ISSN: **1304-7981** Number: **6**, Year: **2014**, Pages: **1-14** **http://jnrs.gop.edu.tr**

Gaziosmanpasa University
Graduate School of Natural and Applied Sciences
Journal of New Results in Science

Received: 14.04.2014 *Editors-in-Chief*: *Naim Çağman*
Accepted: 12.05.2014 *Area Editor*: *Oktay Muhtaroğlu*

On Some New Generalized Difference Sequence Spaces and Their Topological Properties

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Abstract - In this study, we define a new triangle matrix $\widehat{G} = \{g_{nk}^{u,v}(r,s,t)\}$ which derived by using multiplication of weighted mean matrix $G = (g_{nk})$ with triple band matrix $B(r,s,t)$. Also, we examine some topological properties of this new sequence spaces such as Schauder basis, isomorphism and $\alpha-, \beta-\gamma-$ duals . Finally, we characterize the classes $(\mu_1^{u,v}(\widehat{G}) : \mu_2)$ of infinite matrices , where $\mu_1 \in \{c, c_0, \ell_p\}$ and $\mu_2 \in \{\ell_\infty, c, c_0, \ell_p\}$.

Keywords - *Matrix domain of a sequence space, Matrix transformations , Schauder basis, $\alpha-, \beta-$ and $\gamma-$ duals.*

1 Introduction

Let ω be the space of complex sequences. By a sequence space, we understand a linear subspace of the space ω. We write ℓ_∞, c, c_0 and ℓ_p for the classical sequence spaces of all bounded, convergent, null and absolutely p-summable sequences, respectively, where $1 \leq p < \infty$. Also by bs and cs, we denote the spaces of all bounded and convergent series, respectively. We assume throughout unless stated otherwise that $p, q > 1$ with $p^{-1} + q^{-1} = 1$ and use the convention that any term with negative subscript is equal to zero. We denote throughout that the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} where $n, k \in \mathbb{N}$. Then, A defines a matrix mapping from X to Y and is denote by $A : X \to Y$ if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}_{n\in\mathbb{N}}$, the A-transform of x, is in Y where

$$(Ax)_n = \sum_k a_{nk}x_k, \quad (n \in \mathbb{N}) \tag{1}$$

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By $(X:Y)$, denote the class of all matrices A such that $A:X\to Y$. Thus, $A\in(X:Y)$ if and only if the series on the right hand side of (1) converges for each $n\in\mathbb{N}$ and $x\in X$, and we have $Ax=\{(Ax)_n\}_{n\in\mathbb{N}}\in Y$ for all $x\in X$. A sequence $x\in\omega$ is said to be A-summable to l if Ax converges to l, which is called the A-limit of x.

A matrix $A=(a_{nk})$ is called a triangle if $a_{nk}=0$ for $k>n$ and $a_{nn}\neq 0$ for all $n,k\in\mathbb{N}$. It is trivial that $A(Bx)=(AB)x$ holds for the triangle matrices A,B and a sequence x. Further, a triangle matrix U has a unique inverse $U^{-1}=V$ which is also triangle matrix. Then, $x=U(Vx)=V(Ux)$ holds for all $x\in\omega$.

Let $q=(q_k)$ be a sequence of positive reals and write

$$Q_n=\sum_{k=0}^{n}q_k,\quad (n\in\mathbb{N}).$$

Then the Cesàro mean of order one, Riesz mean with respect to the sequence $q=(q_k)$, which are triangle limitation matrices, are respectively defined by the matrices $C=(c_{nk})$ and $R^q=(r^q_{nk})$; where

$$c_{nk}=\begin{cases}\dfrac{1}{n+1}, & (0\le k\le n),\\ 0, & (k>n),\end{cases}\qquad r^q_{nk}=\begin{cases}\dfrac{q_k}{Q_n}, & (0\le k\le n),\\ 0, & (k>n),\end{cases}$$

for all $k,n\in\mathbb{N}$. Also, we define the summation matrix $S=(s_{nk})$, the difference matrix $\Delta=(\Delta^{(1)}_{nk})$, $A^r_u=\{a^r_{nk}(u)\}$ and $\Delta^{(m)}=(\Delta^{(m)}_{nk})$ by

$$s_{nk}=\begin{cases}1, & (0\le k\le n),\\ 0, & (k>n),\end{cases}\qquad \Delta^{(1)}_{nk}=\begin{cases}(-1)^{n-k}, & (n-1\le k\le n),\\ 0, & (0\le k<n-1 \text{ or } k>n),\end{cases}$$

$$a^r_{nk}(u)=\begin{cases}\dfrac{1+r^k}{n+1}u_k, & (0\le k\le n),\\ 0, & (k>n),\end{cases}$$

and

$$\Delta^{(m)}_{nk}=\begin{cases}(-1)^{n-k}\dbinom{m}{n-k}, & (\max\{0,n-m\}\le k\le n)\\ 0, & (0\le k<n-1 \text{ or } k>n)\end{cases}$$

for all $k,n\in\mathbb{N}$.

