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r-τ_{12}-Fuzzy Semiopen Sets and Fuzzy Semi Continuity Mappings in Smooth Bitopological Spaces

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Abstract - In this paper we introduce the notion of r-τ_{12}-fuzzy semiopen sets in smooth supra topological space (X, τ_{12}) which is induced from smooth bitopological space (X, τ_1, τ_2) [1]. We show the present notion of fuzzy semiopen set and the notion of $r(i, j)$-fuzzy semiopen in [25] are independent. In addition by using this new class of r-τ_{12}-fuzzy semiopen sets we constructed a new type of supra fuzzy closure operator which create a new smooth supra topological space τ_{12}^S finer than τ_{12}. Finally, we introduce and study different types of fuzzy semi continuity, which are related to the constructed closure operator and their induced topologies.

Keywords - *Smooth bitopological spaces, smooth supra topological space, r-fuzzy semiopen, supra fuzzy semiclosure operator, fuzzy semi continuous (fuzzy irresolute) maps.*

1 Introduction

The concept of fuzzy sets was introduced by Zadeh in his classical paper [34]. Thereafter many investigation have been carried out in the general theoretical field and also in different application sides, based on this concept. Change [7] used the concept of fuzzy

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sets to introduce fuzzy topological spaces and several authors continued the investigation of such space. A cording to Šostak and Badard, the definition of fuzzy topology is a crisp subfamily of fuzzy sets and fuzziness in the concept of openness of a fuzzy set has not been considered, which seems to be a drawback in the process of fuzzifications of the concept of topological spaces. Therefore, Šostak in [30] introduced a new definition of fuzzy topology as an extension of both crisp topology and Chang's fuzzy topology, in the sense that not only the object were fuzzified, but also the axiomatic. Badard in [5] introduce the concept of smooth structure and gives some rules and shows how such an extension can be realized. Chattopdhyay et al. [8, 9] have redefined the similar concept of fuzzy topology in Šostak sense under the name "gradation of openness". In [10, 23] Ramadan gave a similar definition namely "smooth topology" for lattice $L = [0, 1]$, it has developed in many direction [3, 13, 16, 18, 22, 31, 32]. It worths to mention that the terms fuzzy topology in Šostak sense, gradation of openness and smooth topology are all more or less referring to the same concept. In our paper, we choose the term smooth topology. Lee et al. [20] introduced the concept of smooth bitopological space (smooth bts, for short) as a generalization of smooth topological space and Kandil's fuzzy bitopological space [14].

The so-called supra topology was established by Mashhour et al. [21] (recall that a supra topology on a set X is a collection of subsets of X, which is closed under arbitrary unions). Abd El-Monsef and Ramadan in [2] introduced the concept supra fuzzy topology, followed by Ghanim et al. [12] who introduced the supra fuzzy topology in Šostak sense. Abbas [1] generated the supra fuzzy topology from fuzzy bitopological spaces in Šostak sense as an extension of generated supra fuzzy topology in the sense of Kandil et al. [15].

The concept of fuzzy semiopen sets and fuzzy semicontinuous mapping in fuzzy topological spaces was studied by Azad [4]. Kumar in [29] generalize the concepts of fuzzy semiopen sets, fuzzy semi-continuous mappings into fuzzy bitopological spaces. In [17] and [19] the authors introduced the notion of fuzzy r-semiopen sets and fuzzy r-semicontinuous maps in smooth topological space which are generalization of fuzzy semiopen sets and fuzzy semi-continuous maps in Chang's fuzzy topology. In [25] Ramadan and Abbas introduced the notion of r-fuzzy semiopen in smooth bts. And in [11] El-sheikh characterized the notion of r-fuzzy semiopen sets in [25] and generalized the notions that introduced in [24], [28], [29] to smooth bts. Recently in [33] we introduced the concept of generalized fuzzy closed set in smooth bts.

In this paper we define $r - \tau_{12}$-fuzzy semiopen sets in smooth supra topological space (X, τ_{12}) induced by smooth fuzzy bitopological space (X, τ_1, τ_2) and we study some properties of them, we show the present notion of fuzzy semiopen set and the notion of $r(i, j)$-fuzzy semiopen in [25] are independent. By using this new class of r-τ_{12}-fuzzy semiopen sets we define fuzzy semiclosure operator in smooth supra topological space associated with smooth fuzzy bitopological space, we show that it is supra fuzzy closure operator. Moreover it create a smooth supra topology which is finer than a given smooth supra topology τ_{12} induced by τ_1, τ_2. We investigate some properties of the supra fuzzy semiclosure operator. Finally, we use smooth supra topological spaces which are induced from smooth bitopological spaces and constructed supra semiclosure operators and their induced topologies to introduce and study fuzzy semi continuous (resp., open, closed) mappings and fuzzy irresolute, fuzzy irresolute open (resp., closed) mappings in smooth bitopological spaces.

2 Preliminary

Throughout this paper, let X be a nonempty set, $I = [0, 1]$, $I_0 = (0, 1]$. A fuzzy set μ of X is a mapping from X to I, the family of all fuzzy sets of X is denoted by I^X. For $\alpha \in I$, $\bar{\alpha}(x) = \alpha$ for all $x \in X$. By $\bar{0}$ and $\bar{1}$ we denote constant maps on X with value 0 and 1, respectively. For any fuzzy set $\mu \in I^X$ the complement of μ, denoted by $\bar{1} - \mu$. For $x \in X$ and $t \in I_0$, a fuzzy point x_t is defined by t if $x = y$ and 0 otherwise, for all $y \in X$. Let $Pt(X)$ be the family of all fuzzy points in X. A fuzzy point x_t is said to be belong to a fuzzy set λ, denoted $x_t \in \lambda$ if and only if $\lambda(x) \geq t$. For $\mu, \lambda \in I^X$, μ is called quasi-coincident with λ, denoted by $\mu \, q \, \lambda$, if $\mu(x) + \lambda(x) > 1$ for some $x \in X$, otherwise we write $\mu \, \bar{q} \, \lambda$. And $\mu \, q \, \lambda$ if and only if $\exists x_t; x_t \in \mu, x_t \, q \, \lambda$. FP (resp., FP^*) stand for fuzzy pairwise (resp., fuzzyP^*). The indices $i, j \in \{1, 2\}$ and $i \neq j$.

