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$\mathcal{I}_{*g}$-normal and $\mathcal{I}_{*g}$-regular spaces

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Abstract - $\mathcal{I}_{*g}$-normal and $\mathcal{I}_{*g}$-regular spaces are introduced and various characterizations and properties are given. Characterizations of normal, mildly normal, *g-normal and regular spaces are also given.

Keywords - *$\mathcal{I}_{*g}$-closed sets, $\mathcal{I}_{*g}$-open sets, *g-closed sets, *g-open sets, $\mathcal{I}_{*g}$-normal spaces, $\mathcal{I}_{*g}$-regular spaces.*

1 Introduction and Preliminaries

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If A$\subset$X, cl(A) and int(A) will, respectively, denote the closure and interior of A in (X, τ). A subset A of a space (X, τ) is said to be regular open if A=int(cl(A)) and A is said to be regular closed if A=cl(int(A)). A subset A of a space (X, τ) is said to be semi-open [7] if A$\subset$cl(int(A)). The complement of semi-open set is semi-closed. A subset A of a space (X, τ) is an α-open [15] (resp. preopen [12]) if A$\subset$int(cl(int(A))) (resp. A$\subset$int(cl(A))). The complement of α-open set is α-closed. The α-closure [15] of a subset A of X, denoted by αcl(A), is defined to be the intersection of all α-closed sets containing A. The α-interior [15] of a subset A of X, denoted by αint(A), is defined to be the union of all α-open sets contained in A. The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ. The interior of a subset A in (X, τ^α) is denoted by int$_\alpha$(A). The closure of a subset A in (X, τ^α) is denoted by cl$_\alpha$(A). A subset A of a space (X, τ) is said to be ω-closed [21] if cl(A)$\subset$U whenever A$\subset$U and U is semi-open. The complement of ω-closed set is ω-open. A subset A of a space (X, τ) is said to be $\alpha\hat{g}$-closed [1] (resp. rαg-closed [17]) if cl$_\alpha$(A)$\subset$U whenever A$\subset$U and

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U is ω-open (resp. regular open). A is said to be $\alpha\hat{g}$-open (resp. rαg-open) if X−A is $\alpha\hat{g}$-closed (resp. rαg-closed). A subset A of a space (X, τ) is said to be *g-closed [19] if cl(A)⊆U whenever A⊆U and U is ω-open. A space (X, τ) is said to be *g-normal, if for every disjoint *g-closed sets A and B, there exist disjoint open sets U and V such that A⊂U, B⊂V.

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) A∈$\mathcal{I}$ and B⊂A imply B∈$\mathcal{I}$ and (ii) A∈$\mathcal{I}$ and B∈$\mathcal{I}$ imply A∪B∈$\mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator (.)* : $\wp$(X) → $\wp$(X), called a local function [6] of A with respect to τ and $\mathcal{I}$ is defined as follows: for A⊂X, A*($\mathcal{I}$, τ)={x∈X : U∩A∉$\mathcal{I}$ for every U∈τ(x)} where τ(x)={U∈τ : x∈U}. We will make use of the basic facts about the local functions [5, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator cl*(.) for a topology τ*($\mathcal{I}$, τ), called the $\star$-topology, finer than τ is defined by cl*(A)=A∪A*($\mathcal{I}$, τ) [23]. When there is no chance for confusion, we will simply write A* for A*($\mathcal{I}$, τ) and τ* for τ*($\mathcal{I}$, τ). int*(A) will denote the interior of A in (X, τ*). If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal space. N is the ideal of all nowhere dense subsets in (X, τ). A subset A of an ideal space (X, τ, $\mathcal{I}$) is τ*-closed [5] or $*$-closed (resp. $*$-dense in itself [4]) if A*⊂A (resp. A⊂A*). A subset A of an ideal space (X, τ, $\mathcal{I}$) is $\mathcal{I}_{*g}$-closed [19] if A*⊂U whenever U is ω-open and A⊂U. By Theorem 2.3 of [19], every $*$-closed and hence every closed set is $\mathcal{I}_{*g}$-closed. A subset A of an ideal space (X, τ, $\mathcal{I}$) is said to be $\mathcal{I}_{*g}$-open [19] if X−A is $\mathcal{I}_{*g}$-closed. In this paper, we define $\mathcal{I}_{*g}$-normal, ${}_{*g}\mathcal{I}$-normal and $\mathcal{I}_{*g}$-regular spaces using $\mathcal{I}_{*g}$-open sets and give characterizations and properties of such spaces. Also, characterizations of normal, mildly normal, *g-normal and regular spaces are given.