For a sequence space X, the matrix domain X_A of an infinite matrix A is defined by

$$X_A=\{x=(x_k)\in\omega : Ax\in X\},\tag{2}$$

which is a sequence space. If A is triangle, then one can easily observe that the sequence space X_A and X are linearly isomorphic, i.e., $X_A\cong X$.

By $\mathcal{U}$, we denote for the set of all sequences $u=(u_k)$ such that $u_k\neq 0$ for all $k\in\mathbb{N}$. For $u\in\mathcal{U}$, let $1/u=(1/u_k)$. Let $u,v\in\mathcal{U}$ and define the matrix $G(u,v)=(g_{nk})$ by

$$g_{nk}=\begin{cases}u_nv_k, & (0\le k\le n),\\ 0, & (k>n),\end{cases}$$

for all $k,n\in\mathbb{N}$, where u_n and v_k depend only on n and k, respectively. The $G(u,v)$ matrix is called as generalized weighted mean or factorable matrix.

Let r, s and t be non-zero real numbers, and define the generalized difference matrix $\widehat{B} = B(r,s,t) = (b_{nk})$ by

$$b_{nk} = \begin{cases} r, & (k = n) \\ s, & (k = n-1) \\ t, & (k = n-2) \\ 0, & (0 \leq k < n-1 \text{ or } k > n) \end{cases} \tag{3}$$

for all $n, k \in \mathbb{N}$.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by Wang [1], Ng and Lee [2], Malkowsky [3], Altay and Başar [5], Malkowsky and Savaş [9], Başarır [10], Aydın and Başar [11], Başar et al. [12], Şengönül and Başar [13], Altay [14], Polat and Başar [15], and Malkowsky et al. [16]. Δ, Δ^2 and Δ^m are the transposes of the matrices $\Delta^{(1)}, \Delta^{(2)}$ and $\Delta^{(m)}$, respectively, and $c_0(u,p)$ are the spaces consisting of the sequences $x = (x_k)$ such that $ux = (u_k x_k)$ in the spaces $c_0(p)$ and $c(p)$ for $u \in \mathcal{U}$, respectively, studied by Başarır [10]. Also, the generalized difference matrix $B(r,s,t) = (b_{nk})$ has been used by Sönmez [42] for defining the some new sequence spaces. Finally, the new technique for deducing certain topological properties, for example $AB-, KB-, AD-$ properties, etc., and determining the $\beta-$ and $\gamma-$ duals of the domain of a triangle matrix in a sequence space has been given by Altay and Başar [20].

The main purpose of the present paper is to introduce the sequence space $\mu^{u,v}(\widehat{G})$ and to determine the $\alpha-, \beta-$ and $\gamma-$ duals of this space, where μ denotes the any of the classical spaces ℓ_∞, c, c_0 or ℓ_p, and $\widehat{G} = G(u,v)\widehat{B}$. Furthermore, the Schauder bases for the spaces $c_0^{u,v}(\widehat{G}), c^{u,v}(\widehat{G})$ and $\ell_p^{u,v}(\widehat{G})$ are given, and some topological properties of the spaces $c_0^{u,v}(\widehat{G}), c^{u,v}(\widehat{G}), \ell_\infty^{u,v}(\widehat{G})$ and $\ell_p^{u,v}(\widehat{G})$ are examined. Finally, some classes of matrix mappings on the space $\mu^{u,v}(\widehat{G})$ are characterized.

2 The Difference Sequence Spaces $\mu^{u,v}(\widehat{G})$ of Non-Absolute Type for $\mu \in \{c_0, c, \ell_\infty, \ell_p\}$

In the present section, we introduce the spaces $c_0^{u,v}(\widehat{G}), c^{u,v}(\widehat{G}), \ell_\infty^{u,v}(\widehat{G})$ and $\ell_p^{u,v}(\widehat{G})$ derived by the generalized weighted mean $G(u,v)$ and generalized difference matrix $B(r,s,t)$ and show that these spaces are $BK-$ spaces of non-absolute type which are norm isomorphic to the spaces c_0, c, ℓ_∞ and ℓ_p, respectively. Furthermore, we give the bases of the spaces $c_0^{u,v}(\widehat{G}), c^{u,v}(\widehat{G})$ and $\ell_p^{u,v}(\widehat{G})$.

Recently, using the generalized weighted mean $G(u,v)$, some new sequence spaces have been defined by several authors. For example, Malkowsky and Savaş[9], Başar and Altay [6], Polat, Karakaya and Şimşek [7] and Başarır and Kara [37].