Definition 2.1. *[5, 8, 23, 30] A smooth topology on X is a mapping $\tau : I^X \to I$ which satisfies the following properties.*

1. $\tau(\bar{0}) = \tau(\bar{1}) = 1$,

2. $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, $\forall \, \mu_1, \mu_2 \in I^X$,

3. $\tau(\bigvee_{i \in J} \mu_i) \geq \bigwedge_{i \in J} \tau(\mu_i)$, for any $\{\mu_i : i \in J\} \subseteq I^X$.

The pair (X, τ) is called a smooth topological space. For $r \in I_0$, μ is r-open fuzzy set of X if $\tau(\mu) \geq r$, and μ is r-closed fuzzy set of X if $\tau(\bar{1} - \mu) \geq r$.

In [30], Šostak used the term "fuzzy topology" and in [8], Chattopadhyay et al. used the term "gradation of openness" for a smooth topology τ.

If τ satisfies conditions (1) and (3), then τ is said to be a smooth supra topology and (X, τ) is said to be a smooth supra topological space [12].

Definition 2.2. *[20, 30] A triple (X, τ_1, τ_2) consisting of the set X endowed with smooth topologies τ_1 and τ_2 on X is called a smooth bitopological space (smooth bts, for short). For $\lambda \in I^X$ and $r \in I_0$, r-τ_i-open (respectively, closed) fuzzy set denotes the r-open (respectively, closed) fuzzy set in (X, τ_i), for $i = 1, 2$.*

The concepts of fuzzy closure (resp., interior) for any fuzzy set in smooth topological space is given in the following definition.

Definition 2.3. *[9] Let (X, τ) be a smooth topological space. A fuzzy closure is a mapping $C_\tau : I^X \times I_0 \to I^X$ such that*

$$C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X | \; \mu \geq \lambda, \; \tau(\bar{1} - \mu) \geq r\}, \forall \lambda \in I^X \text{ and } \forall r \in I_0. \quad (1)$$

And, a fuzzy interior is a mapping $I_\tau : I^X \times I_0 \to I^X$ defined as:

$$I_\tau(\lambda, r) = \bigvee\{\mu \in I^X | \; \mu \leq \lambda, \; \tau(\mu) \geq r\}, \forall \lambda \in I^X \text{ and } \forall r \in I_0, \quad (2)$$

satisfies

$$I_\tau(\bar{1} - \lambda, r) = \bar{1} - C_\tau(\lambda, r). \quad (3)$$

Definition 2.4. *[9] A mapping $C : I^X \times I_0 \to I^X$ is called a fuzzy closure operator if, for $\lambda, \mu \in I^X$ and $r, s \in I_0$, the mapping C satisfies the following conditions.*

(C1) $C(\bar{0}, r) = \bar{0}$,

(C2) $\lambda \leq C(\lambda, r)$,

(C3) $C(\lambda, r) \vee C(\mu, r) = C(\lambda \vee \mu, r)$,

(C4) $C(\lambda, r) \leq C(\lambda, s)$ *if* $r \leq s$,

(C5) $C(C(\lambda, r), r) = C(\lambda, r)$.

The fuzzy closure operator C generates a smooth topology $\tau_C : I^X \longrightarrow I$ defined as follows

$$\tau_C(\lambda) = \bigvee\{r \in I |\ C(\bar{1} - \lambda, r) = \bar{1} - \lambda\} \tag{4}$$

such that $C = C_{\tau_C}$. If the map $C : I^X \times I_0 \to I^X$ satisfied the conditions (C1) − (C4) only, then the pair (X, C) is called fuzzy closure space, such that the fuzzy closure operator C and the fuzzy closure C_{τ_C} are not coincide.

If C satisfies conditions (C1),(C2),(C4),(C5) and the following inequality

$(C3)^*$ $\quad C(\lambda, r) \vee C(\mu, r) \leq C(\lambda \vee \mu, r)$,

then C is called supra fuzzy closure operator on X [1]. and it generates a smooth supra topology $\tau_C : I^X \longrightarrow I$ as (4)

By applying (3) in Definition 2.4, the definitions of fuzzy interior operator and supra fuzzy interior operator are obtained.

The following theorems show how to generate a supra fuzzy closure operator from a smooth bts (X, τ_1, τ_2).

Theorem 2.5. *[1] Let (X, τ_1, τ_2) be a smooth bts. For each $\lambda \in I^X, r \in I_0$.*

1. *The mapping $C_{12} : I^X \times I_0 \to I^X$ such that $C_{12}(\lambda, r) = C_{\tau_1}(\lambda, r) \wedge C_{\tau_2}(\lambda, r)$ is a supra fuzzy closure operator on X.*

2. *The mapping $I_{12} : I^X \times I_0 \to I^X$ which is defined as $I_{12}(\lambda, r) = I_{\tau_1}(\lambda, r) \vee I_{\tau_2}(\lambda, r)$ is a supra fuzzy interior operator on X, satisfies $I_{12}(\bar{1} - \lambda, r) = \bar{1} - C_{12}(\lambda, r)$.*

Theorem 2.6. *[1] Let (X, τ_1, τ_2) be a smooth bts and (X, C_{12}) be a supra fuzzy closure space. Define the mapping $\tau_S : I^X \to I$ on X by*

$$\tau_S(\lambda) = \bigvee\{\tau_1(\lambda_1) \wedge \tau_2(\lambda_2) : \lambda = \lambda_1 \vee \lambda_2, \ \ \lambda_1, \lambda_2 \in I^X\}$$

where $\bigvee$ is taken over all families $\{\lambda_1, \lambda_2 \in I^X : \lambda = \lambda_1 \vee \lambda_2\}$. Then

1. $\tau_S = \tau_{C_{12}}$ is the coarsest smooth supra topology on X which is finer than τ_1 and τ_2.

