ISSN: 1304-7981 Number: 5, Year: 2014, Pages: 21-27. http://jnrs.gop.edu.tr

Gaziosmanpasa University
Graduate School of Natural and Applied Sciences
Journal of New Results in Science

Received: 10.02.2014
Accepted: 31.03.2014

Editors-in-Chief: *Naim Çağman*
Area Editor: *Oktay Muhtaroğlu*

On Topological θgs-Quotient Functions

Md.Hanif PAGE [1] (hanif01@yahoo.com)

Department of Mathematics, BVB College of Engineering and Technology,Hubli-580031.Karnataka, India

Abstract - The aim of this paper is to introduce θgs-Quotient function using θgs-closed sets and study their basic properties.

Keywords - *θgs-Quotient function, θgs-closed set.*

1 Introduction

In 1970, Levine [4] introduced the notion of generalized closed set. This notion has been studied extensively in recent years by many toplogists. The investigation of generalized closed sets had led to several new and interesting concepts. After the introduction of generalized closed sets there are many research papers which deal with different types of genaralized closed sets.Recently in [5] the notion of of θ-generalized semi closed (briefly,θgs-closed)set was introduced. The aim of this paper is to introduce θgs-quotient functions and using these new types of functions, several characterizations and its properties have been obtained. Also obtained the relationship between strong and weak form of θgs-quotient functions.

2 Preliminary

Throughout this paper (X, τ), (Y, σ)(or simply X, Y)denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X the closure and interior of A with respect to τ are denoted by $Cl(A)$ and $Int(A)$ respectively.

Definition 2.1. *A subset A of a space X is called*
(1) a semi-open set [3] if $A \subset Cl(Int(A))$.
(2) a semi-closed set [1] if $Int(Cl(Int(A))) \subset A$.

[1]Corresponding Author

Definition 2.2. [2] *A point $x \in X$ is called a semi-θ-cluster point of A if $sCl(U) \cap A \neq \phi$, for each semi-open set U containing x. The set of all semi-θ-cluster point of A is called semi-θ-closure of A and is denoted by $sCl_\theta(A)$. A subset A is called semi-θ-closed set if $sCl_\theta(A) = A$. The complement of semi-θ-closed set is semi-θ-open set.*

Definition 2.3. [5] *A subset A of X is θgeneralized semi-closed(briefly, θgs-closed)set if $sCl_\theta(A) \subset U$ whenever $A \subset U$ and U is open in X. The complement of θgs-closed set is θgeneralized-semi open (briefly,θgs-open).The family of all θgs-closed sets of X is denoted by $\theta GSC(X,\tau)$ and θgs-open sets by $\theta GSO(X,\tau)$.*

Definition 2.4. [8] *A topological space X is called $T_{\theta gs}$-space if every θgs-closed set in it is closed set.*

Definition 2.5. *A function $f: X \to Y$ is called:*
(i) θ-generalized semi-irresolute (briefly,θgs-irresolute)[6] *if $f^{-1}(F)$ is θgs-closed set in X for every θgs-closed set F of Y*
(ii) θ-generalized semi-continuous (briefly,θgs-continuous)[6] *if $f^{-1}(F)$ is θgs-closed set in X for every closed set F of Y.*
(iii) Strongly θ-generalized semi-continuous (briefly,strongly θgs-continuous)[10] *if $f^{-1}(F)$ is closed set of X for each θgs-closed set F of Y.*

Definition 2.6. [7] *A function $f: X \to Y$ is said to be θgs-open (resp., θgs-closed) if $f(V)$ is θgs-open (resp., θgs-closed) in Y for every open set (resp., closed) V in X.*

3 θgs-Quotient Functions

Definition 3.1. A surjective function $f : X \to Y$ is said to be θgs-quotient if f is θgs-continuous and $f^{-1}(V)$ is open in X implies V is θgs-open in Y.

Definition 3.2. A surjective function $f : X \to Y$ is said to be strongly θgs-open if $f(U)$ is θgs-open in Y for each θgs-open set U in X.

Theorem 3.3. If a function $f : X \to Y$ is surjective, θgs-continuous and θgs-open, then f is θgs-quotient funtion.