Following [9, 6, 7, 37], we define the sequences spaces $\mu^{u,v}(\widehat{G})$ for $\mu \in \{c_0, c, \ell_\infty, \ell_p\}$ by

$$\mu^{u,v}(\widehat{G}) = \{x = (x_k) \in \omega : y = ((\widehat{G}x)_k) \in \mu\}$$

where the sequence $y = (y_k)$ is the $\widehat{G} = G(u,v)\widehat{B}-$transform of a sequence $x = (x_k)$, that is,

$$y_k = (\widehat{G}x)_k = u_k\left(\sum_{i=0}^{k} v_i(rx_i + sx_{i-1} + tx_{i-2})\right) \tag{4}$$

for all $k \in \mathbb{N}$. It is natural that the spaces $\mu^{u,v}(\widehat{G})$ may also be defined with the notation of (2) that

$$\mu^{u,v}(\widehat{G}) = \mu_{\widehat{G}}. \tag{5}$$

On the other hands we define the triangle matrix $\widehat{G} = G\widehat{B} = (\widehat{g}_{nk})$ by

$$\widehat{g}_{nk} = \begin{cases} u_n v_k r + u_n v_{k+1} s + u_n v_{k+2} t, & (k < n-1) \\ u_n v_{n-1} r + u_n v_n s, & (k = n-1) \\ u_n v_n r, & (k = n) \\ 0, & (\text{otherwise}) \end{cases} \tag{6}$$

for all $k, n \in \mathbb{N}$. Also, it can be easily seen that, which will be frequently used, the $\widehat{G}-$transform of a sequence $x = (x_k)$ is

$$y_0 = ru_0v_0x_0, \quad y_1 = u_1(rv_0 + sv_1)x_0 + ru_1v_1x_1 \quad \text{and}$$

$$y_k = u_k \sum_{i=0}^{k-2}(rv_i + sv_{i+1} + tv_{i+2})x_i + u_k(rv_{k-1} + sv_k)x_{k-1} + u_k v_k r x_k \quad \text{for} \quad k > 1. \tag{7}$$

The definition in (5) includes the following special cases:
(i) If $v = (\lambda_k - \lambda_{k-1})$ and $u = (1/\lambda_n)$ then $\mu^{u,v}(\widehat{G}) = \mu^{\lambda}(\widehat{B})$ (cf[18]).
(ii) If $v = (\lambda_k - \lambda_{k-1})$, $u = (1/\lambda_n)$, $r = 1$, $s = 1$ and $t = 0$ then $c^{u,v}(\widehat{G}) = c^{\lambda}(B)$ and $c_0^{u,v}(\widehat{G}) = c_0^{\lambda}(B)$ (cf[32]).
(iii) If $v = (\lambda_k - \lambda_{k-1})$, $u = (1/\lambda_n)$, $r = 1$, $s = -1$ and $t = 0$ then $c^{u,v}(\widehat{G}) = c^{\lambda}(\Delta)$ and $c_0^{u,v}(\widehat{G}) = c_0^{\lambda}(\Delta)$ (cf[7]).

Since the proof may also be obtained in the similar way as for the other spaces, to avoid the repetition of the similar statements, we give the proof only for one of those spaces. Now, we may begin with the following theorem which is essential in the study.

Theorem 2.1. *(i) The difference sequence spaces $c_0^{u,v}(\widehat{G}), c^{u,v}(\widehat{G})$ and $\ell_\infty^{u,v}(\widehat{G})$ are $BK-$ spaces with the norm $\|x\|_{c_0^{u,v}(\widehat{G})} = \|x\|_{c^{u,v}(\widehat{G})} = \|x\|_{\ell_\infty^{u,v}(\widehat{G})} = \|\widehat{G}(x)\|_\infty$, that is ,*

$$\|x\|_{c_0^{u,v}(\widehat{G})} = \|x\|_{c^{u,v}(\widehat{G})} = \|x\|_{\ell_\infty^{u,v}(\widehat{G})} = \sup_{n\in\mathbb{N}} \left|\widehat{G}_n(x)\right|.$$

(ii) Let $1 \leq p < \infty$. Then $\ell_p^{u,v}(\widehat{G})$ is a $BK-$ space with the norm $\|x\|_{\ell_p^{u,v}(\widehat{G})} = \|\widehat{G}x\|_p$, that is,

$$\|x\|_{\ell_p^{u,v}(\widehat{G})} = \left(\sum_n |\widehat{G}_n(x)|^p\right)^{1/p}.$$

Proof: Since (5) holds and c_0, c and ℓ_∞ are $BK-$ spaces with respect to their natural norms (see [43, pp. 16-17]) and the matrix $\widehat{G}$ is a triangle, Theorem 4.3.12 Wilansky [44, pp. 63] gives the fact that $c_0^{u,v}(\widehat{G}), c^{u,v}(\widehat{G})$ and $\ell_\infty^{u,v}(\widehat{G})$ are $BK-$ spaces with the given norms. This completes the proof .

