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The Generalized Hyers-Ulam-Rassias Stability of a Quadratic Functional Equation in Fuzzy Banach Spaces

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Abstract - In this paper, our target is to generalize the stability theorem of generalized Hyers-Ulam-Rassias Stability of the quadratic functional equation $f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) +, 4f(x) - 2f(y)$ in fuzzy Banach spaces.

Keywords - *Fuzzy norm, Hyers-Ulam stability, quadratic functional equation, fuzzy Banach space.*

1 Introduction

The idea of studying stability problem of functional equations started with a well-known problem posed by Ulam [13] in 1940 concerning the stability of group homomorphisms. Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $H : G_1 \longrightarrow G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$. In the next

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year Hyers [7] gave a partial acceptable answer to the above said fact. He showed that if $\delta > 0$ and $f : E \longrightarrow E_1$ with E and E_1 Banach spaces, such that $\| f(x + y) - f(x) - f(y) \| \leqslant \delta$ for all $x, y \in E$ then there exists a unique $g : E \longrightarrow E_1$ such that $g(x + y) = g(x) + g(y)$ and $\| f(x) - g(x) \| \leqslant \delta$ for all $x, y \in E$. Aoki discussed the Hyers-Ulam stability theorem in [16]. Rassias much worked on this matter and further generalized the result of Aoki in 1978. He formulated and proved the generalized theorem in [21], which implies Hyers theorem as a special case. The notion of existence of unique additive mapping was initiated by Aoki [16] and recently Maligranda corrected it on allowing that the mapping f satisfy some continuity assumption.

The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1}$$

and therefore the equation (1) is called the quadratic functional equation. Skof [8] proved the Hyers-Ulam stability theorem for (1) for the function $f : E \rightarrow E_1$ where E is a normed space and E_1 is a Banach space. In fact, Cholewa [12], S. Czerwik [15] proved the Hyers-Ulam stability theorem for (1) replacing E_1 by an Abelian group. This result was further generalized by Rassias [22], C. Borelli and Forti [5]. Later on, in the paper [9], the authors further generalized this result for the new quadratic functional equation (1).

Ever since the concept of fuzzy sets was introduced by Zadeh [10] in 1965 to describe the situation in which data are imprecise or vague or uncertain. It has a wide range of application in the field of population dynamics, chaos control, computer programming, medicine, etc. Kramosil and Michalek [11] introduced the concept of fuzzy metric spaces in 1975, which opened an avenue for further development of analysis in such spaces. The idea of fuzzy norm was initiated by Katsaras in [2]. Some mathematician have defined fuzzy norm on vector space from various point of view. Following Cheng and Moderson [14], Bag and Samanta [17] gave an idea of fuzzy norm on a linear space in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [11] and then it is generalized by Samanta et.al. [18]. But according to the discussion of George and Veeramani [1], it can be proved that an ordinary normed linear space is a special case of fuzzy normed linear space. Thereafter many authors [3, 6, 19, 20] tried to generalize the stability theory of several functional equations in fuzzy Banach spaces.

In this paper, our target is to generalize the stability theorem of generalized Hyers-Ulam-Rassias Stability of the quadratic functional equation (1) in Fuzzy Banach Spaces.

2 Preliminary

We quote some definitions and notations which will be needed in the sequel.

Definition 2.1. *[4] A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is continuous t - norm if $*$ satisfies the following conditions :*

(i) $*$ *is commutative and associative;*
(ii) $*$ *is continuous;*
(iii) $a * 1 = a \quad \forall\ a \in [0, 1]$;
(iv) $a * b \leq c * d$ *whenever* $a \leq c$, $b \leq d$ *and* $a, b, c, d \in [0, 1]$.

Through out this article, we further assume that $a * a = a$ for all $a \in [0, 1]$.

Definition 2.2. *[3] The 3-tuple* $(X, N, *)$ *is called a* ***fuzzy normed linear space*** *if* X *is a real linear space,* $*$ *is a continuous* t *- norm and* N *is a fuzzy set in* $X \times (0, \infty)$ *satisfying the following conditions:*
(i) $N(x, t) > 0$;
(ii) $N(x, t) = 1$ *if and only if* $x = 0$;
(iii) $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;
(iv) $N(x, s) * N(y, t) \leqslant N(x + y, s + t)$;
(v) $N(x, \cdot) : (0, \infty) \to (0, 1]$ *is continuous;*
for all $x, y \in X$ *and* $t, s > 0$.
Note that $N(x, t)$ *can be thought of as the degree of nearness between* x *and null vector* 0 *with respect to* t.