2. $C_{12} = C_{\tau_s} = C_{\tau_{C_{12}}}$.

Remark 2.7. *In this paper we will denote to $\tau_{C_{12}}$ by τ_{12}.*

Definition 2.8. *[16] A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ from a smooth bts (X, τ_1, τ_2) to another smooth bts (Y, τ_1^*, τ_2^*) is said to be*

1. *FP-continuous if and only if $\tau_i(f^{-1}(\mu)) \geq \tau_i^*(\mu)$ for each $\mu \in I^Y$ and $i = 1, 2$, (or in other words a mapping f is said to be FP-continuous iff $f : (X, \tau_i) \longrightarrow (Y, \tau_i^*)$) is f-continuous, $i = 1, 2$).*

2. *FP-open (resp., closed) if and only if $\tau_i^*(f(\mu)) \geq \tau_i(\mu)$ (resp., $\tau_i^*(f(\bar{1}-\mu)) \geq \tau_i(\bar{1}-\mu)$) for each $\mu \in I^X$ and $i = 1, 2$.*

3. *FP^*-continuous (resp., FP^*-open, FP^*-closed) if and only if $f : (X, \tau_{12}) \longrightarrow (Y, \tau_{12}^*)$ is F-continuous (resp., F-open, F-closed) [26].*

Definition 2.9. *[17, 19] Let (X, τ) be a smooth topological space, let $\lambda \in I^X$ and $r \in I_0$. Then λ is said to be*

1. *r-fuzzy semiopen set (r-fso set, for short) if there exists r-open fuzzy set μ in X such that $\mu \leq \lambda \leq C_\tau(\mu, r)$.*

2. *r-fuzzy semiclosed set (r-fsc set, for short) if there exists r-closed fuzzy set μ in X such that $I_\tau(\mu, r) \leq \lambda \leq \mu$.*

Definition 2.10. *[19] Let (X, τ) be a smooth topological space. For each $\lambda \in I^X$ and for each $r \in I_0$, the r-fuzzy semiclosure of λ is defined by*
$SC_\tau(\lambda, r) = \bigwedge\{\rho \in I^X \mid \rho \geq \lambda,\ \rho \text{ is } r\text{-fsc set }\}$,
and the r-fuzzy semiinterior of λ is defined by
$SI_\tau(\lambda, r) = \bigvee\{\rho \in I^X \mid \rho \leq \lambda,\ \rho \text{ is } r\text{-fso set }\}$.

Obviously $SC_\tau(\lambda, r)$ is the smallest r-fuzzy semiclosed set which contains λ and $SI_\tau(\lambda, r)$ is the greatest r-fuzzy semiopen set which contained in λ. Also, $SC_\tau(\lambda, r) = \lambda$ for any r-fuzzy semiclosed set μ and $SI_\tau(\mu, r) = \mu$ for any r-fuzzy semiopen set μ. Moreover we have

$$I_\tau(\lambda, r) \leq SI_\tau(\lambda, r) \leq \lambda \leq SC_\tau(\lambda, r) \leq C_\tau(\lambda, r).$$

It is obvious that any r-open (resp., closed) fuzzy set is r-fuzzy semiopen (resp., semiclosed) set. But the converse need not true. The intersection (union) of any two r-fuzzy semiopen (resp., r-fuzzy semiclosed) sets need not to be r-fuzzy semiopen (resp., r-fuzzy semiclosed).

Theorem 2.11. *[17] Let (X, τ) be a smooth topological space. For each $\lambda \in I^X$, and $r \in I_0$ it satisfies the following statements.*
1. *$SC_\tau(\bar{0}, r) = \bar{0}$.*
2. *$SC_\tau(SC_\tau(\lambda, r), r) = SC_\tau(\lambda, r)$.*
3. *$SI_\tau(SI_\tau(\lambda, r), r) = SI_\tau(\lambda, r)$.*
4. *$SI_\tau(\bar{1} - \lambda, r) = \bar{1} - SC_\tau(\lambda, r)$.*

Definition 2.12. *[11, 25] Let (X, τ_1, τ_2) be a smooth bts, let $\lambda \in I^X$ and $r \in I_0$. Then λ is called*

1. *$r(i, j)$-fuzzy semiopen set ($r(i, j)$-fso, for short) if there exists $\upsilon \in I^X$ with $\tau_i(\upsilon) \geq r$ and $\upsilon \leq \lambda \leq C_{\tau_j}(\upsilon, r)$, $i, j = 1, 2$, $i \neq j$.*

2. *$r(i, j)$-fuzzy semiclosed set ($r(i, j)$-fsc, for short) if there exists $\upsilon \in I^X$ with $\tau_i(\bar{1} - \upsilon) \geq r$ and $I_{\tau_j}(\upsilon, r) \leq \lambda \leq \upsilon$, $i, j = 1, 2$, $i \neq j$.*

Definition 2.13. *Let (X, τ) and (Y, τ^*) be smooth topological spaces. A mapping $f : (X, \tau) \longrightarrow (Y, \tau^*)$ is said to be*