Theorem 2.2. *The sequence spaces $c_0^{u,v}(\widehat{G})$, $c^{u,v}(\widehat{G})$, $\ell_\infty^{u,v}(\widehat{G})$ and $\ell_p^{u,v}(\widehat{G})$ of non-absolutely type are norm isomorphic to the spaces c_0, c, ℓ_∞ and ℓ_p , respectively, that is, $c_0^{u,v}(\widehat{G}) \cong c_0$, $c^{u,v}(\widehat{G}) \cong c$, $\ell_\infty^{u,v}(\widehat{G}) \cong \ell_\infty$ and $\ell_p^{u,v}(\widehat{G}) \cong \ell_p$.*

Proof: We prove the theorem for the space $c_0^{u,v}(\widehat{G})$. To prove our assertion we should show the existence of a linear bijection between the spaces $c_0^{u,v}(\widehat{G})$ and c_0. Let $T : c_0^{u,v}(\widehat{G}) \to c_0$ be defined by (4) . Then, $y = T(x) = \widehat{G}(x) \in c_0$ for every $x \in c_0^{u,v}(\widehat{G})$ and the linearity of T is clear. Further, it is trivial that $x = 0$ whenever $Tx = \theta$ and hence T is injective .

Moreover, let $y = (y_k) \in c_0$ and we define the sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k} d_{kj} \sum_{i=j-1}^{j} (-1)^{j-i} \frac{1}{v_j u_i} y_i \tag{8}$$

for $k \in \mathbb{N}$ where $d_{nk} = 0$ for $k > n$ and

$$d_{nk} = \frac{1}{r} \sum_{v=0}^{n-k} \left(\frac{-s + \sqrt{s^2 - 4tr}}{2r} \right)^{n-k-v} \left(\frac{-s - \sqrt{s^2 - 4tr}}{2r} \right)^{v} \tag{9}$$

for $0 \leq k \leq n$. Then we obtain

$$rx_k + sx_{k-1} + tx_{k-2} = \sum_{j=k-1}^{k} (-1)^{k-j} \frac{1}{v_k u_j} y_j \qquad \text{for all } k \in \mathbb{N}.$$

Hence, for every $n \in \mathbb{N}$ we get by (4)

$$\widehat{G}_n(x) = u_n \sum_{k=0}^{n} v_k (rx_k + sx_{k-1} + tx_{k-2}) = u_n \sum_{k=0}^{n} v_k \sum_{j=k-1}^{k} (-1)^{k-j} \frac{1}{v_k u_j} y_j = y_n.$$

This show that $\widehat{G}(x) = y$ and since $y \in c_0$, we conclude that $\widehat{G}(x) \in c_0$. Thus, we deduce that $x \in c_0^{u,v}(\widehat{G})$ and $Tx = y$. Hence T is surjective.

Moreover one can easily see for every $x \in c_0^{u,v}(\widehat{G})$ that

$$\|Tx\|_\infty = \|\widehat{G}x\|_\infty = \|x\|_{c_0^{u,v}(\widehat{G})}$$

which means that T is norm preserving. Consequently T is a linear bijection which show that the spaces $c_0^{u,v}(\widehat{G})$ and c_0 are linearly isomorphic, as desired.

Let $(X, \|.\|)$ be a normed space. A sequence (b_k) of elements of X is called a Schauder basis for X if and only if, for each $x \in X$ there exists a unique sequence (α_k) of scalars such that $x = \sum_k \alpha_k b_k$, i.e.

$$\lim_{n \to \infty} \|x - \sum_{k=0}^{n} \alpha_k b_k\| = 0.$$

Lemma 2.3. *[4] Let T be a triangle and S be its inverse. If $(b(n))$ is a basis of the normed sequence space X, then $(S(b(n)))$ is a basis of X_T.*

The Schauder basis of the sequence spaces $c_0^{u,v}(\widehat{G})$, $c^{u,v}(\widehat{G})$ and $\ell_p^{u,v}(\widehat{G})$ can be derived by using Lemma 2.3, as follows:

Corollary 2.4. *Let $\alpha_k = \widehat{G}_k(x)$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \widehat{G}_k(x) = l$. Define the sequence $b^{(k)} = \{b_n^{(k)}\}_{n\in\mathbb{N}}$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)} = \begin{cases} \dfrac{d_{nk}}{u_k v_k} - \dfrac{d_{n,k+1}}{u_k v_{k+1}}, & (n > k) \\ \dfrac{1}{r u_k v_k}, & (n = k) \\ 0, & (n < k). \end{cases} \tag{10}$$

Then, the following statements hold:

(i) The sequence $\{b_n^{(k)}\}_{n\in\mathbb{N}}$ is a basis for the spaces $c_0^{u,v}(\widehat{G})$ and any $x \in c_0^{u,v}(\widehat{G})$ has a unique representation of the form $x = \sum_k \alpha_k b^{(k)}$.