Example 2.3. *Let* $X = [0, \infty)$, $a * b = ab$ *for every* $a, b \in [0, 1]$ *and* $\|\cdot\|$ *be a norm defined on* X. *Define* $N(x, t) = e^{-\frac{\|x\|}{t}}$ *for all* $x \in X$. *Then clearly* $(X, N, *)$ *is a fuzzy normed linear space.*

Example 2.4. *Let* $(X, \|\cdot\|)$ *be a normed linear space, and let* $a * b = ab$ *or* $a * b = \min\{a, b\}$ *for all* $a, b \in [0, 1]$. *Let* $N(x, t) = \frac{t}{t + \|x\|}$ *for all* $x \in X$ *and* $t > 0$. *Then* $(X, N, *)$ *is a fuzzy normed linear space and this fuzzy norm* N *induced by* $\|\cdot\|$ *is called the standard fuzzy norm.*

Note 2.5. *According to George and Veeramani [1], it can be proved that every fuzzy normed linear space is a metrizable topological space. In fact, also it can be proved that if* $(X, \|\cdot\|)$ *is a normed linear space, then the topology generated by* $\|\cdot\|$ *coincides with the topology generated by the fuzzy norm* N *of example (2.4). As a result, we can say that an ordinary normed linear space is a special case of fuzzy normed linear space.*

Remark 2.6. *In fuzzy normed linear space* $(X, N, *)$, *for all* $x \in X$, $N(x, \cdot)$ *is non- decreasing with respect to the variable* t.

Definition 2.7. *[18] Let* $(X, N, *)$ *be a fuzzy normed linear space. A sequence* $\{x_n\}$ *in* X *is said to be convergent or converge if there exists an* $x \in X$ *such that* $\lim\limits_{n\to\infty} N(x_n - x, t) = 1$. *In this case,* x *is called the limit of the sequence* $\{x_n\}$ *and we denote it by* $N - \lim\limits_{n \to \infty} x_n = x$.

Definition 2.8. *[18] Let* $(X, N, *)$ *be a fuzzy normed linear space. A sequence* $\{x_n\}$ *in* X *is called Cauchy sequence if for each* $\varepsilon > 0$ *and* $t > 0$ *there exists an* $n_0 \in \mathbb{N}$ *such that for all* $n \geqslant n_0$ *and all* $p > 0$, *we have* $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

3 The Generalized Hyers-Ulam-Rassias Stability of the Functional Equation

In this section, let X be a real vector space and (Y, N) be a fuzzy Banach space.

Theorem 3.1. *Let $\psi : X^2 \longrightarrow \mathbb{R}^+$ be a function such that*

$$\sum_{i=0}^{\infty} \frac{\psi(2^i x, 0)}{4^i} \quad converges, \qquad \frac{1}{8}\sum_{i=0}^{\infty} \frac{\psi(2^i x, 0)}{4^i} = \widetilde{\psi}(x, 0)$$

$$and \quad \lim_{n\to\infty} \frac{\psi(2^n x, 2^n y)}{4^n} = 0 \tag{2}$$

for all $x, y \in X$. Let $f : X \to Y$ be a function such that

$$\lim_{t\to\infty} N(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 4f(x) + 2f(y), t\psi(x, y)) = 1 \tag{3}$$

uniformly on $X \times X$. Then

$$Q(x) := N - \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$$

exists for each $x \in X$ and defines a quadratic function $Q : X \to Y$ such that if for some $\delta > 0, \alpha > 0$

$$N(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 4f(x) + 2f(y), \delta\psi(x, y)) \geqslant \alpha \tag{4}$$

for all $x, y \in X$, then

$$N\Big(f(x) - Q(x), \delta\widetilde{\psi}(x, 0)\Big) \geqslant \alpha \tag{5}$$

for all $x \in X$. Furthermore, the function $Q : X \to Y$ is a unique function such that

$$\lim_{t\to\infty} N\Big(f(x) - Q(x), t\widetilde{\psi}(x, 0)\Big) = 1 \tag{6}$$

uniformly on X.