An ideal $\mathcal{I}$ is said to be codense [3] if τ∩$\mathcal{I}$={∅}. $\mathcal{I}$ is said to be completely codense [3] if PO(X)∩$\mathcal{I}$={∅}, where PO(X) is the family of all preopen sets in (X, τ). Every completely codense ideal is codense but not conversely [3]. The following lemmas will be useful in the sequel.

Lemma 1.1 ([20], Theorem 6)**.** *Let (X, τ, $\mathcal{I}$) be an ideal space. If $\mathcal{I}$ is completely codense, then $\tau^*\subset\tau^\alpha$.*

Lemma 1.2 ([19], Theorem 2.16)**.** *Let (X, τ, $\mathcal{I}$) be an ideal space where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

1. *X is normal.*
2. *For any disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{*g}$-open sets U and V such that A⊂U, B⊂V.*
3. *For a closed set A and an open set V containing A, there exists an $\mathcal{I}_{*g}$-open set U such that A⊂U⊂cl*(U)⊂V.*

Lemma 1.3. *If (X, τ, $\mathcal{I}$) is an ideal space and A⊂X, then the following hold.*

1. *If $\mathcal{I}$={∅}, then A is $\mathcal{I}_{*g}$-closed if and only if A is *g-closed [[19], Corollary 2.3].*
2. *If $\mathcal{I}$=N, then A is $\mathcal{I}_{*g}$-closed if and only if A is $\alpha\hat{g}$-closed [19].*

Lemma 1.4 ([19], Theorem 2.2)**.** *If (X, τ, $\mathcal{I}$) is an ideal space and A⊂X, then the following are equivalent.*

1. *A is $\mathcal{I}_{*g}$-closed.*

2. *cl*(A)⊂U whenever A⊂U and U is ω-open in X.*

Lemma 1.5 ([19], Theorem 2.12)**.** *Let (X, τ, $\mathcal{I}$) be an ideal space and A⊂X. Then A is $\mathcal{I}_{*g}$-open if and only if F⊂int*(A) whenever F is ω-closed and F⊂A.*

Lemma 1.6 ([19], Theorem 2.15)**.** *Let (X, τ, $\mathcal{I}$) be an ideal space. Then every subset of X is $\mathcal{I}_{*g}$-closed if and only if every ω-open set is ∗-closed.*

Proposition 1.7. *[21] Every open set is ω-open but not conversely.*

2 $\mathcal{I}_{*g}$-normal and ${}_{*g}\mathcal{I}$-normal Spaces

An ideal space (X, τ, $\mathcal{I}$) is said to be an $\mathcal{I}_{*g}$-normal space if for every pair of disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{*g}$-open sets U and V such that A⊂U and B⊂V. Since every open set is an $\mathcal{I}_{*g}$-open set, every normal space is $\mathcal{I}_{*g}$-normal. The following Example 2.1 shows that an $\mathcal{I}_{*g}$-normal space is not necessarily a normal space. Theorem 2.2 below gives characterizations of $\mathcal{I}_{*g}$-normal spaces. Theorem 2.3 below shows that the two concepts coincide for completely codense ideal spaces.

Example 2.1. *Let X={a, b, c}, τ={∅, {b}, {a, b}, {b, c}, X} and $\mathcal{I}$={∅, {b}}. Then ∅*=∅, ({a, b})*={a}, ({b, c})*={c}, ({b})*=∅ and X*={a, c}. Here every ω-open set is ∗-closed and so, by Lemma 1.6, every subset of X is $\mathcal{I}_{*g}$-closed and hence every subset of X is $\mathcal{I}_{*g}$-open. This implies that (X, τ, $\mathcal{I}$) is $\mathcal{I}_{*g}$-normal. Now {a} and {c} are disjoint closed subsets of X which are not separated by disjoint open sets and so (X, τ) is not normal.*

Theorem 2.2. *Let (X, τ, $\mathcal{I}$) be an ideal space. Then the following are equivalent.*

1. *X is $\mathcal{I}_{*g}$-normal.*

2. *For every pair of disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{*g}$-open sets U and V such that A⊂U and B⊂V.*

3. *For every closed set A and an open set V containing A, there exists an $\mathcal{I}_{*g}$-open set U such that A⊂U⊂cl*(U)⊂V.*

Proof. (1)⇒(2). The proof follows from the definition of $\mathcal{I}_{*g}$-normal spaces.