(ii) The sequence $\{b_n^{(k)}\}_{n\in\mathbb{N}}$ is a basis for the spaces $\ell_p^{u,v}(\widehat{G})$ and any $x \in \ell_p^{u,v}(\widehat{G})$ has a unique representation of the form $x = \sum_k \alpha_k b^{(k)}$.

(iii) The sequence $\{b, b^{(0)}, b^{(0)}, b^{(1)}, ...\}$ is a basis for the space $c^{u,v}(\widehat{G})$, where $b = (b_k) = \left(\sum_{j=0}^{k} d_{kj} \right)$, and any $x \in c^{u,v}(\widehat{G})$ has a unique representation of the form

$$x = lb + \sum_k [\alpha_k - l] b^{(k)}.$$

3 The α-, β- and γ-Duals of the Spaces $\mu^{u,v}(\widehat{G})$ of Non-Absolute Type for $\mu \in \{c_0, c, \ell_\infty, \ell_p\}$

In this section, we determine the α-, β- and γ-duals of the generalized difference sequence spaces $c_0^{u,v}(\widehat{G}), c^{u,v}(\widehat{G}), \ell_\infty^{u,v}(\widehat{G})$ and $\ell_p^{u,v}(\widehat{G})$ of non-absolute type.

Firstly, we must give the definition of α-, β- and γ-duals of a sequences space. For the sequence spaces λ and μ, define the set $S(\lambda, \mu)$ by

$$S(\lambda, \mu) = \{z = (z_k) \in w : xz = (x_k z_k) \in \mu \text{ for all } x \in \lambda\}. \tag{11}$$

With the notation of (11), the α-, β- and γ-duals of a sequences space λ, which are respectively denoted by

$$\lambda^\alpha = S(\lambda, \ell_1), \quad \lambda^\beta = S(\lambda, cs) \quad \text{and} \quad \lambda^\gamma = S(\lambda, bs).$$

Lemma 3.1. *(i) $A \in (c_0 : \ell_1) = (c : \ell_1) = (\ell_\infty : \ell_1)$ if and only if*

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} a_{nk} \right| < \infty.$$

(ii) Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $A \in (\ell_p : \ell_1)$ if and only if

$$\sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in N} a_{nk} \right|^q < \infty.$$

Lemma 3.2. *$A \in (c_0 : c)$ if and only if*

$$\lim_{n \to \infty} a_{nk} = \alpha_k \text{ for each fixed } k \in \mathbb{N}, \tag{12}$$

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}| < \infty. \tag{13}$$

Lemma 3.3. *$A \in (c : c)$ if and only if (12) and (13) hold, and*

$$\lim_{n \to \infty} \sum_k a_{nk} \quad \text{exists.} \tag{14}$$

Lemma 3.4. *$A \in (\ell_\infty : c)$ if an only if (12) holds and*

$$\lim_{n \to \infty} \sum_k |a_{nk}| = \sum_k |\alpha_k|.$$

Lemma 3.5. *Let $1 < p < \infty$. Then, $A \in (\ell_p : c)$ if and only if (12) hold and*

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}|^q < \infty \qquad \left(\frac{1}{p} + \frac{1}{q} = 1 \right). \tag{15}$$

Lemma 3.6. *$A \in (c : \ell_\infty) = (c_0 : \ell_\infty) = (\ell_\infty : \ell_\infty)$ if and only if (13) holds.*

Lemma 3.7. *Let $1 < p < \infty$. Then, $A \in (\ell_p : \ell_\infty)$ if and only if (15) holds.*

Now we consider the following sets:

$$\begin{aligned}
f_1 &= \left\{ a = (a_n) \in w : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} f_{nk} \right| < \infty \right\}, \\
f_2 &= \left\{ a = (a_n) \in w : \sum_{j=k}^{\infty} d_{jk} a_j \text{ exists for each } k \in \mathbb{N}. \right\}, \\
f_3 &= \left\{ a = (a_n) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} |\widehat{g}_k(n)|^q < \infty \right\}, \\
f_4 &= \left\{ a = (a_n) \in w : \sup_{n \in \mathbb{N}} \left| \frac{1}{r} \frac{a_n}{u_n v_n} \right| < \infty \right\}, \\
f_5 &= \left\{ a = (a_n) \in w : \lim_{n \to \infty} \sum_{k=0}^{n} \left[\sum_{j=0}^{k} d_{kj} \right] a_k \quad \text{exists} \right\}, \\
f_6 &= \left\{ a = (a_n) \in w : \sup_{N \in \mathcal{F}} \sum_{k=0}^{\infty} \left| \sum_{n \in N} f_{nk} \right|^q < \infty \right\}, \\
f_7 &= \left\{ a = (a_n) \in w : \lim_{n \to \infty} \sum_k |v_{nk}| = \sum_k |\lim_{n \to \infty} v_{nk}| \right\},
\end{aligned}$$