Proof: Since $f : X \to Y$ is θgs-continuous, it is enough to prove $f^{-1}(V)$ is open in X implies V is θgs-open in Y. Let $f^{-1}(V)$ is an open set in X. Since f is θgs-open, surjective implies $f(f^{-1}(V)) = V$ is a θgs-open in Y. Therefore f is θgs-quotient function.

Theorem 3.4. If $f : (X, \tau^{\theta gs}) \to (Y, \sigma^{\theta gs})$ be a θgs-quotient function, then the function $f : (X, \tau) \to (Y, \sigma)$ is θgs-quotient funtion.

Proof: Let V be any open set in (Y, σ) then V is a θgs-open set in (Y, σ) and $V \in \sigma^{\theta gs}$. Then $f^{-1}(V)$ is open in (X, τ). Because f is quotient function, that is $f^{-1}(V)$ is θgs-open set in (X, τ).Suppose $f^{-1}(V)$ is open in (X, τ), that is $f^{-1}(V) \in \tau^{\theta gs}$. Since f is quotient function, $V \in \sigma^{\theta gs}$ and V is a θgs-open set in Y, σ. This shows that $f : (X, \tau) \to (Y, \sigma)$ is θgs-quotient funtion.

4 Strong form of θgs-Quotient Functions

Definition 4.1. A surjective function $f : X \to Y$ is said to be strongly θgs-quotient if f is θgs-continuous and $f^{-1}(V)$ is θgs-open in X implies V is θgs-open in Y.

Theorem 4.2. Every strongly θgs-quotient function is θgs-quotient function.

Remark 4.3. The converse need not be true in general.

Example 4.4. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, c\}\}$ be topologies on X and Y respectively. We have $\theta GSO(X) = \{X, \phi, \{b, c\}, \{a\}\}$ and $\theta GSO(Y) = \{Y, \phi, \{b\}, \{c\}, \{b, c\}\}$. Define a function $f : X \to Y$ by $f(a) = c$,$f(b) = b$ and $f(c) = a$. Then f is θgs-quotient but not strongly θgs-quotient. Because for set {c} in Y, $f^{-1}(\{c\}) = \{a\}$ is θgs-open in X and {a} is not θgs-open in Y.

Theorem 4.5. Every strongly θgs-quotient function is θgs-open.

Proof: Let V be a open set in X.Since every open set is θgs-open set and hence V is θgs-open in X. That is $f^{-1}(f(V))$ is θgs-open in X. Since f is strongly θgs-quotient, $f(V)$ is open and hence θgs-open in Y. This shows that f is θgs-open.

Remark 4.6. The converse need not be true in general.

Example 4.7. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{c\}, \{b, c\}\}$ be topologies on X and Y respectively. We have $\theta GSO(X) = \{X, \phi, \{a, c\}, \{b\}\}$ and $\theta GSO(Y) = \{Y, \phi, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}\}$. Define a function $f : X \to Y$ by $f(a) = b$,$f(b) = c$ and $f(c) = a$. Then f is θgs-open but not strongly θgs-quotient. Because for set V={b} θgs-open in Y, $f^{-1}(\{b\}) = \{a\}$ is not θgs-open in X.

Theorem 4.8. Every strongly θgs-quotient function is strongly θgs-open.

Proof: Let $f : X \to Y$ be strongly θgs-quotient function. Let V be a θgs-open in X. That is $f^{-1}(f(V))$ is θgs-open in X. Since f is strongly θgs-quotient, $f(V)$ is open and hence θgs-open in Y. This shows that f is strongly θgs-open.

Remark 4.9. The converse need not be true in general.

Example 4.10. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, c\}\}$ be topologies on X and Y respectively. We have $\theta GSO(X) = \{X, \phi, \{b, c\}, \{a\}\}$ and $\theta GSO(Y) = \{Y, \phi, \{b\}, \{c\}, \{b, c\}\}$. Define a function $f : X \to Y$ by $f(a) = c$,$f(b) = b$ and $f(c) = a$. Then f is strongly θgs-open but not strongly θgs-quotient. Because for set {c} in Y, $f^{-1}(\{c\}) = \{a\}$ is θgs-open in X and {a} is not θgs-open in Y.