1. *fuzzy semicontinuous (fs-continuous, for short) iff $f^{-1}(\mu)$ is r-fso set in X for each $\mu \in I^Y$, $\tau^*(\mu) \geq r$ [17].*

2. *fuzzy semiopen (fs-open, for short)(resp., semiclosed (fs-closed, for short)) iff $f(\lambda)$ is r-fso (resp., r-fsc) set in Y for each $\mu \in I^X$, $\tau(\mu) \geq r$ (resp., $\tau(\bar{1} - \mu) \geq r$) [17].*

3. *fuzzy irresolute (f-irresolute, for short) iff $f^{-1}(\mu)$ is r-fso set in X for each μ is r-fso set in Y [24].*

4. *fuzzy irresolute open (f-irresolute open, for short) (resp., irresolute closed (f-irresolute closed, for short)) iff $f(\lambda)$ is r-fso(resp., r-fsc) set in Y for each μ is r-fso(resp., r-fsc) set in X [24].*

3 C_{12}^S-supra fuzzy semiclosure operator

In this section we use smooth supra topological space (X, τ_{12}) which induced from smooth bts (X, τ_1, τ_2), to introduce and study the concept of fuzzy semiopen sets in smooth bts (X, τ_1, τ_2). By using this new class of fuzzy semiopen sets we introduce the supra fuzzy semiclosure operator.

Definition 3.1. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then λ is called*

1. *r-τ_{12}-fuzzy semiopen set (r-τ_{12}-fso set, for short) in X if there is $\mu \in I^X$ with $\tau_{12}(\mu) \geq r$ such that $\mu \leq \lambda \leq C_{12}(\mu, r)$.*

2. *r-τ_{12}-fuzzy semiclosed set (r-τ_{12}-fsc set , for short) in X if there is $\mu \in I^X$ with $\tau_{12}(\bar{1} - \mu) \geq r$ such that $I_{12}(\mu, r) \leq \lambda \leq \mu$.*

Proposition 3.2. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$.*
1. *If $\tau_{12}(\lambda) \geq r$ then λ is r-τ_{12}-fso set.*
2. *If $\tau_{12}(\bar{1} - \lambda) \geq r$ then λ is r-τ_{12}-fsc set.*
3. *If $\tau_1(\lambda) \geq r$ or $\tau_2(\lambda) \geq r$ then λ is r-τ_{12}-fso set.*
4. *If $\tau_1(\bar{1} - \lambda) \geq r$ or $\tau_2(\bar{1} - \lambda) \geq r$ then λ is r-τ_{12}-fsc set.*

Proof. (1) Let $\lambda \in I^X$ such that $\tau_{12}(\lambda) \geq r$. Since, $\lambda \leq \lambda$ and $\lambda \leq C_{12}(\lambda, r)$ then $\lambda \leq \lambda \leq C_{12}(\lambda, r)$, implies λ is r-τ_{12}-fso set. To proof (2), let $\lambda \in I^X$ such that $\tau_{12}(\bar{1} - \lambda) \geq r$. Since $I_{12}(\lambda, r) \leq \lambda \leq \lambda$ then λ is r-τ_{12}-fsc set. Finally, the proof of (3) and (4) are obtained from Theorem 2.6 that is τ_{12} finer than τ_i, $i = 1, 2$ and then by using part (1) and (2) respectively. □

Remark 3.3. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$.*
1. *The converse of Proposition 3.2 is not true for all it's parts.*
2. *If λ is r-fso set in (X, τ_1) or (X, τ_2) then λ need not r-τ_{12}-fso set, and conversely.*
3. *If λ is $r(i, j)$-fso set in (X, τ_1, τ_2) then λ need not r-τ_{12}-fso set, and conversely (that is mean the concept of $r(i, j)$-fso set and r-τ_{12}-fso set are independent).*

Now, we give an Examples to explain Remark 3.3.

Example 3.4. *Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2 \in I^X$ as follows:*

$$\lambda_1 = a_{\frac{1}{2}} \vee b_{\frac{1}{3}} \quad , \qquad \lambda_2 = a_{\frac{1}{3}} \vee b_{\frac{1}{2}}$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ 0.2 & \text{if } \lambda = \lambda_1 \\ 0 & o.w \end{cases} , \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ 0.3 & \text{if } \lambda = \lambda_2 \\ 0 & o.w \end{cases}$$

The associated smooth supra topological space of (X, τ_1, τ_2), is defined as follows: $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ 0.2 & \text{if } \lambda = \lambda_1 \\ 0.3 & \text{if } \lambda = \lambda_2 \\ 0.2 & \text{if } \lambda = \lambda_1 \vee \lambda_2 \\ 0 & o.w \end{cases}$$

To show the converse of Proposition 3.2 part (1) is not true, let $\rho = a_{0.4} \vee b_{\frac{1}{2}} \in I^X$, it is clear that ρ is 0.3-τ_{12}-fso set, since there exists $\lambda_2 \in I^X$ such that $\tau_{12}(\lambda_2) \geq 0.3$ and $\lambda_2 \leq \rho \leq C_{12}(\lambda_2, 0.3) = a_{\frac{2}{3}} \vee b_{\frac{1}{2}}$. But ρ is not 0.3-open fuzzy set since $\tau_{12}(\rho) = 0 \not\geq 0.3$. Also, the converse of Proposition 3.2 part (3) is not true, since there exists $\rho = a_{0.4} \vee b_{\frac{1}{2}} \in I^X$ is 0.3-τ_{12}-fso set, but $\tau_1(\rho) = 0 \not\geq 0.3$ and $\tau_2(\rho) = 0 \not\geq 0.3$.

