

Contributions to the Theory of Soft Sets

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Abstract - Molodtsov introduced the concept of soft set as a new mathematical tool for dealing with problems containing uncertainties. In the literature, different kinds of operations of soft sets are defined and used in theory and applications. Some researchers showed that some of these definitions have a few gaps. In this paper, we make contributions to the theory of soft sets in which we first make some modifications to the operations of soft sets and fill in these gaps. Until now, to define soft sets and their properties, the authors have used different subsets of the parameters set for each soft set. Whereas, we use a single parameter set for all of soft sets in our approach. We then compare the definitions with those defined before.

1 Introduction

tersection, soft union, soft complement, soft products.

Keywords - Soft sets, soft in-

Problems with uncertainties are a major issue in many fields such as economics, engineering, environment and so on. One of the theories that deal with uncertainties is the soft set theory which is introduced by Molodtsov [4] as a new mathematical tool this.

In recent years, works on soft set theory and its applications have been progressing rapidly. In the literature, different kinds of operations of soft sets are defined and used in the works on soft set theory and its applications. Well known operations of soft sets and their properties are given by Maji et al. [3]. But some of these definitions and their properties have a few gaps, which have been pointed out by Ali *et al.* [1] and Yang [9]. To make some modifications to the operations of soft sets and fill in these gaps, Ali *et al.* [1], Cagman and Enginoglu [2], Neog and Sut [5], Pai and Miao [6], Sezgin and Atagun [7], Xia and Zuhua [8], and Zhu and Wen [10] make contributions.

A soft set is a mapping of parameters set into the set of all subsets of the universe. Until now, to define soft sets, the authors have used different subsets of the parameters set for each soft set. This has led to the emergence of some problems in the operations and properties of soft sets. The aim of this paper is to make contributions to the theory of soft sets, in which we first give a new approach to the soft sets by using a single parameter set for all of soft sets. This new approach makes the operations and their properties clear and free of the difficulties mentioned above. We then compare the definitions with other recently defined operations.

2 Definition of Soft Sets

Let U be an initial universe set and let E be a set of parameters. Then, to use an adequate parametrization, Molodtsov [4] gave the definition of soft sets as follows;

Definition 2.1. [4] A pair (F, E) is called a soft set (over U) if and only if F is a mapping of E into the set of all subsets of the set U.

Where F is called approximate function and E is called parameters set of the soft set (F, E). Each parameter $x \in E$ may be a fuzzy word, fuzzy sentence, real number and so on. For $x \in E$, the set F(x) is called x-approximation of the soft set (F, E)which may be arbitrary, some of them may be empty and some may have a nonempty intersection.

From now on, the soft set (F, E) will be shortly denoted by F_E . A soft set F_A over U can be written as a set of ordered pairs,

$$F_E = \{(x, F(x)) : x \in E\}$$

Elements of a soft set with empty approximation are normally not listed.

A soft set is a mapping of parameters set into the set of all subsets of the universe. Until now, to define the soft sets and their operations, the authors have used different subsets of the parameters set for each soft set. To illustrate, let us consider following convenient example, which is an expansion of Molodtsov's example given in [4].

Example 2.2. Assume that there are six houses in the universe $U = \{h_l, h_2, h_3, h_4, h_5, h_6\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$ is the set of parameters. Here, e_i (i = 1, 2, 3, 4, 5) stand for the parameters "expensive", "beautiful", "wooden", "cheap", and "in green surroundings" respectively. In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on. Then, following soft sets describe the attractiveness of the houses which a couple is going to buy.

Suppose that according to Mr. X's, $F(e_1) = \{h_2, h_3, h_4\}$, $F(e_2) = U$, $F(e_3) = U$, $F(e_4) = \emptyset$, and $F(e_5) = \{h_1, h_2, h_5, h_6\}$. If Mr. X considers a subset $A = \{e_1, e_4, e_5\}$ or $B = \{e_2, e_3, e_4, e_5\}$ of E, then soft set F_A or F_B can be written by

$$F_A = \{(e_1, \{h_2, h_3, h_4\}), (e_5, \{h_1, h_2, h_5, h_6\})\} \text{ or } F_B = \{(e_2, U), (e_3, U), (e_5, \{h_1, h_2, h_5, h_6\})\}$$