Definition 4.11. A surjective function $f : X \to Y$ is said to be completely θgs-quotient if f is θgs-irresolute and $f^{-1}(V)$ is θgs-open set in X implies V is an open set in Y.

Theorem 4.12. Every completely θgs-quotient function is a strongly θgs-quotient function.

Proof: Let V be an open set in Y then it is θgs-open in Y. Since f is θgs-irresloute, $f^{-1}(f(V))$ is θgs-open set in X. Since f is completely θgs-quotient function, V is an open set in Y. Hence f is strongly θgs-open function.

Remark 4.13. The converse need not be true in general.

Example 4.14. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{c\}$ $\{a, b\}, \{a, c\}\}$ be topologies on X and Y respectively. We have $\theta GSO(X) = \{X, \phi, \{a\}, \{b, c\}\}$ and $\theta GSO(Y) = \{Y, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Define a function $f : X \to Y$ by $f(a) = a, f(b) = b,\ f(c) = c$. Then f is strongly θgs-open but not completely θgs-quotient.Because for an open set $V = \{c\}$ in Y, $f^{-1}(\{c\}) = \{c\}$ is not θgs-open in X.

Theorem 4.15. Every completely θgs-quotient function is a θgs-irresolute.

Proof: Follows from the definition

Remark 4.16. The converse need not be true in general.

Example 4.17. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, c\}\}$ be topologies on X and Y respectively. we have $\theta GSO(X) = \{X, \phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$ and $\theta GSO(Y) = \{Y, \phi, \{b\}, \{c\}, \{b, c\}\}$. Define a function $f : X \to Y$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$. Then f is θgs-irresolute but not completely θgs-quotient function. Because for an open set $\{a\}$ in Y, $f^{-1}(\{a\}) = \{a\}$ is not θgs-open set in X.

Theorem 4.18. Every completely θgs-quotient function is a strongly θgs-open function.

Proof: Let $f : X \to Y$ be completely θgs-quotient function. Let V be a θgs-open in X. That is $f^{-1}(f(V))$ is θgs-open in X. Since f is completely θgs-quotient, this implies that $f(V)$ is open in Y and thus θgs-open in Y. Hence f is completely θgs-open.

Remark 4.19. The converse need not be true in general.

Example 4.20. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{c\}, \{b, c\}\}$ be topologies on X and Y respectively. We have $\theta GSO(X) = \{X, \phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$ and $\theta GSO(Y) = \{Y, \phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b\}\}$. Define a function $f : X \to Y$ by $f(a) = a$, $f(b) = b$, $f(c) = c$. Then f is strongly θgs-open but not completely θgs-quotient. Since f is not θgs-irresolute i.e. for θgs-open {a,c} in Y, $f^{-1}(\{a, c\}) = \{a, c\}$ is not θgs-open in X.

Theorem 4.21. Let X, Y both are $T_{\theta gs}$-spaces and $f : X \to Y$ be surjective function. Then following are equivalent
(i) f is a completely θgs-quotient function.
(ii) f is a strongly θgs-quotient function.
(iii) f is a θgs-quotient function.

Proof: (i)⇒(ii) Suppose (i) holds.Clearly f is θgs-continuous, because every θgs-irresolute function is θgs-continuous. Let $f^{-1}(V)$ is θgs-open, by (i) V is open set. Since every open set is θgs-open, implies V is θgs-open. Therefore(ii) holds.
(ii)⇒(iii) Suppose (ii) holds. Therefore f is θgs-continuous.Let $f^{-1}(V)$ is open, and hence it is $f^{-1}(V)$ is θgs-open. By (ii) V is θgs-open set. Therefore (iii) holds.
(iii)⇒ (i) Suppose (iii) holds. Let V be a θgs-open set in Y and Y is $T_{\theta gs}$-space, implies V is an open set in Y. Since f is θgs-continuous, implies $f^{-1}(V)$ is θgs-open in X. This implies f is θgs-irresolute. Suppose $f^{-1}(V)$ is θgs-open in X. Since X is $T_{\theta gs}$-space, $f^{-1}(V)$ is open in X. By (iii) V is θgs-open in Y. Since Y is a $T_{\theta gs}$-space, V is open in Y. Hence (i) hold.