(2)⇒(3). Let A be a closed set and V be an open set containing A. Since A and X−V are disjoint closed sets, there exist disjoint $\mathcal{I}_{*g}$-open sets U and W such that A⊂U and X−V⊂W. Again, U∩W=∅ implies that U∩int*(W)=∅ and so cl*(U)⊂X−int*(W). Since X−V is ω-closed and W is $\mathcal{I}_{*g}$-open, X−V⊂W implies that X−V⊂int*(W) and so X−int*(W)⊂V. Thus, we have A⊂U⊂cl*(U)⊂X−int*(W)⊂V which proves (3).

(3)⇒(1). Let A and B be two disjoint closed subsets of X. By hypothesis, there exists an $\mathcal{I}_{*g}$-open set U such that A⊂U⊂cl*(U)⊂X−B. If W=X−cl*(U), then U and W are the required disjoint $\mathcal{I}_{*g}$-open sets containing A and B respectively. So, (X, τ, $\mathcal{I}$) is $\mathcal{I}_{*g}$-normal.

Theorem 2.3. *Let (X, τ, $\mathcal{I}$) be an ideal space where $\mathcal{I}$ is completely codense. If (X, τ, $\mathcal{I}$) is $\mathcal{I}_{*g}$-normal, then it is a normal space.*

Proof. Suppose that $\mathcal{I}$ is completely codense. By Theorem 2.2, (X, τ, $\mathcal{I}$) is $\mathcal{I}_{*g}$-normal if and only if for each pair of disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{*g}$-open sets U and V such that A⊂U and B⊂V if and only if X is normal, by Lemma 1.2.

Theorem 2.4. *Let (X, τ, $\mathcal{I}$) be an $\mathcal{I}_{*g}$-normal space. If F is closed and A is a *g-closed set such that A∩F=∅, then there exist disjoint $\mathcal{I}_{*g}$-open sets U and V such that A⊂U and F⊂V.*

Proof. Since A∩F=∅, A⊂X−F where X−F is ω-open. Therefore, by hypothesis, cl(A)⊂X−F. Since cl(A)∩F=∅ and X is $\mathcal{I}_{*g}$-normal, there exist disjoint $\mathcal{I}_{*g}$-open sets U and V such that cl(A)⊂U and F⊂V.

The following Corollaries 2.5 and 2.6 give properties of normal spaces. If $\mathcal{I}$={∅} in Theorem 2.4, then we have the following Corollary 2.5, the proof of which follows from Theorem 2.3 and Lemma 1.3, since {∅} is a completely codense ideal. If $\mathcal{I}$=N in Theorem 2.4, then we have the Corollary 2.6 below, since $\tau^*(N)=\tau^\alpha$ and $\mathcal{I}_{*g}$-open sets coincide with $\alpha\hat{g}$-open sets.

Corollary 2.5. *Let (X, τ) be a normal space. If F is a closed set and A is a *g-closed set disjoint from F, then there exist disjoint *g-open sets U and V such that A⊂U and F⊂V.*

Corollary 2.6. *Let (X, τ, $\mathcal{I}$) be a normal ideal space where $\mathcal{I}$=N. If F is a closed set and A is a *g-closed set disjoint from F, then there exist disjoint $\alpha\hat{g}$-open sets U and V such that A⊂U and F⊂V.*

Theorem 2.7. *Let (X, τ, $\mathcal{I}$) be an ideal space which is $\mathcal{I}_{*g}$-normal. Then the following hold.*

1. *For every closed set A and every *g-open set B containing A, there exists an $\mathcal{I}_{*g}$-open set U such that A⊂int*(U)⊂U⊂B.*
2. *For every *g-closed set A and every open set B containing A, there exists an $\mathcal{I}_{*g}$-closed set U such that A⊂U⊂cl*(U)⊂B.*

Proof. (1) Let A be a closed set and B be a *g-open set containing A. Then A∩(X−B)=∅, where A is closed and X−B is *g-closed. By Theorem 2.4, there exist disjoint $\mathcal{I}_{*g}$-open sets U and V such that A⊂U and X−B⊂V. Since U∩V=∅, we have U⊂X−V. By Lemma 1.5, A⊂int*(U). Therefore, A⊂int*(U)⊂U⊂X−V⊂B. This proves (1).

(2) Let A be a *g-closed set and B be an open set containing A. Then X−B is a closed set contained in the *g-open set X−A. By (1), there exists an $\mathcal{I}_{*g}$-open set V such that X−B⊂int*(V)⊂V⊂X−A. Therefore, A⊂X−V⊂cl*(X−V)⊂B. If U=X−V, then A⊂U⊂cl*(U)⊂B and so U is the required $\mathcal{I}_{*g}$-closed set.