Suppose that according to Mrs. X's, $G(e_1) = \emptyset$, $G(e_2) = \{h_3, h_6\}$, $G(e_3) = U$, $G(e_4) = \emptyset$, and $G(e_5) = \{h_2, h_4, h_6\}$. If Mrs. X considers a subset $A = \{e_1, e_4, e_5\}$ or $B = \{e_2, e_3, e_4, e_5\}$ of E, then soft set G_A or G_B can be written by

$$\begin{array}{rcl} G_A & = & \{(e_5, \{h_2, h_4, h_6\})\} \ or \\ G_B & = & \{(e_2, \{h_3, h_6\}), (e_3, U), (e_5, \{h_2, h_4, h_6\})\} \end{array}$$

3 Operations of Soft Sets

In this section, we give contributions to the theory of soft sets in which we will use only a single parameter set for all soft sets. This new approach make the operations and their properties clear and free of the difficulties mentioned in the first section.

From now on, the single parameters set to be used is denoted by E to define the soft sets and their operations and therefore the subscript E can be deleted from the soft sets F_E , *i.e.*, a soft set F_E will be denoted shortly by F unless this will cause confusion. In general, soft sets and their approximate functions are denoted by F, G, H, ..., and soft sets and their approximate functions are used interchangeably.

Example 3.1. Let us consider Example 2.2. Then, two soft sets F and G describe the attractiveness of the houses which a couple is going to buy.

Mr. X's soft set F can be written by

$$F = \{(e_1, \{h_2, h_3, h_4\}), (e_2, U), (e_3, U), (e_5, \{h_1, h_2, h_5, h_6\})\}$$

Mrs. Y's soft set G can be written by

$$G = \{(e_2, \{h_3, h_6\}), (e_3, U), (e_5, \{h_2, h_4, h_6\})\}$$

Definition 3.2. A soft sets F over U is said to be a null soft set, denoted by Φ , if $F(x) = \emptyset$ for all $x \in E$.

Definition 3.3. A soft sets F over U is said to be absolute soft set, denoted by \mathbb{U} , if F(x) = U for all $x \in E$.

Remark 3.4. Definitions 3.2 and 3.3 were first given by Maji et al. ([3], Definition 2.7 and 2.8) which are inadequate according to the definition of soft sets ([3], Definition 2.1). For two different soft sets F_A and F_B over U have own absolute and null soft sets which may be different. It means that there are more than one absolute and null soft sets in the soft set theory where the definitions can be given with different subsets of parameters set.

For example, let us consider Example 2.2. Then, If $A = \{e_3\}$, then $F_A = \tilde{A}$ and $G_A = \tilde{A}$, where \tilde{A} is the absolute soft set. If $B = \{e_2, e_3\}$, then $F_B = \tilde{B}$ and $G_A \neq \tilde{B}$, where \tilde{B} is the absolute soft set. If $C = \{e_4\}$, then $F_C = \Phi_C$ and $G_C = \Phi_C$, where Φ_C is the null soft set. If $D = \{e_1, e_4\}$, then $F_D \neq \Phi_C$ and $G_D = \Phi_D$, where Φ_D is the null soft set. In this case, what can we say about $\tilde{A}^{\tilde{c}} = ?$, $\tilde{B}^{\tilde{c}} = ?$, $\Phi^{\tilde{c}}_C = ?$ and $\Phi^{\tilde{c}}_D = ?$.

Definition 3.5. Let $F, G \in S$. Then, F is a soft subset of G, denoted by $F \subseteq G$, if $F(x) \subseteq G(x)$ for all $x \in E$.

Proposition 3.6. *If* $F \in S$ *, then*

- 1. $\Phi \subseteq F$
- 2. $F \subseteq F$
- 3. $F \subseteq \mathbb{U}$

Definition 3.7. Let $F, G \in \mathbb{S}$. Then, F equal to G, denoted by F = G, if F(x) = G(x) for all $x \in E$.

Proposition 3.8. If $F, G, H \in \mathbb{S}$, then

- 1. $F \subseteq G, G \subseteq F \Leftrightarrow F = G$
- 2. $F \subseteq G, G \subseteq H \Rightarrow F \subseteq H$

Definition 3.9. Let $F, G \in S$. Then, soft union of F and G, denoted by $F \widetilde{\cup} G$, is a soft set over U whose approximate function is defined by

$$F \tilde{\cup} G : E \to P(U), \quad (F \tilde{\cup} G)(x) = F(x) \cup G(x)$$

Proposition 3.10. *If* $F \in S$ *, then*

- 1. $F \tilde{\cup} F = F$
- 2. $F\tilde{\cup}\Phi = F$
- 3. $F \tilde{\cup} \mathbb{U} = \mathbb{U}$