5 Applications

Theorem 5.1. If $f : X \to Y$ is an open surjective, θgs-irresolute and $g : Y \to Z$ is a θgs-quotient function, then $g \circ f : X \to Z$ is θgs-quotient function.

Proof: Let U be an open set in Z.Since g is a θgs-quotient function, implies $g^{-1}(U)$ is a θgs-open in Y. Also, since f is θgs-irresolute, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is θgs-open in X. Therefore $g \circ f$ is θgs-continuous.Assume $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in X for some subset U in Z. Since f is an open and surjective, implies $f(f^{-1}(g^{-1}(U)) = g^{-1}(U)$ is open in Y and since g is a θgs-quotient function, implies U is a θgs-open set in Y. This shows that, $g \circ f$ is θgs-quotient function.

Theorem 5.2. If $h : X \to Y$ is a θgs-quotient function and $g : Y \to Z$ is a continuous function that is constant on each set $h^{-1}(y)$,for $y \in Y$, then g induces a θgs-continuous function $f : Y \to Z$ such that $f \circ h$=g .

Proof: Since g is constant on $h^{-1}(y)$,for $y \in Y$, the set g($h^{-1}(y)$) is a one point set in Z. If f(y) denote this point, then it clear that f is θgs-continuous . For if we let V be any open set in Z, then $g^{-1}(V)$ is an open set as g is continuous . But $g^{-1}(V)$=$h^{-1}(f^{-1}(V))$ is open is X. Since h is θgs-quotient function, f^{-1}(V) is a θgs-open set. Hence f is θgs-continuous function.

Theorem 5.3. If $f : X \to Y$ be strongly θgs-open surjective and θgs-irresolute function and $g : Y \to Z$ be strongly θgs-quotient function, then $g \circ f : X \to Z$ is strongly θgs-quotient function.

Proof: Let V be an open set in Z. Since g is strongly θgs-quotient function,then $g^{-1}(V)$ is a θgs-open set. Also, since f is θgs-irresolute, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(U)$ is θgs-open set in X. Assume $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is θgs-open in X for some subset V in Z. Then $f^{-1}(g^{-1}(V))$ is θgs-open set in X. Since f is strongly θgs-open, f($f^{-1}(g^{-1}(V))$) is θgs-open set in Y. It follows that $g^{-1}(V)$ is θgs-open set in Y. This gives that V is an open set in Y.Thus $g \circ f$ is strongly θgs-quotient function.

Theorem 5.4. If $f : X \to Y$ be a θgs-quotient function where X and Y are $T_{\theta gs}$-spaces. Then $g : Y \to Z$ be strongly θgs-continuous function if and only if the composite function $g \circ f : X \to Z$ is strongly θgs-continuous function.

Proof: Let g be a stongly θgs-continuous function and U be any θgs-open set in Z. Then $g^{-1}(U)$ is a open set in Y. Then $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is θgs-open in X. Since X is $T_{\theta gs}$-space, then $f^{-1}(g^{-1}(U))$ is open set in X. Thus composite function is strongly θgs-continuous.
Conversely, let the composite function $g \circ f$ be strongly θgs-continuous. Then for any θgs-open set U in Z, $f^{-1}(g^{-1}(U))$ is open in X. Since f is θgs-quotient function, it implies that $g^{-1}(U)$ is θgs-open set in Y. Since Y is $T_{\theta gs}$-space, $g^{-1}(U)$ is open in Y. Hence g is strongly θgs-continuous function.

Theorem 5.5. If $f : X \to Y$ be a surjective strongly θgs-open and θgs-irresolute function and $g : Y \to Z$ be completely θgs-quotient then $g \circ f : X \to Z$ is completely θgs-quotient function.

Proof: Let V be θgs-open set in Z. Then $g^{-1}(V)$ is a θgs-open set in Y because g is completely θgs-quotient function. Since f is θgs-irresolute $f^{-1}(g^{-1}(V))$ is θgs-open set in X. Then $g \circ f$ is θgs-irresolute. Suppose $(g \circ f)^{-1}(V)$ is θgs-open set in X for $V \subseteq Z$. That is $f^{-1}(g^{-1}(V))$ is open set in X. Since f is strongly θgs-open, f($f^{-1}(g^{-1}(V))$) is θgs-open set in Y. Thus $f^{-1}(V)$ is θgs-open set in Y. Since g is a completely θgs-quotient function, V is an open set in Z. Hence $g \circ f$ is completly θgs-quotient function.