The following Corollaries 2.8 and 2.9 give some properties of normal spaces. If $\mathcal{I}$={∅} in Theorem 2.7, then we have the following Corollary 2.8. If $\mathcal{I}$=N in Theorem 2.7, then we have the Corollary 2.9 below.

Corollary 2.8. *Let (X, τ) be a normal space. Then the following hold.*

1. *For every closed set A and every *g-open set B containing A, there exists a *g-open set U such that A⊂int(U)⊂U⊂B.*

2. *For every *g-closed set A and every open set B containing A, there exists a *g-closed set U such that* $A \subset U \subset cl(U) \subset B$.

Corollary 2.9. *Let (X, τ) be a normal space. Then the following hold.*

1. *For every closed set A and every *g-open set B containing A, there exists an* $\alpha\hat{g}$*-open set U such that* $A \subset int_\alpha(U) \subset U \subset B$.

2. *For every *g-closed set A and every open set B containing A, there exists an* $\alpha\hat{g}$*-closed set U such that* $A \subset U \subset cl_\alpha(U) \subset B$.

An ideal space (X, τ, $\mathcal{I}$) is said to be $_{*g}\mathcal{I}$-normal if for each pair of disjoint $\mathcal{I}_{*g}$-closed sets A and B, there exist disjoint open sets U and V in X such that A⊂U and B⊂V. Since every closed set is $\mathcal{I}_{*g}$-closed, every $_{*g}\mathcal{I}$-normal space is normal. But a normal space need not be $_{*g}\mathcal{I}$-normal as the following Example 2.10 shows. Theorems 2.11 and 2.13 below give characterizations of $_{*g}\mathcal{I}$-normal spaces.

Example 2.10. *Let* $X=\{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. *Every* ω*-open set is* $*$*-closed and so every subset of X is* $\mathcal{I}_{*g}$*-closed. Now* $A=\{a, b\}$ *and* $B=\{c\}$ *are disjoint* $\mathcal{I}_{*g}$*-closed sets, but they are not separated by disjoint open sets. So (X,* τ*,* $\mathcal{I}$*) is not* $_{*g}\mathcal{I}$*-normal. But (X,* τ*,* $\mathcal{I}$*) is normal.*

Theorem 2.11. *In an ideal space (X,* τ*,* $\mathcal{I}$*), the following are equivalent.*

1. *X is* $_{*g}\mathcal{I}$*-normal.*

2. *For every* $\mathcal{I}_{*g}$*-closed set A and every* $\mathcal{I}_{*g}$*-open set B containing A, there exists an open set U of X such that* $A \subset U \subset cl(U) \subset B$.

Proof. (1)⇒(2). Let A be an $\mathcal{I}_{*g}$-closed set and B be an $\mathcal{I}_{*g}$-open set containing A. Since A and X−B are disjoint $\mathcal{I}_{*g}$-closed sets, there exist disjoint open sets U and V such that A⊂U and X−B⊂V. Now U∩V=∅ implies that cl(U)⊂X−V. Therefore, A⊂U⊂cl(U)⊂X−V⊂B. This proves (2).

(2)⇒(1). Suppose A and B are disjoint $\mathcal{I}_{*g}$-closed sets, then the $\mathcal{I}_{*g}$-closed set A is contained in the $\mathcal{I}_{*g}$-open set X−B. By hypothesis, there exists an open set U of X such that A⊂U⊂cl(U)⊂X−B. If V=X−cl(U), then U and V are disjoint open sets containing A and B respectively. Therefore, (X, τ, $\mathcal{I}$) is $_{*g}\mathcal{I}$-normal.

If $\mathcal{I}=\{\emptyset\}$, then $_{*g}\mathcal{I}$-normal spaces coincide with *g-normal spaces and so if we take $\mathcal{I}=\{\emptyset\}$, in Theorem 2.11, then we have the following characterization for *g-normal spaces.

Corollary 2.12. *In a space (X,* τ*), the following are equivalent.*

1. *X is *g-normal.*

2. *For every *g-closed set A and every *g-open set B containing A, there exists an open set U of X such that* $A \subset U \subset cl(U) \subset B$.