Now, to show part (2) in Remark 3.3 , let $\eta = a_{\frac{1}{2}} \vee b_{\frac{2}{3}} \in I^X$ it's clear that η is 0.2-fso set in (X, τ_1). But not 0.2-τ_{12}-fso set in (X, τ_{12}).

Finally, to explain part (3) in Remark 3.3 , let $\upsilon = a_{\frac{2}{3}} \vee b_{\frac{1}{2}} \in I^X$ it is clear that υ is 0.2(1, 2)-fso set in (X, τ_1, τ_2). But υ is not 0.2-τ_{12}-fso set, since $\forall\ \lambda \in I^X$; $\tau_{12}(\lambda) \geq 0.2$ such that $\lambda \leq \eta$, implies $\eta \not\leq C_{12}(\lambda, 0.2)$.

Example 3.5. *Let $X = \{a, b, c\}$. Define $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in I^X$ as follows:*

$$\lambda_1 = a_1, \quad \lambda_2 = b_1 \vee c_1, \quad \lambda_3 = b_1, \quad \lambda_4 = a_1 \vee c_1$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ \frac{1}{4} & \text{if } \lambda = \lambda_1 \\ \frac{1}{2} & \text{if } \lambda = \lambda_2 \\ 0 & o.w \end{cases} , \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ \frac{1}{4} & \text{if } \lambda = \lambda_3 \\ \frac{1}{2} & \text{if } \lambda = \lambda_4 \\ 0 & o.w \end{cases}$$

The associated smooth supra topological space of (X, τ_1, τ_2), is defined as follows: $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ \frac{1}{4} & \text{if } \lambda = \lambda_1,\ \lambda_3 \\ \frac{1}{2} & \text{if } \lambda = \lambda_2,\ \lambda_4 \\ \frac{1}{4} & \text{if } \lambda = \lambda_1 \vee \lambda_3 = a_1 \vee b_1 \\ 0 & o.w \end{cases}$$

To explain the conversely of part (2) and (3), respectively in Remark 3.3. Consider $\lambda = a_1 \vee b_1$ is $\frac{1}{4}$-τ_{12}-fso set but it is not $\frac{1}{4}$-τ_i-fso set and not $r(i,j)$-fso set in (X, τ_1, τ_2), $i = 1, 2$.

Theorem 3.6. *Let (X, τ_1, τ_2) be a smooth bts, let $\lambda \in I^X$ and $r \in I_0$. Then the following statements are equivalent.*

1. *λ is r-τ_{12}-fso set.*
2. *$\lambda \leq C_{12}(I_{12}(\lambda, r), r)$.*
3. *$C_{12}(\lambda, r) = C_{12}(I_{12}(\lambda, r), r)$.*
4. *$\bar{1} - \lambda$ is r-τ_{12}-fsc set.*
5. *$I_{12}(C_{12}(\bar{1} - \lambda, r), r) \leq \bar{1} - \lambda$.*
6. *$I_{12}(\bar{1} - \lambda, r) = I_{12}(C_{12}(\bar{1} - \lambda, r), r)$.*

Proof. (1) $\Longrightarrow$ (2) Let λ is r-τ_{12}-fso set. Then there exists $\mu \in I^X$, with $\tau_{12}(\mu) \geq r$ such that $\mu \leq \lambda \leq C_{12}(\mu, r)$. Since $\mu \leq \lambda$ then $I_{12}(\mu, r) \leq I_{12}(\lambda, r)$, this mean $\mu \leq I_{12}(\lambda, r)$, implies $C_{12}(\mu, r) \leq C_{12}(I_{12}(\lambda, r), r)$ and since $\lambda \leq C_{12}(\mu, r)$. Then, we have $\lambda \leq C_{12}(I_{12}(\lambda, r), r)$.

(2) $\Longrightarrow$ (3) Since $\lambda \leq C_{12}(I_{12}(\lambda, r), r)$. Then, we have $C_{12}(\lambda, r) \leq C_{12}(I_{12}(\lambda, r), r)$. On the other hand since $I_{12}(\lambda, r) \leq \lambda$, then $C_{12}(I_{12}(\lambda, r), r) \leq C_{12}(\lambda, r)$. Hence $C_{12}(\lambda, r) = C_{12}(I_{12}(\lambda, r), r)$.

(3) $\Longrightarrow$ (1) Let $\mu = I_{12}(\lambda, r)$ implies $\mu \leq \lambda$, then we have $\mu \leq \lambda \leq C_{12}(\lambda, r) = C_{12}(I_{12}(\lambda, r), r) = C_{12}(\mu, r)$, implies $\mu \leq \lambda \leq C_{12}(\mu, r)$. Hence λ is r-τ_{12}-fso set.

The implications, (1) $\Longleftrightarrow$ (4), (2) $\Longleftrightarrow$ (5), (3) $\Longleftrightarrow$ (6) follow immediately by taking the complement of two sides. □

Theorem 3.7. *Let (X, τ_1, τ_2) be a smooth bts and $r \in I_0$. Then*

1. *any union of r-τ_{12}-fso sets is r-τ_{12}-fso set.*
2. *any intersection of r-τ_{12}-fsc sets is r-τ_{12}-fsc set.*