Theorem 5.6. If $f : X \to Y$ be a strongly θgs-quotient function and $g : Y \to Z$ be completely θgs-quotient function then $g \circ f : X \to Z$ is completely θgs-quotient function.

Proof: Let V be θgs-open set in Z. Then $g^{-1}(V)$ is a θgs-open set in Y. Since g is completely θgs-quotient function $f^{-1}(g^{-1}(V))$ is also a θgs-open set in X. Since f is strongly θgs-quotient function,$(g \circ f)^{-1}(V)$ is θgs-open set in X. Hence $g \circ f$ is θgs-irresolute. Let $(g \circ f)^{-1}(V)$ be a θgs-open set in X for $V \subseteq Z$ that is, $f^{-1}(g^{-1}(V))$ is θgs-open set in X. Then $g^{-1}(V)$ is θgs-open set in Y. Since g is a completely θgs-quotient function, V is an open set in Z. Hence $g \circ f$ is completly θgs-quotient function.

Theorem 5.7. The composition of two completely θgs-quotient functions is is completely θgs-quotient function.

Proof: Let $f : X \to Y$ and $g : Y \to Z$ be two completely θgs-quotient functions. Let V be θgs-open set in Z. Since g is completely θgs-quotient function,$g^{-1}(V)$ is a θgs-open set in Y. Since f is completely θgs-quotient function, $f^{-1}(g^{-1}(V))$ is θgs-open set in X. That is $(g \circ f)^{-1}(V)$ is θgs-open set in X. Hence $g \circ f$ is θgs-irresolute. Let $(g \circ f)^{-1}(V)$ be a θgs-open set in X.Then $f^{-1}(g^{-1}(V))$ is θgs-open set in X. Since f is completely θgs-quotient function, $g^{-1}(V)$ is open set in Y.Then $g^{-1}(V)$ is open set in Y. Since g is a completely θgs-quotient function, V is an open set in Z. Hence $g \circ f$ is completly θgs-quotient function.

References

[1] S. G. Crossley and S. K. Hildebrand, *Semi-Topological properties*, *Fund. Math.*, **74**,(1972), 233-254.

[2] G.Di. Maio, T. Noiri, *On s-closed spaces*, *Indian. J. Pure and Appl.Math.*, **18**,(1987), 226-233.

[3] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, *mer. Math.Monthly*, **70**,(1963), 36-41

[4] N. Levine, *Generalized closed sets in topology*, *Rend. Circlo. Mat. Palemro.*, **19(2)**,(1970), 89-96

[5] Govindappa Navavalgi and Md. Hanif Page, *On θgs-Neighbourhoods*, *Indian Journal of Mathematics and Mathematical Sciences*, Vol.4, No.1,(June-2008), 21-31.

[6] Govindappa Navavalgi and Md. Hanif Page, *On θgs-continuity and θgs-Irresoluteness*,*Internatonal Journal of Mathematics and Computer Sciences and Technology*, Vol.1, No.1,(Jan-June-2008), 95-101.

[7] Govindappa Navavalgi and Md. Hanif Page, *On θgs-Open and θgs-Closed functions*,*Proycecciones Journal of Mathematics*, Vol.28,(April-2009), 111-123.

[8] Govindappa Navavalgi and Md. Hanif Page, *On Some separation axioms via θgs-open sets*,*Bulletin of Allahabad Mathematical Society*, Vol.25,Part 1,(2010), 13-22.

[9] Md. Hanif Page, *On Some more properties of θgs-Neighbourhodds*,*American Journal of Applied Mathematics and Mathematical Analysis*, Vol.1, No.2,(2012).

[10] Md. Hanif Page, *On strongly θgs-continuous functions*,*International Journal of Advances in Management, Technology and Engineering Sciences*, Vol.II, Issue 3, Dec-2012, 55-58.