Theorem 2.13. *In an ideal space (X,* τ*,* $\mathcal{I}$*), the following are equivalent.*

1. *X is* $_{*g}\mathcal{I}$*-normal.*

2. *For each pair of disjoint $\mathcal{I}_{*g}$-closed subsets A and B of X, there exists an open set U of X containing A such that cl(U)∩B=∅.*

3. *For each pair of disjoint $\mathcal{I}_{*g}$-closed subsets A and B of X, there exist an open set U containing A and an open set V containing B such that cl(U)∩cl(V)=∅.*

Proof. (1)⇒(2). Suppose that A and B are disjoint $\mathcal{I}_{*g}$-closed subsets of X. Then the $\mathcal{I}_{*g}$-closed set A is contained in the $\mathcal{I}_{*g}$-open set X−B. By Theorem 2.11, there exists an open set U such that A⊂U⊂cl(U)⊂X−B. Therefore, U is the required open set containing A such that cl(U)∩B=∅.

(2)⇒(3). Let A and B be two disjoint $\mathcal{I}_{*g}$-closed subsets of X. By hypothesis, there exists an open set U of X containing A such that cl(U)∩B=∅. Also, cl(U) and B are disjoint $\mathcal{I}_{*g}$-closed sets of X. By hypothesis, there exists an open set V of X containing B such that cl(U)∩cl(V)=∅.

(3)⇒(1). The proof is clear.

If $\mathcal{I}$={∅}, in Theorem 2.13, then we have the following characterizations for *g-normal spaces.

Corollary 2.14. *Let (X, τ) be a space. Then the following are equivalent.*

1. *X is *g-normal.*

2. *For each pair of disjoint *g-closed subsets A and B of X, there exists an open set U of X containing A such that cl(U)∩B=∅.*

3. *For each pair of disjoint *g-closed subsets A and B of X, there exist an open set U containing A and an open set V containing B such that cl(U)∩cl(V)=∅.*

Theorem 2.15. *Let (X, τ, $\mathcal{I}$) be an $_{*g}\mathcal{I}$-normal space. If A and B are disjoint $\mathcal{I}_{*g}$-closed subsets of X, then there exist disjoint open sets U and V such that cl*(A)⊂U and cl*(B)⊂V.*

Proof. Suppose that A and B are disjoint $\mathcal{I}_{*g}$-closed sets. By Theorem 2.13(3), there exist an open set U containing A and an open set V containing B such that cl(U)∩cl(V)=∅. Since A is $\mathcal{I}_{*g}$-closed, A⊂U implies that cl*(A)⊂U. Similarly cl*(B)⊂V.

If $\mathcal{I}$={∅}, in Theorem 2.15, then we have the following property of disjoint *g-closed sets in *g-normal spaces.

Corollary 2.16. *Let (X, τ) be a *g-normal space. If A and B are disjoint *g-closed subsets of X, then there exist disjoint open sets U and V such that cl(A)⊂U and cl(B)⊂V.*

Theorem 2.17. *Let (X, τ, $\mathcal{I}$) be an $_{*g}\mathcal{I}$-normal space. If A is an $\mathcal{I}_{*g}$-closed set and B is an $\mathcal{I}_{*g}$-open set containing A, then there exists an open set U such that A⊂cl*(A)⊂U⊂int*(B)⊂B.*

Proof. Suppose A is an $\mathcal{I}_{*g}$-closed set and B is an $\mathcal{I}_{*g}$-open set containing A. Since A and X−B are disjoint $\mathcal{I}_{*g}$-closed sets, by Theorem 2.15, there exist disjoint open sets U and V such that cl*(A)⊂U and cl*(X−B)⊂V. Now, X−int*(B)=cl*(X−B)⊂V implies that X−V⊂int*(B). Again, U∩V=∅ implies U⊂X−V and so A⊂cl*(A)⊂U⊂X−V⊂ int*(B)⊂B.

If $\mathcal{I}$={∅}, in Theorem 2.17, then we have the following Corollary 2.18.

Corollary 2.18. *Let (X, τ) be a *g-normal space. If A is a *g-closed set and B is a *g-open set containing A, then there exists an open set U such that A$\subset$cl(A)$\subset$U$\subset$int(B)$\subset$B.*

The following Theorem 2.19 gives a characterization of normal spaces in terms of *g-open sets which follows from Lemma 1.2 if $\mathcal{I}=\{\emptyset\}$.

Theorem 2.19. *Let (X, τ) be a space. Then the following are equivalent.*

1. *X is normal.*
2. *For any disjoint closed sets A and B, there exist disjoint *g-open sets U and V such that A$\subset$U and B$\subset$V.*
3. *For any closed set A and open set V containing A, there exists a *g-open set U such that A$\subset$U$\subset$cl(U)$\subset$V.*

The rest of the section is devoted to the study of mildly normal spaces in terms of $\mathcal{I}_{*g}$-open sets, $\mathcal{I}_g$-open sets and $\mathcal{I}_{rg}$-open sets. A space (X, τ) is said to be a mildly normal space [22] if disjoint regular closed sets are separated by disjoint open sets. A subset A of a space (X, τ) is said to be αg-closed [11] if cl_α(A)$\subset$U whenever A$\subset$U and U is open. A subset A of a space (X, τ) is said to be g-closed [8] (resp. rg-closed [18]) if cl(A)$\subset$U whenever A$\subset$U and U is open (resp. regular open) in X. The complements of the above closed sets are called their respective open sets.