Proof. For a given $\varepsilon > 0$, by (3), there exists some $t_0 > 0$ such that

$$N(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 4f(x) + 2f(y), t\psi(x, y)) \geq 1 - \varepsilon \tag{7}$$

for all $t \geqslant t_0$. Putting $x = y = 0$ in (1) we get $f(0) = 0$. Putting $y = 0$ in (7), we get

$$N(2f(2x) - 8f(x), t\psi(x, 0)) \geqslant 1 - \varepsilon$$

$$\Rightarrow N\left(f(x) - \frac{f(2x)}{4}, \frac{t}{8}\psi(x, 0)\right) \geqslant 1 - \varepsilon \tag{8}$$

for all $x \in X$ and for all $t \geqslant t_0$. By induction on positive integer n, we show that

$$N\left(f(x) - \frac{f(2^n x)}{4^n}, \frac{t}{8}\sum_{i=0}^{n-1}\frac{\psi(2^i x, 0)}{4^i}\right) \geqslant 1 - \varepsilon \tag{9}$$

for all $x \in X$ and for all $t \geqslant t_0$.
From (8), we see that (9) is true for $n = 1$. Let us assume that (9) hold for $n = k$, where $k \in \mathbb{N}$. Then we have

$$N\left(f(x) - \frac{f(2^k x)}{4^k}, \frac{t}{8}\sum_{i=0}^{k-1}\frac{\psi(2^i x, 0)}{4^i}\right) \geqslant 1 - \varepsilon \tag{10}$$

for all $x \in X$ and for all $t \geqslant t_0$. Now,

$$\begin{aligned}
& N\left(f(x) - \frac{f(2^{k+1} x)}{4^{k+1}}, \frac{t}{8}\sum_{i=0}^{k}\frac{\psi(2^i x, 0)}{4^i}\right) \\
&= N\left(f(x) - \frac{f(2^k x)}{4^k} + \frac{f(2^k x)}{4^k} - \frac{f(2^{k+1} x)}{4^{k+1}}, \right. \\
&\qquad\qquad \left. \frac{t}{8}\sum_{i=0}^{k-1}\frac{\psi(2^i x, 0)}{4^i} + \frac{t}{8}\frac{\psi(2^k x, 0)}{4^k}\right) \\
&\geqslant N\left(f(x) - \frac{f(2^k x)}{4^k}, \frac{t}{8}\sum_{i=0}^{k-1}\frac{\psi(2^i x, 0)}{4^i}\right) * \\
&\qquad\qquad N\left(\frac{f(2^k x)}{4^k} - \frac{f(2^{k+1} x)}{4^{k+1}}, \frac{t}{8}\frac{\psi(2^k x, 0)}{4^k}\right) \\
&= N\left(f(x) - \frac{f(2^k x)}{4^k}, \frac{t}{8}\sum_{i=0}^{k-1}\frac{\psi(2^i x, 0)}{4^i}\right) * \\
&\qquad\qquad N\left(f(2^k x) - \frac{f(2 \cdot 2^k x)}{4}, \frac{t}{8}\psi(2^k x, 0)\right) \\
&\geqslant (1 - \varepsilon) * (1 - \varepsilon) = 1 - \varepsilon \quad by \quad (10).
\end{aligned}$$

This completes the proof of (9). Letting $t = t_0$ and replacing n and x by p and $2^n x$ in (9) respectively, we get

$$N\left(f(2^n x) - \frac{f(2^{n+p}x)}{4^p}, \frac{t_0}{8}\sum_{i=0}^{p-1}\frac{\psi(2^{n+i}x, 0)}{4^i}\right) \geqslant 1 - \varepsilon \tag{11}$$

for all $n \geqslant 0$ and for all $p > 0$. Again, we see that

$$\sum_{i=0}^{p-1}\frac{\psi(2^{n+i}x, 0)}{4^i} = \sum_{i=n}^{n+p-1}\frac{\psi(2^i x, 0)}{4^{i-n}}. \tag{12}$$