Proof. (1) Let $\{\lambda_\alpha | \alpha \in \Lambda\}$ be a family of r-τ_{12}-fso sets. Then, for each $\alpha \in \Lambda$ there exists $\mu_\alpha \in I^X$ with $\tau_{12}(\mu_\alpha) \geq r$ such that $\mu_\alpha \leq \lambda_\alpha \leq C_{12}(\mu_\alpha, r)$. Since $\{\mu_\alpha | \alpha \in \Lambda\}$ is r-$\tau_{12}-$fuzzy open sets, then $\tau_{12}(\bigvee_{\alpha \in \Lambda} \mu_\alpha) \geq \bigwedge_{\alpha \in \Lambda} \tau_{12}(\mu_\alpha)$. Then, $\bigvee_{\alpha \in \Lambda} \mu_\alpha$ is r-$\tau_{12}-$fuzzy open set. Let $\mu = \bigvee_{\alpha \in \Lambda} \mu_\alpha$ such that $\bigvee_{\alpha \in \Lambda} \mu_\alpha \leq \bigvee_{\alpha \in \Lambda} \lambda_\alpha \leq \bigvee_{\alpha \in \Lambda} C_{12}(\mu_\alpha, r) \leq C_{12}(\bigvee_{\alpha \in \Lambda} \mu_\alpha)$. Implies $\mu \leq \bigvee_{\alpha \in \Lambda} \lambda_\alpha \leq C_{12}(\mu, r)$. Thus $\bigvee_{\alpha \in \Lambda} \lambda_\alpha$ r-τ_{12}-fso set.

(2) Let $\{\lambda_\alpha | \alpha \in \Lambda\}$ be a family of r-τ_{12}-fsc sets. For each $\alpha \in \Lambda$, since λ_α is r-τ_{12}-fsc set, then $\bar{1} - \lambda_\alpha$ is r-τ_{12}-fso set, from part (1) we get $\bigvee_{\alpha \in \Lambda} \bar{1} - \lambda_\alpha$ is r-τ_{12}-fso set, implies $\bar{1} - (\bigvee_{\alpha \in \Lambda} \bar{1} - \lambda_\alpha) = \bigwedge_{\alpha \in \Lambda} \lambda_\alpha$ is r-τ_{12}-fsc set. □

Remark 3.8. *1. The intersection of any two r-τ_{12}-fso sets need not to be r-τ_{12}-fso set .*

2. The union of any two r-τ_{12}-fsc sets need not to be r-τ_{12}-fsc set .

In Example 3.4, let $\rho_1 = a_{0.4} \vee b_{\frac{1}{2}}$ and $\rho_2 = a_{\frac{1}{2}} \vee b_{0.4} \in I^X$, such that ρ_1 and ρ_2 are 0.2-τ_{12}-fso sets, since $\exists\ \lambda_2 \leq \rho_1 \leq C_{12}(\lambda_2, 0.2) = a_{\frac{1}{2}} \vee b_{\frac{1}{2}}$ and $\exists\ \lambda_1 \leq \rho_2 \leq C_{12}(\lambda_1, 0.2) = a_{\frac{1}{2}} \vee b_{\frac{1}{2}}$. But $\rho_1 \wedge \rho_2 = a_{0.4} \vee b_{0.4}$ is not 0.2-τ_{12}-fso set, since $\bar{0}$ is the only 0.2-τ_{12}-open fuzzy set such that $\bar{0} \leq \rho_1 \wedge \rho_2 \nleq C_{12}(\bar{0}, 0.2) = \bar{0}$.

Proposition 3.9. *Let (X, τ_1, τ_2) be a smooth bts, let $\lambda \in I^X$ and $r \in I_0$. Then*
1. *$I_{12}(\lambda, r)$ is r-τ_{12}-fso set.*
2. *$C_{12}(\lambda, r)$ is r-τ_{12}-fsc set.*
3. *if λ is r-τ_{12}-fso set and $I_{12}(\lambda, r) \leq \mu \leq C_{12}(\lambda, r)$, then μ is r-τ_{12}-fso set.*
4. *if λ is r-τ_{12}-fsc set and $I_{12}(\lambda, r) \leq \mu \leq C_{12}(\lambda, r)$, then μ is r-τ_{12}-fsc set.*

Proof. The proof of part(1) and (2) are direct.

(3) Let λ is r-τ_{12}-fso set, implies there exists $\eta \in I^X$ with $\tau_{12}(\eta) \geq r$ such that $\eta \leq \lambda \leq C_{12}(\eta, r)$. It implies $\eta = I_{12}(\eta, r) \leq I_{12}(\lambda, r)$ and since $\lambda \leq C_{12}(\eta, r)$, then $C_{12}(\lambda, r) \leq C_{12}(\eta, r)$. Thus, $\eta \leq \mu \leq C_{12}(\eta, r)$. therefore, μ is r-τ_{12}-fso set.

(4) Let λ is r-τ_{12}-fsc set implies $\bar{1} - \lambda$ is r-τ_{12}-fso set. Since $I_{12}(\lambda, r) \leq \mu \leq C_{12}(\lambda, r)$, then by take the complement of the last inequality we get, $I_{12}(\bar{1} - \lambda, r) \leq \bar{1} - \mu \leq C_{12}(\bar{1} - \lambda, r)$. And by applying (3) we have $\bar{1} - \mu$ is r-τ_{12}-fso set which is mean μ is r-τ_{12}-fsc set. □

Next the concepts of fuzzy semiclosure (resp., semiinterior) are given in following definition.

Definition 3.10. *Let (X, τ_1, τ_2) be a smooth bts, for $\lambda \in I^X$ and $r \in I_0$. The fuzzy semiclosure $C^S_{12}(\lambda, r)$ of a fuzzy set λ is defined by*

$$C^S_{12}(\lambda, r) = \bigwedge\{\rho \in I^X \mid \rho \geq \lambda,\ \rho \textit{ is } r\text{-}\tau_{12}\text{-}fsc \textit{ set}\},$$

and the fuzzy semiinterior $I^S_{12}(\lambda, r)$ of a fuzzy set λ is defined by

$$I^S_{12}(\lambda, r) = \bigvee\{\rho \in I^X \mid \rho \leq \lambda,\ \rho \textit{ is } r\text{-}\tau_{12}\text{-}fso \textit{ set}\}.$$

The following proposition gives the basic properties of C^S_{12} and I^S_{12}.