A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be $\mathcal{I}_g$-closed [14] if A*$\subset$U whenever A$\subset$U and U is open. A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be a regular generalized closed set with respect to an ideal $\mathcal{I}$ ($\mathcal{I}_{rg}$-closed) [14] if A*$\subset$U whenever A$\subset$U and U is regular open. A is called $\mathcal{I}_g$-open (resp. $\mathcal{I}_{rg}$-open) if X$-$A is $\mathcal{I}_g$-closed (resp. $\mathcal{I}_{rg}$-closed). Clearly, every $\mathcal{I}_{*g}$-closed set is $\mathcal{I}_g$-closed and every $\mathcal{I}_g$-closed set is $\mathcal{I}_{rg}$-closed but the separate converses are not true. Theorem 2.21 below gives characterizations of mildly normal spaces. Corollary 2.22 below gives characterizations of mildly normal spaces in terms of $\alpha\hat{g}$-open, αg-open and rαg-open sets. Corollary 2.23 below gives characterizations of mildly normal spaces in terms of *g-open, g-open and rg-open sets. The following Lemma 2.20 is essential to prove Theorem 2.21.

Lemma 2.20. *[14] Let (X, τ, $\mathcal{I}$) be an ideal space. A subset A$\subset$X is $\mathcal{I}_{rg}$-open if and only if F$\subset$int*(A) whenever F is regular closed and F$\subset$A.*

Theorem 2.21. *Let (X, τ, $\mathcal{I}$) be an ideal space where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

1. *X is mildly normal.*
2. *For disjoint regular closed sets A and B, there exist disjoint $\mathcal{I}_{*g}$-open sets U and V such that A$\subset$U and B$\subset$V.*
3. *For disjoint regular closed sets A and B, there exist disjoint $\mathcal{I}_g$-open sets U and V such that A$\subset$U and B$\subset$V.*
4. *For disjoint regular closed sets A and B, there exist disjoint $\mathcal{I}_{rg}$-open sets U and V such that A$\subset$U and B$\subset$V.*

5. *For a regular closed set A and a regular open set V containing A, there exists an $\mathcal{I}_{rg}$-open set U of X such that A⊂ U⊂ cl*(U)⊂ V.*

6. *For a regular closed set A and a regular open set V containing A, there exists an ∗-open set U of X such that A⊂ U⊂ cl*(U)⊂ V.*

7. *For disjoint regular closed sets A and B, there exist disjoint ∗-open sets U and V such that A⊂ U and B⊂ V.*

Proof. (1)⇒(2). Suppose that A and B are disjoint regular closed sets. Since X is mildly normal, there exist disjoint open sets U and V such that A⊂U and B⊂V. But every open set is an $\mathcal{I}_{*g}$-open set. This proves (2).

(2)⇒(3). The proof follows from the fact that every $\mathcal{I}_{*g}$-open set is an $\mathcal{I}_g$-open set.

(3)⇒(4). The proof follows from the fact that every $\mathcal{I}_g$-open set is an $\mathcal{I}_{rg}$-open set.

(4)⇒(5). Suppose A is a regular closed and B is a regular open set containing A. Then A and X−B are disjoint regular closed sets. By hypothesis, there exist disjoint $\mathcal{I}_{rg}$-open sets U and V such that A⊂U and X−B⊂V. Since X−B is regular closed and V is $\mathcal{I}_{rg}$-open, by Lemma 2.20, X−B⊂int*(V) and so X−int*(V)⊂B. Again, U∩V=∅ implies that U∩int*(V)=∅ and so cl*(U)⊂X−int*(V)⊂B. Hence U is the required $\mathcal{I}_{rg}$-open set such that A⊂U⊂cl*(U)⊂B.

(5)⇒(6). Let A be a regular closed set and V be a regular open set containing A. Then there exists an $\mathcal{I}_{rg}$-open set G of X such that A⊂G⊂cl*(G)⊂V. By Lemma 2.20, A⊂int*(G). If U=int*(G), then U is an ∗-open set and A⊂U⊂cl*(U)⊂cl*(G)⊂V. Therefore, A⊂U⊂cl*(U)⊂V.