Since $\sum_{i=0}^{\infty}\frac{\psi(2^i x, 0)}{4^i}$ converges, for a given $\delta > 0$, there exits $n_0 \in \mathbb{N}$ such that

$$\frac{t_0}{8}\sum_{i=n}^{n+p-1}\frac{\psi(2^i x, 0)}{4^i} < \delta \tag{13}$$

for all $n \geqslant n_0$ and for all $p > 0$. Now,

$$\begin{aligned}
&N\left(\frac{f(2^n x)}{4^n} - \frac{f(2^{n+p}x)}{4^{n+p}}, \delta\right) \\
&\geqslant N\left(\frac{f(2^n x)}{4^n} - \frac{f(2^{n+p}x)}{4^{n+p}}, \frac{t_0}{8}\sum_{i=n}^{n+p-1}\frac{\psi(2^i x, 0)}{4^i}\right) \quad \text{by (13)} \\
&= N\left(f(2^n x) - \frac{f(2^{n+p}x)}{4^p}, \frac{t_0}{8}\sum_{i=n}^{n+p-1}\frac{\psi(2^i x, 0)}{4^{i-n}}\right) \\
&= N\left(f(2^n x) - \frac{f(2^{n+p}x)}{4^p}, \frac{t_0}{8}\sum_{i=0}^{p-1}\frac{\psi(2^{n+i}x, 0)}{4^i}\right) \quad \text{by (12)} \\
&\geqslant 1-\varepsilon \quad \text{by (11)}
\end{aligned}$$

for all $n \geqslant n_0$ and for all $p > 0$. Hence the sequence $\left\{\frac{f(2^n x)}{4^n}\right\}$ is Cauchy in Y. Since Y is a fuzzy Banach space, therefore the sequence $\left\{\frac{f(2^n x)}{4^n}\right\}$ converges to some $Q(x) \in Y$. So we can define a function $Q : X \to Y$ by $Q(x) := N - \lim_{n\to\infty}\frac{f(2^n x)}{4^n}$, namely for each $t > 0$ and $x \in X$,

$$\lim_{n\to\infty} N\left(\frac{f(2^n x)}{4^n} - Q(x), t\right) = 1.$$

Now we show that Q satisfies (1). Let $x, y \in X$. Since $\lim_{n\to\infty}\frac{\psi(2^n x, 2^n y)}{4^n} = 0$, therefore for fixed $t > 0$ and $0 < \varepsilon < 1$ there exists $n_1 \geqslant n_0$ such that

$$t_0\,\frac{\psi(2^n x, 2^n y)}{4^n} < \frac{t}{7} \tag{14}$$

for all $n \geqslant n_1$. Now,

$$N\,(\,Q\,(2x+y)\,+\,Q\,(2x-y)\,-\,2Q\,(x+y)\,-\,2Q\,(x-y)\,-\,4Q\,(x)\,+\,2Q\,(y)\;,\;t\,)$$

$$\geqslant\; N\left(Q\,(2\,x+y)\,-\,\frac{f\,(2^n\,(2\,x+y))}{4^n}\;,\;\frac{t}{7}\right)\,*\,N\left(Q\,(2\,x-y)\,-\,\frac{f\,(2^n\,(2\,x-y))}{4^n}\;,\;\frac{t}{7}\right)\,*$$
$$N\left(\frac{2f\,(2^n\,(x+y))}{4^n}\,-\,2\,Q\,(x+y)\;,\;\frac{t}{7}\right)\,*\,N\left(\frac{2f\,(2^n\,(x-y))}{4^n}\,-\,2\,Q\,(x-y)\;,\;\frac{t}{7}\right)\,*$$
$$N\left(\frac{4f\,(2^n\,x)}{4^n}\,-\,4\,Q\,(x)\;,\;\frac{t}{7}\right)\,*\,N\left(2\,Q\,(y)\,-\,\frac{2f\,(2^n\,y)}{4^n}\;,\;\frac{t}{7}\right)\,*\,N\left(\frac{f\,(2^n\,(2\,x+y))}{4^n}\,+\right.$$
$$\left.\frac{f\,(2^n\,(2\,x-y))}{4^n}\,-\,\frac{2f\,(2^n\,(\,x+y))}{4^n}\,-\,\frac{2f\,(2^n\,(\,x-y))}{4^n}\,-\,\frac{4f\,(2^n\,x)}{4^n}\,+\,\frac{2f\,(2^n\,y)}{4^n}\;,\;\frac{t}{7}\right). \tag{15}$$