Definition 3.11. Let $F, G \in S$. Then, soft intersection of F and G, denoted by $F \cap G$, is a soft set over U whose approximate function is defined by

 $F \cap G : E \to P(U), \quad (F \cap G)(x) = F(x) \cap G(x)$

Proposition 3.12. *If* $F \in S$ *, then*

- 1. $F \cap F = F$
- 2. $F \tilde{\cap} \Phi = \Phi$
- 3. $F \cap \mathbb{U} = F$

Proposition 3.13. If $F, G \in \mathbb{S}$, then

- 1. $F \cap G = G \cap F$
- 2. $F \tilde{\cup} G = G \tilde{\cup} F$

Proposition 3.14. If $F, G, H \in \mathbb{S}$, then

- 1. $F \cap (G \cap H) = (F \cap G) \cap H$
- 2. $F\tilde{\cup}(G\tilde{\cup}H) = (F\tilde{\cup}G)\tilde{\cup}H$
- 3. $F \cap (G \cup H) = (F \cap G) \cup (F \cap H)$
- 4. $F\tilde{\cup}(G\tilde{\cap}H) = (F\tilde{\cup}G)\tilde{\cap}(F\tilde{\cup}H)$

Definition 3.15. Let $F, G \in S$. Then, soft difference of F and G, denoted by $F \setminus G$, is a soft set over U whose approximate function is defined by

$$F \setminus G : E \to P(U), \quad (F \setminus G)(x) = F(x) \setminus G(x)$$

Proposition 3.16. *If* $F \in S$ *, then*

- 1. $F \tilde{\setminus} F = \Phi$
- 2. $F \tilde{\langle} \Phi = F$
- 3. $\Phi \tilde{\setminus} F = \Phi$
- 4. $F \widetilde{\mathbb{U}} = \Phi$

Definition 3.17. Let $F \in S$. Then, soft complement of F, denoted by $F^{\tilde{c}}$, is a soft set over U whose approximate function is defined by

$$F^{\tilde{c}}: E \to P(U), \quad F^{\tilde{c}}(x) = U \setminus F(x)$$

Note that to keep clear of confusion, for the soft sets operations we use classical sets operations with tilde.

Proposition 3.18. *If* $F \in S$ *, then*

- 1. $(F^{\tilde{c}})^{\tilde{c}} = F$
- 2. $\Phi^{\tilde{c}} = \mathbb{U}$
- 3. $F \tilde{\cup} F^{\tilde{c}} = \mathbb{U}$
- 4. $F \tilde{\cap} F^{\tilde{c}} = \Phi$

Proposition 3.19. If $F, G \in \mathbb{S}$, then $F \cap G^{\tilde{c}} = F \setminus G$.

Proof: The proofs can be made by using the approximate functions. For all $x \in E$,

$$(F \cap G^c)(x) = F(x) \cap G^c(x)$$

$$= F(x) \cap (U \setminus G(x))$$

$$= F(x) \cap (U \cap (G(x))^c)$$

$$= F(x) \cap (G(x))^c$$

$$= F(x) \setminus G(x)$$

$$= (F \setminus G)(x)$$

Proposition 3.20. If $F, G \in S$, then the following De Morgan's types of results are true.

- 1. $(F\tilde{\cup}G)^{\tilde{c}} = F^{\tilde{c}}\tilde{\cap}G^{\tilde{c}}$
- 2. $(F \tilde{\cap} G)^{\tilde{c}} = F^{\tilde{c}} \tilde{\cup} G^{\tilde{c}}$

Proof: Let us prove first one, the other can be made similarly. For all $x \in E$,

$$(F \tilde{\cup} G)^{\tilde{c}}(x) = U \setminus (F \tilde{\cup} G)(x)$$

$$= [(F \tilde{\cup} G)(x)]^{c}$$

$$= [F(x) \cup G(x)]^{c}$$

$$= (F(x))^{c} \cap (G(x))^{c}$$

$$= (U \setminus F(x)) \cap (U \setminus G(x))$$

$$= F^{\tilde{c}}(x) \cap G^{\tilde{c}}(x)$$

$$= (F^{\tilde{c}} \cap G^{\tilde{c}})(x)$$

Example 3.21. Let us consider Example 3.1 where two soft sets F and G are given as;

$$F = \{(e_1, \{h_2, h_3, h_4\}), (e_2, U), (e_3, U), (e_5, \{h_1, h_2, h_5, h_6\})\}$$

$$G = \{(e_2, \{h_3, h_6\}), (e_3, U), (e_5, \{h_2, h_4, h_6\})\}$$

Then, we can find the following:

(

$$\begin{split} & F \cap G = \{(e_2, \{h_3, h_6\}), (e_3, U), (e_5, \{h_2, h_6\})\} \\ & F \cup G = \{(e_1, \{h_2, h_3, h_4\}), (e_2, U), (e_3, U), (e_5, \{h_1, h_2, h_4, h_5, h_6\})\} \\ & F \setminus G = \{(e_1, \{h_2, h_3, h_4\}), (e_2, \{h_1, h_2, h_4, h_5\}), (e_5, \{h_1, h_5\})\} \\ & G \setminus F = \{(e_5, \{h_4\})\} \\ & F^{\tilde{c}} = \{(e_1, \{h_1, h_5, h_6\}), (e_4, U\}), (e_5, \{h_3, h_4\})\} \\ & G^{\tilde{c}} = \{(e_1, U), (e_2, \{h_1, h_2, h_4, h_5\}), (e_4, U\}), (e_5, \{h_1, h_3, h_5\})\} \\ & F^{\tilde{c}} \cap G^{\tilde{c}} = \{(e_1, U), (e_2, \{h_1, h_2, h_4, h_5\}), (e_4, U\}), (e_5, \{h_1, h_3, h_4, h_5\})\} \\ & F^{\tilde{c}} \cup G^{\tilde{c}} = \{(e_1, U), (e_2, \{h_1, h_2, h_4, h_5\}), (e_4, U\}), (e_5, \{h_1, h_3, h_4, h_5\})\} \\ & F \cap G)^{\tilde{c}} = \{(e_1, U), (e_2, \{h_1, h_2, h_4, h_5\}), (e_4, U\}), (e_5, \{h_1, h_3, h_4, h_5\})\} \\ & F \cup G)^{\tilde{c}} = \{(e_1, \{h_1, h_5, h_6\}), (e_4, U\}), (e_5, \{h_3\})\} \end{split}$$

Until now, we defined the binary operations of soft sets which depend on an approximate function of one variable. Now, we define products of the soft sets which are binary operations of soft sets depending on an approximate function of two variables.

Definition 3.22. Let $F, G \in S$. Then, AND-product of two soft sets F and G, denoted by $F \wedge G$, is a binary operation of the soft sets over U whose approximate function is defined by

$$F \wedge G : E \times E \to P(U), \quad (F \wedge G)(x, y) = F(x) \cap G(y)$$

Definition 3.23. Let $F, G \in S$. Then, OR-product of two soft sets F and G, denoted by $F \vee G$, is a binary operation of the soft sets over U whose approximate function is defined by

$$F \lor G : E \times E \to P(U), \quad (F \lor G)(x, y) = F(x) \cup G(y)$$

Note that \wedge and \vee are not commutative.

Proposition 3.24. If $F, G, H \in S$, then

- 1. $F \lor (G \lor F_C) = (F \lor G) \lor H$
- 2. $F \wedge (G \wedge H) = (F \wedge G) \wedge H$

Proposition 3.25. If $F, G, H \in \mathbb{S}$, then the following De Morgan's types of results are true.

- 1. $(F \lor G)^{\tilde{c}} = F^{\tilde{c}} \land G^{\tilde{c}}$
- 2. $(F \wedge G)^{\tilde{c}} = F^{\tilde{c}} \vee G^{\tilde{c}}$

Proof: Let us prove first one, the other can be made similarly. For all $x, y \in E$,

4 Comparision of the Definitions

In this section, we compare our definitions with the definitions given by Cagman and Enginoglu [2] and Maji et al. [3].

In this paper,	In [2],	In [3],
Our approach	by Cagman and Enginoglu	by Maji et al.
$F = \{(x, F(x)) : x \in E\}$	$F_A = \{(x, F(x)) : x \in E\}$	$F_A = \{(x, F(x)) : x \in A\}$
where	where	where
${\cal E}$ is a universe of parameters	$A \subseteq E$	$A \subseteq E$
and	and	and
$F: E \to P(U)$	$F: \left\{ \begin{array}{l} A \to P(U) \\ A^c \to \{ \emptyset \} \end{array} \right.$	$F: A \to P(U)$

Let us first compare the definitions of soft sets in Table 1.

Table 1: Definition of the soft sets

We can compere the union of soft sets as in Table 2.