where the matrices $F = (f_{nk})$ and $V = (v_{nk})$ are defined as follows,

$$f_{nk} = \begin{cases} \dfrac{d_{nk}a_n}{u_k v_k} - \dfrac{d_{n,k+1}a_n}{u_k v_{k+1}}, & (k < n) \\ \dfrac{a_n}{r u_n v_n}, & (k = n) \\ 0, & (k > n) \end{cases}$$

and

$$v_{nk} = \begin{cases} \widehat{g}_k(n), & (k < n) \\ \dfrac{a_n}{r u_n v_n}, & (k = n) \\ 0, & (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$ and the $\widehat{g}_k(n)$ is defined as follows

$$\widehat{g}_k(n) = \frac{1}{u_k}\left[\frac{\sum_{j=k}^{n} d_{jk}a_j}{v_k} - \frac{\sum_{j=k+1}^{n} d_{j,k+1}a_j}{v_{k+1}}\right] \qquad \text{for } k < n.$$

Theorem 3.8. *(i)* $\{c_0^{u,v}(\widehat{G})\}^{\alpha} = \{c^{u,v}(\widehat{G})\}^{\alpha} = \{\ell_\infty u, v(\widehat{G})\}^{\alpha} = f_1$.
(ii) Let $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then,* $\{\ell_p^{\lambda}(\widehat{B}u, v(\widehat{G}))\}^{\alpha} = f_6$.

Proof: We prove the theorem for the space $c_0^{u,v}(\widehat{G})$. Let $a = (a_n) \in w$. Then, we obtain the equality

$$\begin{aligned} a_n x_n &= \sum_{k=0}^{n} d_{nk} \sum_{j=k-1}^{k} (-1)^{k-j} \frac{1}{v_k u_j} a_n y_j \\ &= \sum_{k=0}^{n-1} \left(\frac{d_{nk}}{v_k} - \frac{d_{n,k+1}}{v_{k+1}}\right) \frac{a_n}{u_k} y_k + \frac{a_n}{r u_n v_n} y_n \\ &= F_n(y) \end{aligned} \tag{16}$$

by relation (8). Thus we observe by (16) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in c_0^{u,v}(\widehat{G})$ if and only if $Fy \in \ell_1$ whenever $y = (y_k) \in c_0$. This means that the sequence $a = (a_n) \in \{c_0^{u,v}(\widehat{G})\}^{\alpha}$ if and only if $F \in (c_0 : \ell_1)$. Therefore we obtain by Lemma 3.1 with F instead of A that $a = (a_n) \in \{c_0^{u,v}(\widehat{G})\}^{\alpha}$ if and only if

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} f_{nk} \right| < \infty$$

which leads us to the consequence that $\{c_0^{u,v}(\widehat{G})\}^{\alpha} = f_1$. This completes the proof.

Theorem 3.9. *(i)* $\{c_0^{u,v}(\widehat{G})\}^{\beta} = f_2 \cap f_3 \cap f_4 \quad$ *(with* $q = 1$*).*
(ii) $\{c^{u,v}(\widehat{G})\}^{\beta} = f_2 \cap f_3 \cap f_4 \cap f_5 \quad$ *(with* $q = 1$*).*
(iii) $\{\ell_\infty^{u,v}(\widehat{G})\}^{\beta} = f_2 \cap f_4 \cap f_7$
(iv) Let $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then,* $\{\ell_p^{u,v}(\widehat{G})\}^{\beta} = f_2 \cap f_3 \cap f_4$.

Proof: Consider the equality

$$\begin{aligned}
\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n}\left[\sum_{j=0}^{k} d_{kj}\sum_{i=j-1}^{i}(-1)^{j-i}\frac{1}{v_j u_i}y_i\right]a_k \\
&= \sum_{k=0}^{n}\left(\frac{y_k}{u_k}-\frac{y_{k-1}}{u_{k-1}}\right)\left(\frac{1}{v_k}\sum_{j=k}^{n} d_{jk}a_j\right) \\
&= \sum_{k=0}^{n-1}\frac{1}{u_k}\left[\frac{\sum_{j=k}^{n} d_{jk}a_j}{v_k}-\frac{\sum_{j=k+1}^{n} d_{j,k+1}a_j}{v_{k+1}}\right]y_k+\frac{1}{r}\frac{a_n}{u_n v_n}y_n \\
&= \sum_{k=0}^{n-1}\widehat{g}_k(n)y_k+\frac{1}{r}\frac{a_n}{u_n v_n}y_n \\
&= V_n(y); \quad (n\in\mathbb{N}). \qquad (17)
\end{aligned}$$

