\}}$-open set but not a E-set.*

Definition 4.22. *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called F-set if A $= M \cup N$ where M is regular closed and N is pre*$_{\mathcal{I}}$-open.*

Remark 4.23. *Every pre*$_{\mathcal{I}}$-open (resp. regular closed) set is F-set but not conversely.*

Example 4.24. *Let X, τ and $\mathcal{I}$ be as in Example 4.19. Then $\{b, c, d\}$ is a F-set but not a pre*$_{\mathcal{I}}$-open set. Also $\{a, b\}$ is a F-set but not a regular closed set.*

Corollary 4.25. *For a subset H of $(X, \tau, \mathcal{I})$, the following are equivalent.*

1. *H is pre*$_{\mathcal{I}}$-open.*

2. *H is a F-set and weakly $\mathcal{I}_{rg}$-open.*

The following Example shows that the concepts of weakly $\mathcal{I}_{rg}$-open set and F-set are independent.

Example 4.26. *Let X, τ and $\mathcal{I}$ be as in Example 4.19. Then $\{a, d\}$ is a F-set but not a weakly $\mathcal{I}_{rg}$-open set. Also $\{c\}$ is a weakly $\mathcal{I}_{rg}$-open set but not a F-set.*

5 Quasi-pre*$_{\mathcal{I}}$-normal Spaces

Definition 5.1. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be quasi-pre*$_{\mathcal{I}}$-normal if for every pair of disjoint π-closed subsets A, B of X, there exist disjoint pre*$_{\mathcal{I}}$-open sets U, V of X such that $A \subseteq U$ and $B \subseteq V$.*

Theorem 5.2. *The following properties are equivalent for a space $(X, \tau, \mathcal{I})$.*

1. *X is quasi-pre*$_{\mathcal{I}}$-normal;*

2. *for any disjoint π-closed sets A and B, there exist disjoint weakly $\mathcal{I}_{\pi g}$-open sets U, V of X such that $A \subseteq U$ and $B \subseteq V$;*

3. *for any π-closed set A and any π-open set B containing A, there exists a weakly $\mathcal{I}_{\pi g}$-open set U such that $A \subseteq U \subseteq cl^*(int(U)) \subseteq B$.*

Proof. $(1) \Rightarrow (2)$: The proof is obvious.

$(2) \Rightarrow (3)$: Let A be any π-closed set of X and B any π-open set of X such that $A \subseteq$ B. Then A and X/B are disjoint π-closed sets of X. By (2), there exist disjoint weakly $\mathcal{I}_{\pi g}$-open sets U, V of X such that $A \subseteq U$ and $X/B \subseteq V$. Since V is weakly $\mathcal{I}_{\pi g}$-open set, by Theorem 4.12, $X/B \subseteq int^*(cl(V))$ and $U \cap int^*(cl(V)) = \emptyset$. Therefore we obtain $cl^*(int(U)) \subseteq cl^*(int(X/V))$ and hence $A \subseteq U \subseteq cl^*(int(U)) \subseteq$ B.

$(3) \Rightarrow (1)$: Let A and B be any disjoint π-closed sets of X. Then $A \subseteq X/B$ and X/B is π-open and hence there exists a weakly $\mathcal{I}_{\pi g}$-open set G of X such that $A \subseteq G \subseteq cl^*(int(G)) \subseteq X/B$. Put $U = int^*(cl(G))$ and $V = X|cl^*(int(G))$. Then U and V are disjoint pre*$_{\mathcal{I}}$-open sets of X such that $A \subseteq U$ and $B \subseteq V$. Therefore X is quasi-pre*$_{\mathcal{I}}$-normal.

Definition 5.3. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be weakly $\mathcal{I}_{\pi g}$-continuous if $f^{-1}(V)$ is weakly $\mathcal{I}_{\pi g}$-closed in X for every closed set V of Y.*

Definition 5.4. *A function f : (X, τ, $\mathcal{I}$) $\to$ (Y, σ, $\mathcal{J}$) is called weakly $\mathcal{I}_{\pi g}$-irresolute if $f^{-1}(V)$ is weakly $\mathcal{I}_{\pi g}$-closed in X for every weakly $\mathcal{J}_{\pi g}$-closed of Y.*

Definition 5.5. *A function f : (X, τ) $\to$ (Y, σ) is said to be m-π-closed [6] if f(V) is π-closed in Y for every π-closed set V of X.*

Theorem 5.6. *Let f : X $\to$ Y be a weakly $\mathcal{I}_{\pi g}$-continuous m-π-closed injection. If Y is quasi-normal, then X is quasi-pre*$_{\mathcal{I}}$-normal.*

Proof. Let A and B be disjoint π-closed sets of X. Since f is m-π-closed injection, f(A) and f(B) are disjoint π-closed sets of Y. By the quasi-normality of Y, there exist disjoint open sets U and V such that f(A) $\subseteq$ U and f(B) $\subseteq$ V. Since f is weakly $\mathcal{I}_{\pi g}$-continuous, then $f^{-1}(U)$ and $f^{-1}(V)$ are weakly $\mathcal{I}_{\pi g}$-open sets such that A $\subseteq$ $f^{-1}(U)$ and B $\subseteq$ $f^{-1}(V)$. Therefore X is quasi-pre*$_{\mathcal{I}}$-normal by Theorem 5.2.

Theorem 5.7. *Let f : X $\to$ Y be a weakly $\mathcal{I}_{\pi g}$-irresolute m-π-closed injection. If Y is quasi-pre*$_{\mathcal{I}}$-normal, then X is quasi-pre*$_{\mathcal{I}}$-normal.*

Proof. Let A and B be disjoint π-closed sets of X. Since f is m-π-closed injection, f(A) and f(B) are disjoint π-closed sets of Y. Since Y is quasi-pre*$_{\mathcal{I}}$-normal, by Theorem 5.2, there exist disjoint weakly $\mathcal{J}_{\pi g}$-open sets U and V such that f(A) $\subseteq$ U and f(B) $\subseteq$ V. Since f is weakly $\mathcal{I}_{\pi g}$-irresolute, then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint weakly $\mathcal{I}_{\pi g}$-open sets of X such that A $\subseteq$ $f^{-1}(U)$ and B $\subseteq$ $f^{-1}(V)$. Therefore X is quasi-pre*$_{\mathcal{I}}$-normal.

6   Conclusion

Topology as a field of mathematics is concerned with all questions directly or indirectly related to open/closed sets. Therefore, generalization of open/closed sets is one of the most important subjects in topology. Topology plays a significant role in quantum physics, high energy physics and superstring theory. In this paper, new type of generalization of closed sets called weakly $\mathcal{I}_{\pi g}$-closed sets was investigated, and some of their properties and characterizations in ideal topological spaces were obtained. Moreover, some notions of the sets and functions in topological spaces and ideal topolgical spaces are highly developed and used extensively in many practical and engineering problems.

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