Proposition 3.11. *Let (X, τ_1, τ_2) be a smooth bts, let $\mu, \lambda \in I^X$ and $r \in I_0$. Then*
1. *$I^S_{12}(\bar{1} - \lambda, r) = \bar{1} - C^S_{12}(\lambda, r)$.*
2. *if $\mu \leq \lambda$, then $I^S_{12}(\mu, r) \leq I^S_{12}(\lambda, r)$.*
3. *if $\mu \leq \lambda$, then $C^S_{12}(\mu, r) \leq C^S_{12}(\lambda, r)$.*
4. *λ is r-τ_{12}-fso set if and only if $I^S_{12}(\lambda, r) = \lambda$.*
5. *λ is r-τ_{12}-fsc set if and only if $C^S_{12}(\lambda, r) = \lambda$.*

Proof. (1) From Definition 3.10, we have

$$\begin{aligned} \bar{1} - C^S_{12}(\lambda, r) &= \bar{1} - \bigwedge\{\rho \in I^X \mid \rho \geq \lambda,\ \rho \textit{ is } r\text{-}\tau_{12}\text{-}fsc \textit{ set}\} \\ &= \bigvee\{\bar{1} - \rho \in I^X \mid \bar{1} - \rho \leq \bar{1} - \lambda,\ \bar{1} - \rho \textit{ is } r\text{-}\tau_{12}\text{-}fso \textit{ set}\} \\ &= I^S_{12}(\bar{1} - \lambda, r) \end{aligned}$$

(2) Let $x_t \in I^S_{12}(\mu, r)$. Then there exists $\rho \in I^X$ such that $x_t \in \rho$, $\rho \leq \mu$ and ρ is r-τ_{12}-fso set. Since $\rho \leq \mu \leq \lambda$, then there exists ρ is r-τ_{12}-fso set such that $\rho \leq \lambda$ and $x_t \in \rho$. Hence $x_t \in I^S_{12}(\lambda, r)$.

(3) Follows by taking the complement of (2) and using (1).

(4) Suppose λ is r-τ_{12}-fso set. From Definition of $I^S_{12}(\lambda, r)$, we have, $I^S_{12}(\lambda, r) \leq \lambda$. On the other hand. Since λ is r-τ_{12}-fso set and $\lambda \leq \lambda$, then $\lambda \leq I^S_{12}(\lambda, r)$. Thus $I^S_{12}(\lambda, r) = \lambda$.

Conversely, follows direct from Definition of $I^S_{12}(\lambda, r)$.

(5) Taking $\bar{1} - \lambda$ as a r-τ_{12}-fso then apply (4), we get the result. □

Definition 3.12. *Let (X, τ_1, τ_2) be a smooth bts, let $x_t \in Pt(X)$ and $r \in I_0$. Then*
$Q_{\tau_{12}}(x_t, r) = \{\mu \in I^X |\ x_t\ q\ \mu, \tau_{12}(\mu) \geq r\}$.
$\mathcal{N}^S_{\tau_{12}}(x_t, r) = \{\mu \in I^X \mid x_t \in \mu,\ \mu \textit{ is } r\text{-}\tau_{12}\text{-}fso \textit{ set}\}$.
$\mathcal{Q}^S_{\tau_{12}}(x_t, r) = \{\mu \in I^X \mid x_t\ q\ \mu,\ \mu \textit{ is } r\text{-}\tau_{12}\text{-}fso \textit{ set}\}$.

Theorem 3.13. *A fuzzy set λ in a smooth bts (X, τ_1, τ_2) is r-τ_{12}-fso set if and only if for each fuzzy point $x_t \in \lambda$ there exists $\mu \in \mathcal{N}^S_{\tau_{12}}(x_t, r)$ such that $\mu \leq \lambda$.*

Proof. ($\Longrightarrow$) It is obvious.

($\Longleftarrow$) Since, for each fuzzy point $x_t \in \lambda$ there exists $\mu_i \in \mathcal{N}^S_{\tau_{12}}(x_t, r)$ such that $\mu_i \leq \lambda$, then from Theorem 3.7 part(1), $\bigvee \mu_i$ is r-τ_{12}-fso set. That means for each fuzzy point $x_t \in \lambda$ there exists $\mu_i \in \mathcal{N}^S_{\tau_{12}}(x_t, r)$ such that $x_t \in \mu_i \leq \bigvee \mu_i$, implies $\lambda \leq \bigvee \mu_i$. On the other hand let $x_s \in \bigvee \mu_i$ implies $x_s \in \mu_i$ for some i, and since $\mu_i \leq \lambda$, then $x_s \in \mu_i \leq \lambda$, this yields $\bigvee \mu_i \leq \lambda$. Thus $\lambda = \bigvee \mu_i$ and hence λ is r-τ_{12}-fso set. □

Theorem 3.14. *Let (X, τ_1, τ_2) be a smooth bts, let $\lambda \in I^X$ and $r \in I_0$. Then $x_t \in C^S_{12}(\lambda, r)$ if and only if for each $\mu \in \mathcal{Q}^S_{\tau_{12}}(x_t, r)$, $\mu\ q\ \lambda$.*

Proof. Let $x_t \in C^S_{12}(\lambda, r)$ and suppose there exists $\mu \in \mathcal{Q}^S_{\tau_{12}}(x_t, r)$ such that $\mu\ \bar{q}\ \lambda$, implies $\lambda \leq \bar{1} - \mu$ and since μ is r-τ_{12}-fso set, then $\bar{1} - \mu$ is r-τ_{12}-fsc set, implies $C^S_{12}(\lambda, r) \leq C^S_{12}(\bar{1} - \mu, r) = \bar{1} - \mu$, this yields $C^S_{12}(\lambda, r)(x) \leq \bar{1} - \mu(x) < t$, thus $x_t \notin C^S_{12}(\lambda, r)$ which is a contradiction.