Then we deduce by (17) that $ax=(a_kx_k)\in cs$ whenever $x=(x_k)\in c_0^{u,v}(\widehat{G})$ if and only if $Vy\in c$ whenever $y=(y_k)\in c_0$. This means that $a=(a_k)\in\{c_0^{u,v}(\widehat{G})\}^{\beta}$ if and only if $V\in(c_0:c)$. Therefore, by using Lemma 3.2, we obtain :

$$\sum_{j=k}^{\infty} d_{jk}a_j \qquad \text{exists for each } k\in\mathbb{N}, \qquad (18)$$

$$\sup_{n\in\mathbb{N}}\sum_{k=0}^{n-1}|\widehat{g}_k(n)|<\infty, \qquad (19)$$

$$\sup_{k\in\mathbb{N}}\left|\frac{1}{r}\frac{a_n}{u_n v_n}y_n\right|<\infty. \qquad (20)$$

Hence, we conclude that $\{c_0^{u,v}(\widehat{G})\}^{\beta}=f_2\cap f_3\cap f_4$.

Finally, we ended up this section with the following theorem which determines the γ-duals of sequence spaces $c_0^{u,v}(\widehat{G}), c^{u,v}(\widehat{G}), \ell_\infty^{u,v}(\widehat{G})$ and $\ell_p^{u,v}(\widehat{G})$.

Theorem 3.10. *(i)* $\{\mu^{u,v}(\widehat{G})\}^{\gamma}=f_3\cap f_4$ *(with* $q=1$*), where* $\mu\in\{c_0,c,\ell_\infty\}$.

(ii) Let $1<p<\infty$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Then,* $\{\ell_p^{\lambda}(\widehat{B})\}^{\gamma}=f_3^{\lambda}\cap f_4^{\lambda}$.

4 Some Matrix Transformations Related to the Spaces $\mu^{u,v}(\widehat{G})$ of Non-Absolute Type for $\mu\in\{c_0,c,\ell_\infty,\ell_p\}$

In this final section, we state some results which characterize various matrix mappings on the spaces $c_0^{u,v}(\widehat{G}), c^{u,v}(\widehat{G}), \ell_\infty^{u,v}(\widehat{G})$ and $\ell_p^{u,v}(\widehat{G})$. We shall write throughout for brevity that

$$\widehat{a}_{nk}(m)=\frac{1}{u_k}\left[\frac{\sum_{j=k}^{m} d_{jk}a_{j,k}}{v_k}-\frac{\sum_{j=k+1}^{m} d_{j,k+1}a_{j,k+1}}{v_{k+1}}\right] \qquad \text{for } k<m$$

and

$$\widehat{a}_{nk}=\frac{1}{u_k}\left[\frac{\sum_{j=k}^{\infty} d_{jk}a_{j,k}}{v_k}-\frac{\sum_{j=k+1}^{\infty} d_{j,k+1}a_{j,k+1}}{v_{k+1}}\right]$$

for all $k, m, n \in \mathbb{N}$ provided the series on the right hand to be convergent.

Theorem 4.1. *Let λ be any given sequence space and $\mu \in \{c_0, c, \ell_\infty, \ell_p\}$. Then, $A = (a_{nk}) \in (\mu^{u,v}(\widehat{G}) : \lambda)$ if and only if $B \in (\mu : \lambda)$ and*

$$B^{(n)} \in (\mu : c) \tag{21}$$

for every fixed $n \in \mathbb{N}$, where $b_{nk} = \widehat{a}_{nk}$ and $B^{(n)} = (b^{(n)}_{mk})$

$$b^{(n)}_{mk} = \begin{cases} \widehat{a}_{nk}(m), & (k < n) \\ \dfrac{a_{nm}}{ru_m v_m}, & (k = n) \\ 0, & (k > n) \end{cases}$$

for all $k, m \in \mathbb{N}$.

Proof: This result can be proved similarly as the proof of Theorem 3.1 in [8].