In this paper	In [2]	In [3]
$F\tilde{\cup}G = H$	$F_A \tilde{\cup} G_B = H_{A \cup B}$	$F_A \tilde{\cup} G_B = H_{A \cup B}$
$H: E \to P(U)$	$H: \left\{ \begin{array}{l} A \cup B \to P(U) \\ (A \cup B)^c \to \{\emptyset\} \end{array} \right.$	$H: A \cup B \to P(U)$
$H(x) = F(x) \cup G(x)$	$H(x) = F(x) \cup G(x)$	$H(x) = \begin{cases} F(x), & \text{if } x \in A - B, \\ G(x), & \text{if } x \in B - A, \\ F(x) \cup G(x), & \text{if } x \in A \cap B. \end{cases}$

Table 2: Union of the soft sets

We can compere the intersection of soft sets as in the in Table 3.

In this paper	In [2]	In [3]
$F \tilde{\cap} G = H$	$F_A \tilde{\cap} G_B = H_{A \cap B}$	$F_A \tilde{\cap} G_B = H_{A \cap B}$
$H: E \to P(U)$	$H: \left\{ \begin{array}{l} A\cap B \to P(U) \\ (A\cap B)^c \to \{\emptyset\} \end{array} \right.$	$H:A\cap B\to P(U)$
$H(x) = F(x) \cap G(x)$	$H(x) = F(x) \cap G(x)$	H(x) = F(x) or $G(x)$

Table 3: Intersection of the soft sets

We can compere the complement of soft sets as in Table 4.

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In this paper	In [2]	In [3]
$F^{\tilde{c}}$ is complement of F	$(F_A)^{\tilde{c}} = F_A^{\tilde{c}}$	$(F_A)^{\tilde{c}} = F^{\tilde{c}}_{\rceil A}$
$F^{\tilde{c}}: E \to P(U)$	$F^{\tilde{c}}: \left\{ \begin{array}{l} A \to P(U) \\ A^c \to \{\emptyset\} \end{array} \right.$	$F^{\tilde{c}}:]A \to P(U)$
$F^{\tilde{c}}(x) = U - F(x)$	$F^{\tilde{c}}(x) = U - F(x)$	$F^{\tilde{c}}(x) = U - F(\neg x)$

Table 4: Complement of the soft sets

Where $\neg x$ is not $x, \neg(\neg x) = x$ and $\rceil A = \{\neg x : x \in A\}$. We can compare the AND-product of soft sets as in the in Table 5.

In this paper	In [2]	In [3]
$F \wedge G = H$	$F_A \wedge G_B = H_{A \wedge B}$	$F_A \wedge G_B = H_{A \wedge B}$
$H: E \times E \to P(U)$	$H: \left\{ \begin{array}{l} A \times B \to P(U) \\ (A \times B)^c \to \{ \emptyset \} \end{array} \right.$	$H: A \times B \to P(U)$
$H(x,y) = F(x) \cap G(y)$	$H(x,y) = F(x) \cap G(y)$	$H(x,y) = F(x) \cap G(y)$

Table 5: AND-product of the soft sets

We can compere the OR-product of soft sets as in the in Table 6.

In this paper	In [2]	In [3]
$F \lor G = H$	$F_A \lor G_B = H_{A \lor B}$	$F_A \lor G_B = H_{A \lor B}$
$H: E \times E \to P(U)$	$H: \left\{ \begin{array}{l} A \times B \to P(U) \\ (A \times B)^c \to \{ \emptyset \} \end{array} \right.$	$H: A \times B \to P(U)$
$H(x,y) = F(x) \cup G(y)$	$H(x,y) = F(x) \cup G(y)$	$H(x,y) = F(x) \cup G(y)$

Table 6: OR-product of the soft sets

5 Conclusion

To deal with uncertainties, the soft set theory has been applied to many fields from theoretical to practical. In the theory and applications, authors have defined and used different kinds of operations of soft sets, some of which have a few gaps. Most of them have been pointed out by Ali *et al.* [1]. They also have made some modifications to the operations of soft sets to fill in these gaps under some conditions. In addition, they have given definition of extended intersection, restricted intersection, restricted union, restricted difference, relative null soft set, relative whole soft set, relative complement, and have given their properties under some restrictions.

In this paper, we make contributions to the theory of soft sets, in which we use only a single parameter set for all soft sets. This new approach makes operations and their properties clear and free of the difficulties without conditions. We then compare the definitions with other recently defined operations. We are confident that this work will be beneficial for future studies on soft set theory.

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