Replacing $x\,,\,y$ by $2^n\,x\,,\;2^n\,y$ respectively in (7) and for $t\,=\,t_0$, we get

$$N\left(\frac{f\,(2^n\,(2x+y))}{4^n}\,+\,\frac{f\,(2^n\,(2x-y))}{4^n}\,-\,\frac{2f\,(2^n\,(x+y))}{4^n}\,-\,\frac{2f\,(2^n\,(x-y))}{4^n}\right.$$
$$\left.-\,\frac{4f\,(2^n x)}{4^n}\,+\,\frac{2f\,(2^n\,y)}{4^n}\;,\;t_0\,\frac{\psi\,(2^n x\;,\;2^n y)}{4^n}\right)\,\geqslant\,1\,-\,\varepsilon\,. \tag{16}$$

The first six terms on the RHS of (15) tend to 1 as $n\,\to\,\infty$ and the last term $\geqslant\,1-\varepsilon$ by (14) and (16). Thus,

$$N\,(\,Q\,(2x+y)\,+\,Q\,(2x-y)\,-\,2Q\,(x+y)\,-\,2Q\,(x-y)\,-\,4Q\,(x)\,+\,2Q\,(y)\;,\;t\,)$$

$\geqslant\,1-\varepsilon$ for all $t\,>\,0$,

$\Rightarrow\;N\,(\,Q\,(2x+y)\,+\,Q\,(2x-y)\,-\,2\,Q\,(x+y)\,-\,2\,Q\,(x-y)\,-\,4\,Q\,(x)\,+\,2\,Q\,(y)\;,\;t\,)$

$=\,1$ for all $t\,>\,0$,

$\Rightarrow\;Q\,(2x+y)+Q\,(2x-y)-2\,Q\,(x+y)-2\,Q\,(x-y)-4\,Q\,(x)+2\,Q\,(y)$

$=\,0$ for all $x\,,\,y\,\in\,X$.

Hence Q satisfies (1), i.e., Q is quadratic. Let (4) hold for some $\delta\,>\,0\;,\;\alpha\,>\,0$. By the same reasoning as the beginning of the proof, we can deduce from (4) that

$$N\left(f\,(x)\,-\,\frac{f\,(2^n\,x\,)}{4^n}\;,\;\frac{\delta}{8}\sum_{i\,=\,0}^{n\,-\,1}\frac{\psi\,(2^i\,x\,,\,0\,)}{4^i}\right)\,\geqslant\,\alpha \tag{17}$$

for all $n \in \mathbb{N}$. Let $t\,>\,0$. We have

$$N\left(f\,(x)\,-\,Q\,(x)\;,\;\frac{\delta}{8}\,\sum_{i\,=\,0}^{n\,-\,1}\frac{\psi\,(2^i\,x\,,\,0\,)}{4^i}\,+\,t\right)$$

$$\geqslant N\left(f(x) - \frac{f(2^n x)}{4^n}\,,\ \frac{\delta}{8}\sum_{i=0}^{n-1}\frac{\psi(2^i x\,,\,0)}{4^i}\right) * N\left(\frac{f(2^n x)}{4^n} - Q(x)\ ,\ t\right).$$

Taking limit as $n \to \infty$, we get

$$N\left(f(x) - Q(x)\ ,\ \frac{\delta}{8}\sum_{i=0}^{\infty}\frac{\psi(2^i x\,,\,0)}{4^i} + t\right) \geqslant \alpha * 1 = \alpha \quad \text{by} \quad (17).$$

For the continuity of $N(x\,,\cdot)$ and considering $t \to 0$, we get

$$N\left(f(x) - Q(x)\ ,\ \delta\,\widetilde{\psi}(x\,,\,0)\right) \geqslant \alpha \quad \text{for all} \quad x \in X.$$