(6)⇒(7). Let A and B be disjoint regular closed subsets of X. Then X−B is a regular open set containing A. By hypothesis, there exists an ∗-open set U of X such that A⊂U⊂cl*(U)⊂X−B. If V=X−cl*(U), then U and V are disjoint ∗-open sets of X such that A⊂U and B⊂V.

(7)⇒(1). Let A and B be disjoint regular closed sets of X. Then there exist disjoint ∗-open sets U and V such that A⊂U and B⊂V. Since $\mathcal{I}$ is completely codense, by Lemma 1.1, $\tau^*\subset\tau^\alpha$ and so U, V∈τ^α. Hence A⊂U⊂int(cl(int(U)))=G and B⊂V⊂int(cl(int(V)))=H. G and H are the required disjoint open sets containing A and B respectively. This proves (1).

If $\mathcal{I}$=N, in the above Theorem 2.21, then $\mathcal{I}_{rg}$-closed sets coincide with rαg-closed sets and so we have the following Corollary 2.22.

Corollary 2.22. *Let (X, τ) be a space. Then the following are equivalent.*

1. *X is mildly normal.*

2. *For disjoint regular closed sets A and B, there exist disjoint $\alpha\hat{g}$-open sets U and V such that A⊂ U and B⊂ V.*

3. *For disjoint regular closed sets A and B, there exist disjoint αg-open sets U and V such that A⊂ U and B⊂ V.*

4. *For disjoint regular closed sets A and B, there exist disjoint rαg-open sets U and V such that A⊂ U and B⊂ V.*

5. *For a regular closed set A and a regular open set V containing A, there exists an rαg-open set U of X such that A⊂ U⊂ cl_α(U)⊂ V.*

6. *For a regular closed set A and a regular open set V containing A, there exists an α-open set U of X such that $A\subset U\subset cl_\alpha(U)\subset V$.*

7. *For disjoint regular closed sets A and B, there exist disjoint α-open sets U and V such that $A\subset U$ and $B\subset V$.*

If $\mathcal{I}=\{\emptyset\}$ in the above Theorem 2.21, we get the following Corollary 2.23.

Corollary 2.23. *Let (X, τ) be a space. Then the following are equivalent.*

1. *X is mildly normal.*

2. *For disjoint regular closed sets A and B, there exist disjoint *g-open sets U and V such that $A\subset U$ and $B\subset V$.*

3. *For disjoint regular closed sets A and B, there exist disjoint g-open sets U and V such that $A\subset U$ and $B\subset V$.*

4. *For disjoint regular closed sets A and B, there exist disjoint rg-open sets U and V such that $A\subset U$ and $B\subset V$.*

5. *For a regular closed set A and a regular open set V containing A, there exists an rg-open set U of X such that $A\subset U\subset cl(U)\subset V$.*

6. *For a regular closed set A and a regular open set V containing A, there exists an open set U of X such that $A\subset U\subset cl(U)\subset V$.*

7. *For disjoint regular closed sets A and B, there exist disjoint open sets U and V such that $A\subset U$ and $B\subset V$.*

3 $\mathcal{I}_{*g}$-regular Spaces

An ideal space (X, τ, $\mathcal{I}$) is said to be an $\mathcal{I}_{*g}$-regular space if for each pair consisting of a point x and a closed set B not containing x, there exist disjoint $\mathcal{I}_{*g}$-open sets U and V such that x∈U and B⊂V. Every regular space is $\mathcal{I}_{*g}$-regular, since every open set is $\mathcal{I}_{*g}$-open. The following Example 3.1 shows that an $\mathcal{I}_{*g}$-regular space need not be regular. Theorem 3.2 gives a characterization of $\mathcal{I}_{*g}$-regular spaces.

Example 3.1. *Consider the ideal space (X, τ, $\mathcal{I}$) of Example 2.1. Then $\emptyset^*=\emptyset$, $(\{b\})^*=\emptyset$, $(\{a, b\})^*=\{a\}$, $(\{b, c\})^*=\{c\}$ and $X^*=\{a, c\}$. Since every ω-open set is $*$-closed, every subset of X is $\mathcal{I}_{*g}$-closed and so every subset of X is $\mathcal{I}_{*g}$-open. This implies that (X, τ, $\mathcal{I}$) is $\mathcal{I}_{*g}$-regular. Now, $\{c\}$ is a closed set not containing $a\in X$, $\{c\}$ and a are not separated by disjoint open sets. So (X, τ, $\mathcal{I}$) is not regular.*

Theorem 3.2. *In an ideal space (X, τ, $\mathcal{I}$), the following are equivalent.*

1. *X is $\mathcal{I}_{*g}$-regular.*

2. *For every closed set B not containing $x\in X$, there exist disjoint $\mathcal{I}_{*g}$-open sets U and V such that $x\in U$ and $B\subset V$.*

3. *For every open set V containing x∈X, there exists an $\mathcal{I}_{*g}$-open set U of X such that x⊂U⊂cl*(U)⊂V.*

Proof. (1) and (2) are equivalent by the definition.