We will have several consequences by using Theorem 4.1. But we must give firstly some relations which are important for corollaries:

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty \tag{22}$$

$$\lim_{n\to\infty} a_{nk} = 0 \quad \text{for each fixed } k \in \mathbb{N} \tag{23}$$

$$\lim_{n\to\infty} a_{nk} = \alpha_k \quad \text{exists for each fixed } k \in \mathbb{N} \tag{24}$$

$$\lim_{n\to\infty} \sum_k a_{nk} \quad \text{exists} \tag{25}$$

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} a_{nk} \right| < \infty \tag{26}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = 0 \tag{27}$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k |\alpha_k| \tag{28}$$

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}|^q < \infty \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right) \tag{29}$$

$$\sup_{N\in\mathcal{F}} \sum_k \left| \sum_{n\in N} a_{nk} \right|^q < \infty \tag{30}$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = 0 \tag{31}$$

Now, we can give the corollaries:

Corollary 4.2. *$A = (a_{nk})$ be any infinite matrix. Then the following statements hold:*
(i) $A = (a_{nk}) \in (c_0^{u,v}(\widehat{G}) : \ell_\infty)$ if and only if (22) holds with $\widehat{a}_{nk}$ instead of a_{nk} and (21) also holds.

(ii) $A = (a_{nk}) \in (c_0^{u,v}(\widehat{G}) : c)$ *if and only if (24) and (25) hold with* $\widehat{a}_{nk}$ *instead of* a_{nk} *and (21) also holds.*
(iii) $A = (a_{nk}) \in (c_0^{u,v}(\widehat{G}) : c_0)$ *if and only if (22) and (23) hold with* $\widehat{a}_{nk}$ *instead of* a_{nk} *and (21) also holds.*
(iv) $A = (a_{nk}) \in (c_0^{u,v}(\widehat{G}) : \ell_1)$ *if and only if (26) holds with* $\widehat{a}_{nk}$ *instead of* a_{nk} *and (21) also holds.*

Corollary 4.3. $A = (a_{nk})$ *be any infinite matrix. Then the following statements hold:*
(i) $A = (a_{nk}) \in (c^{u,v}(\widehat{G}) : \ell_\infty)$ *if and only if (22) holds with* $\widehat{a}_{nk}$ *instead of* a_{nk} *and (21) also holds.*
(ii) $A = (a_{nk}) \in (c^{u,v}(\widehat{G}) : c)$ *if and only if (22), (24) and (25) hold with* $\widehat{a}_{nk}$ *instead of* a_{nk} *and (21) also holds.*
(iii) $A = (a_{nk}) \in (c^{u,v}(\widehat{G}) : c_0)$ *if and only if (23) and (27) hold with* $\widehat{a}_{nk}$ *instead of* a_{nk} *and (21) also holds.*
(iv) $A = (a_{nk}) \in (c^{u,v}(\widehat{G}) : \ell_1)$ *if and only if (26) holds with* $\widehat{a}_{nk}$ *instead of* a_{nk} *and (21) also holds.*

Corollary 4.4. $A = (a_{nk})$ *be any infinite matrix. Then the following statements hold:*
(i) $A = (a_{nk}) \in (\ell_\infty^{u,v}(\widehat{G}) : \ell_\infty)$ *if and only if (22) holds with* $\widehat{a}_{nk}$ *instead of* a_{nk} *and (21) also holds.*
(ii) $A = (a_{nk}) \in (\ell_\infty^{u,v}(\widehat{G}) : c)$ *if and only if (24) and (28) hold with* $\widehat{a}_{nk}$ *instead of* a_{nk} *and (21) also holds.*
(iii) $A = (a_{nk}) \in (\ell_\infty^{u,v}(\widehat{G}) : c_0)$ *if and only if (31) holds with* $\widehat{a}_{nk}$ *instead of* a_{nk} *and (21) also holds.*
(iv) $A = (a_{nk}) \in (\ell_\infty^{u,v}(\widehat{G}) : \ell_1)$ *if and only if (26) holds with* $\widehat{a}_{nk}$ *instead of* a_{nk} *and (21) also holds.*

Corollary 4.5. *Let* $A = (a_{nk})$ *be any infinite matrix,* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*. Then the following statements hold:*
(i) $A = (a_{nk}) \in (\ell_p^{u,v}(\widehat{G}) : \ell_\infty)$ *if and only if (29) holds with* $\widehat{a}_{nk}$ *instead of* a_{nk} *and (21) also holds.*
(ii) $A = (a_{nk}) \in (\ell_p^{u,v}(\widehat{G}) : c)$ *if and only if (24) and (29) hold with* $\widehat{a}_{nk}$ *instead of* a_{nk} *and (21) also holds.*
(iii) $A = (a_{nk}) \in (\ell_p^{u,v}(\widehat{G}) : c_0)$ *if and only if (23) and (29) hold with* $\widehat{a}_{nk}$ *instead of* a_{nk} *and (21) also holds.*
(iv) $A = (a_{nk}) \in (\ell_p^{u,v}(\widehat{G}) : \ell_1)$ *if and only if (30) holds with* $\widehat{a}_{nk}$ *instead of* a_{nk} *and (21) also holds.*

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