To prove the uniqueness let us assume Q' be another function satisfying (1) and (6). For a given $\varepsilon > 0$ and fixed $c > 0$, by (6) for Q and Q' we can find some $t_0 > 0$ such that

$$\begin{aligned} N\left(f(x) - Q(x)\ ,\ t\,\widetilde{\psi}(x\,,0)\right) &\geqslant 1 - \varepsilon \\ N\left(f(x) - Q'(x)\ ,\ t\,\widetilde{\psi}(x\,,0)\right) &\geqslant 1 - \varepsilon \end{aligned} \tag{18}$$

for all $x \in X$ and $t \geqslant t_0$.
Since $\sum_{i=0}^{\infty}\frac{\psi(2^i x\,,0)}{4^i}$ converges, for some fixed $x \in X$, there exists some $n_0 \in \mathbb{N}$ such that

$$\frac{t_0}{8}\sum_{i=n}^{\infty}\frac{\psi(2^i x\,,\,0)}{4^i} < \frac{c}{2} \tag{19}$$

for all $n \geqslant n_0$. Now,

$$\sum_{i=n}^{\infty}\frac{\psi(2^i x\,,\,0)}{4^i} = \frac{1}{4^n}\sum_{i=0}^{\infty}\frac{\psi(2^{n+i} x\,,\,0)}{4^i}$$

$$\Rightarrow \quad \frac{t_0}{8\cdot 4^n}\sum_{i=0}^{\infty}\frac{\psi(2^{n+i} x\,,\,0)}{4^i} < \frac{c}{2} \tag{20}$$

for all $n \geqslant n_0$.
From (1) we get $f(2x) = 4f(x)$ for $y = 0$, by induction, it implies that

$$f(2^n x) = 4^n f(x). \tag{21}$$

Since Q and Q' satisfy (1), it follows from (21) that

$$Q(2^n x) = 4^n Q(x) \quad \text{and} \quad Q'(2^n x) = 4^n Q'(x).$$

Now, $N\left(Q(x) - Q'(x)\ ,\ c\right)$

$$\geqslant N\left(Q(x) - \frac{f(2^n x)}{4^n}\,,\ \frac{c}{2}\right) * N\left(\frac{f(2^n x)}{4^n} - Q'(x)\,,\ \frac{c}{2}\right)$$
$$= N\left(f(2^n x) - 4^n Q(x)\,,\ 4^n \frac{c}{2}\right) * N\left(f(2^n x) - 4^n Q'(x)\,,\ 4^n \frac{c}{2}\right)$$
$$= N\left(f(2^n x) - Q(2^n x)\,,\ 4^n \frac{c}{2}\right) * N\left(f(2^n x) - Q'(2^n x)\,,\ 4^n \frac{c}{2}\right)$$
$$\geqslant N\left(f(2^n x) - Q(2^n x)\,,\ \frac{t_0}{8} \sum_{i=0}^{\infty} \frac{\psi(2^{n+i} x, 0)}{4^i}\right) *$$
$$N\left(f(2^n x) - Q'(2^n x)\,,\ \frac{t_0}{8} \sum_{i=0}^{\infty} \frac{\psi(2^{n+i} x, 0)}{4^i}\right)$$
$$\geqslant (1-\varepsilon) * (1-\varepsilon) = 1-\varepsilon \text{ by } (18).$$

It follows that $N(Q(x) - Q'(x)\,,\ c) = 1$ for all $c > 0$. Thus $Q(x) = Q'(x)$ for all $x \in X$. This completes the proof of the theorem. $\square$

Corollary 3.2. *Let* $\theta > 0$ *and* p *be a real number with* $0 < p < 2$. *Let* $f : X \to Y$ *be a function such that*

$$\lim_{t\to\infty} N(f(2x+y) + f(2x-y) - 2f(x+y) -$$
$$2f(x-y) - 4f(x) + 2f(y)\,,\ t\,\theta\,(\|x\|^{p} + \|y\|^{p})) = 1$$

uniformly on $X \times X$. *Then*

$$Q(x) := N - \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$$

exists for each $x \in X$ *and defines a quadratic function* $Q : X \to Y$ *such that if for some* $\delta > 0$, $\alpha > 0$

$$N(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) -$$
$$4f(x) + 2f(y)\,,\ \delta\,\theta\,(\|x\|^{p} + \|y\|^{p})) \geqslant \alpha$$

for all $x, y \in X$, *then*

$$N\left(f(x) - Q(x)\,,\ \frac{\delta\,\theta}{2(4-2^p)} \cdot \|x\|^{p}\right) \geqslant \alpha$$

for all $x \in X$. *Furthermore, the function* $Q : X \to Y$ *is a unique function such that*

$$\lim_{t\to\infty} N\left(f(x) - Q(x)\,,\ \frac{t\,\theta}{2(4-2^p)} \cdot \|x\|^{p}\right) = 1$$

uniformly on X.