(2)⇒(3). Let V be an open subset such that x∈V. Then X−V is a closed set not containing x. Therefore, there exist disjoint $\mathcal{I}_{*g}$-open sets U and W such that x∈U and X−V⊂W. Now, X−V⊂W implies that X−V⊂int*(W) and so X−int*(W)⊂V. Again, U∩W=∅ implies that U∩int*(W)=∅ and so cl*(U)⊂X−int*(W). Therefore, x∈U⊂cl*(U)⊂V. This proves (3).

(3)⇒(1). Let B be a closed set not containing x. By hypothesis, there exists an $\mathcal{I}_{*g}$-open set U such that x∈U⊂cl*(U)⊂X−B. If W=X−cl*(U), then U and W are disjoint $\mathcal{I}_{*g}$-open sets such that x∈U and B⊂W. This proves (1).

Theorem 3.3. *If (X, τ, $\mathcal{I}$) is an $\mathcal{I}_{*g}$-regular, T_1-space where $\mathcal{I}$ is completely codense, then X is regular.*

Proof. Let B be a closed set not containing x∈X. By Theorem 3.2, there exists an $\mathcal{I}_{*g}$-open set U of X such that x∈U⊂cl*(U)⊂X−B. Since X is a T_1-space, {x} is ω-closed and so {x}⊂int*(U), by Lemma 1.5. Since $\mathcal{I}$ is completely codense, $\tau^*\subset\tau^\alpha$ and so int*(U) and X−cl*(U) are τ^α-open sets. Now, x∈int*(U)⊂int(cl(int(int*(U))))=G and B⊂X−cl*(U)⊂int(cl(int(X−cl*(U))))=H. Then G and H are disjoint open sets containing x and B respectively. Therefore, X is regular.

If $\mathcal{I}$=N in Theorem 3.2, then we have the following Corollary 3.4 which gives characterizations of regular spaces, the proof of which follows from Theorem 3.3.

Corollary 3.4. *If (X, τ) is a T_1-space, then the following are equivalent.*

1. *X is regular.*
2. *For every closed set B not containing x∈X, there exist disjoint $\alpha\hat{g}$-open sets U and V such that x∈U and B⊂V.*
3. *For every open set V containing x∈X, there exists an $\alpha\hat{g}$-open set U of X such that x∈U⊂cl_α(U)⊂V.*

If $\mathcal{I}$={∅} in Theorem 3.2, then we have the following Corollary 3.5 which gives characterizations of regular spaces.

Corollary 3.5. *If (X, τ) is a T_1-space, then the following are equivalent.*

1. *X is regular.*
2. *For every closed set B not containing x∈X, there exist disjoint *g-open sets U and V such that x∈U and B⊂V.*
3. *For every open set V containing x∈X, there exists a *g-open set U of X such that x∈U⊂cl_α(U)⊂V.*

Theorem 3.6. *If every ω-open subset of an ideal space (X, τ, $\mathcal{I}$) is ∗-closed, then (X, τ, $\mathcal{I}$) is $\mathcal{I}_{*g}$-regular.*

Proof. Suppose every ω-open subset of X is $*$-closed. Then by Lemma 1.6, every subset of X is $\mathcal{I}_{*g}$-closed and hence every subset of X is $\mathcal{I}_{*g}$-open. If B is a closed set not containing x, then {x} and B are the required disjoint $\mathcal{I}_{*g}$-open sets containing x and B respectively. Therefore, (X, τ, $\mathcal{I}$) is $\mathcal{I}_{*g}$-regular.

The following Example 3.7 shows that the reverse direction of the above Theorem 3.6 is not true.

Example 3.7. *Consider the real line $\mathcal{R}$ with the usual topology. Let $\mathcal{I}=\{\emptyset\}$. Then $\mathcal{R}$ is regular and hence $\mathcal{I}_{*g}$-regular. But open sets are not closed and hence open sets are not $*$-closed. Thus ω-open sets are not $*$-closed.*

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