Proof.

Define $\psi(x, y) = \theta(\|x\|^{p} + \|y\|^{p})$.

$$\widetilde{\psi}(x, 0) = \frac{1}{8}\sum_{i=0}^{\infty}\frac{\psi(2^{i}x, 0)}{4^{i}} = \frac{1}{8}\theta\|x\|^{p}\sum_{i=0}^{\infty}\left(\frac{2^{p}}{4}\right)^{i} = \frac{1}{8}\theta\|x\|^{p}\frac{1}{1-\frac{2^{p}}{4}}$$

$$= \frac{\theta}{2(4-2^{p})}\|x\|^{p}.$$

□

Corollary 3.3. *Let* $\theta > 0$ *and* $f : X \to Y$ *be a function such that*

$$\lim_{t\to\infty} N(f(2x+y)+f(2x-y) - 2f(x+y) - 2f(x-y) - 4f(x) + 2f(y), t\theta) = 1$$

uniformly on $X \times X$. *Then*

$$Q(x) := N - \lim_{n\to\infty}\frac{f(2^{n}x)}{4^{n}}$$

exists for each $x \in X$ *and defines a quadratic function* $Q : X \to Y$ *such that if for some* $\delta > 0$, $\alpha > 0$

$$N(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 4f(x) + 2f(y), \delta\theta) \geqslant \alpha$$

for all $x, y \in X$, *then*

$$N\left(f(x) - Q(x), \frac{\delta\theta}{6}\right) \geqslant \alpha \quad \text{for all } x \in X.$$

Furthermore, the function $Q : X \to Y$ *is a unique function such that*

$$\lim_{t\to\infty} N\left(f(x) - Q(x), \frac{t\theta}{6}\right) = 1$$

uniformly on X.

Proof.

Define $\psi(x, y) = \theta$.

$$\widetilde{\psi}(x, 0) = \frac{1}{8}\sum_{i=0}^{\infty}\frac{\psi(2^{i}x, 0)}{4^{i}} = \frac{\theta}{8}\sum_{i=0}^{\infty}\left(\frac{1}{4}\right)^{i} = \frac{\theta}{6}.$$

□

Theorem 3.4. *Let* $\psi : X^2 \to \mathbb{R}^+$ *be a function such that*

$$\sum_{i=0}^{\infty} 4^i \psi\left(\frac{x}{2^i}, 0\right) \text{ converges and } \quad \frac{1}{8} \sum_{i=0}^{\infty} 4^i \psi\left(\frac{x}{2^i}, 0\right) = \widetilde{\psi}(x, 0),$$

$$\text{and } \lim_{n \to \infty} 4^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \quad \text{for all } x, y \in X.$$

Let $f : X \to Y$ *be a function such that*

$$\lim_{t\to\infty} N(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 4f(x) + 2f(y), t\psi(x, y)) = 1$$

uniformly on $X \times X$. *Then*

$$Q(x) := N - \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$$

exists for each $x \in X$ *and defines a quadratic function* $Q : X \to Y$ *such that if for some* $\delta > 0$, $\alpha > 0$

$$N(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 4f(x) + 2f(y), \delta\psi(x, y)) \geqslant \alpha$$

for all $x, y \in X$, *then*

$$N\Big(f(x) - Q(x), \delta \widetilde{\psi}(x, 0)\Big) \geqslant \alpha$$

for all $x \in X$. *Furthermore, the function* $Q : X \to Y$ *is a unique function such that*

$$\lim_{t\to\infty} N\Big(f(x) - Q(x), t\widetilde{\psi}(x, 0)\Big) = 1 \quad \text{uniformly on } X.$$

4 Conclusion

In this paper, we have made an effort to to generalize the stability theorem of generalized Hyers-Ulam-Rassias Stability of the quadratic functional equation (1) in Fuzzy Banach Spaces. It can be further generalized in fuzzy soft Banach spaces. A lot of works regarding Hyers-Ulam-Rassias Stability of several functional equations have been done in Banach Spaces and Fuzzy Banach Spaces. But almost no work has been done in cone Banach spaces and fuzzy cone Banach spaces. So, we think the Theorem (3.1) can be tried to prove cone Banach spaces and fuzzy cone Banach spaces.

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