Conversely, Suppose $x_t \notin C^S_{12}(\lambda, r)$, then $\exists\ \rho$ is r-τ_{12}-fsc set such that $\rho \geq \lambda$ and $x_t \notin \rho$, implies $x_t\ q\ \bar{1} - \rho$, and since $\rho \geq \lambda$ then, $\lambda\ \bar{q}\ \bar{1} - \rho$ such that $\bar{1} - \rho$ is r-τ_{12}-fso set. That mean, there exists $\bar{1} - \rho \in \mathcal{Q}^S_{\tau_{12}}(x_t, r)$ such that $\bar{1} - \rho\ \bar{q}\ \lambda$ which is a contradiction. Hence $x_t \in C^S_{12}(\lambda, r)$. □

Theorem 3.15. *Let (X, τ_1, τ_2) be a smooth bts. Then*

1. C^S_{12} is a supra fuzzy closure operator such that $C^S_{12}(\lambda, r) \leq C_{12}(\lambda, r)$, for all $\lambda \in I^X$ and $r \in I_0$.

2. I^S_{12} is a supra fuzzy interior operator such that $I_{12}(\lambda, r) \leq I^S_{12}(\lambda, r)$, for all $\lambda \in I^X$ and $r \in I_0$.

Proof. We show (1) and in a similar way one can obtain (2). To prove (1), we need to satisfy conditions $(C1), (C2), (C3)^*, (C4)$ and $(C5)$ in Definition 2.4.

$(C1)$ Since $\bar{0}$ is r-τ_{12}-fsc, then by Proposition 3.11 part (5), $C^S_{12}(\bar{0}, r) = \bar{0}$.

$(C2)$ Follows immediately from the Definition of C^S_{12}.

$(C3)^*$ Since $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda \vee \mu$, then from Proposition 3.11 part (3),

$$C^S_{12}(\lambda, r) \leq C^S_{12}(\lambda \vee \mu, r) \text{ and } C^S_{12}(\mu, r) \leq C^S_{12}(\lambda \vee \mu, r).$$

This implies, $C^S_{12}(\lambda, r) \vee C^S_{12}(\mu, r) \leq C^S_{12}(\lambda \vee \mu, r)$.

(C4) Assume that $C^S_{12}(\lambda, r) > C^S_{12}(\lambda, s)$ for $r \leq s$. Then, there exists $x_t \in Pt(X)$ such that $x_t \in C^S_{12}(\lambda, r)$ and $x_t \notin C^S_{12}(\lambda, s)$. This means, there exists $\mu \in \mathcal{Q}^S_{\tau_{12}}(x_t, s)$ such that $\mu \ \bar{q} \ \lambda$. Since $r \leq s$, then $\mu \in \mathcal{Q}^S_{\tau_{12}}(x_t, r)$ implies $x_t \notin C^S_{12}(\lambda, r)$, which is a contradiction.

$(C5)$ From (C2), we have that

$$C^S_{12}(\lambda, r) \leq C^S_{12}(C^S_{12}(\lambda, r), r).$$

Suppose now $C^S_{12}\ (C^S_{12}\ (\lambda, r), r) > C^S_{12}(\lambda, r)$, then there exists $x_t \in Pt(X)$ such that $x_t \in C^S_{12}(C^S_{12}(\lambda, r), r)$ and $x_t \notin C^S_{12}(\lambda, r)$. Hence, $\exists$ r-τ_{12}-fsc μ, such that $x_t \notin \mu$ and $\mu \geq \lambda$. Since $C^S_{12}(\lambda, r)(x) \leq \mu(x) < t$, then from Proposition 3.11 part (3) and (5), we have that

$$C^S_{12}(C^S_{12}(\lambda, r), r)(x) \leq C^S_{12}(\mu, r)(x) = \mu(x) < t.$$

Consequently, we get $x_t \notin C^S_{12}(C^S_{12}(\lambda, r), r)$ which is a contradiction. Thus C^S_{12} is a supra fuzzy closure operator. Since every r-τ_{12}-closed fuzzy set is r-τ_{12}-fsc, then $C^S_{12}(\lambda, r) \leq C_{12}(\lambda, r)$. □

Theorem 3.16. *Let (X, τ_1, τ_2) be a smooth bts. Define a mapping $\tau^S_{12} : I^X \longrightarrow I$ on X by*

$$\tau^S_{12}(\lambda) = \bigvee\{r \in I |\ C^S_{12}(\bar{1} - \lambda, r) = \bar{1} - \lambda\} = \bigvee\{r \in I |\ I^S_{12}(\lambda, r) = \lambda\}.$$

Then τ^S_{12} is a smooth supra topology on X such that $\tau_{12}(\lambda) \leq \tau^S_{12}(\lambda)$, for all $\lambda \in I^X$. The pair (X, τ^S_{12}) is called semi smooth supra topological space (S-smooth supra topological space, for short).

Proof. By Theorem 3.15, C^S_{12} is supra fuzzy closure operator. Thus by Definition 2.4, τ^S_{12} is a smooth supra fuzzy topology on X. Now, to prove $\tau_{12}(\lambda) \leq \tau^S_{12}(\lambda)$. By Theorem 3.15, since $C^S_{12}(\lambda, r) \leq C_{12}(\lambda, r)$. So, if $C_{12}(\bar{1} - \lambda, r) = \bar{1} - \lambda$, then $C^S_{12}(\bar{1} - \lambda, r) = \bar{1} - \lambda$. Thus $\tau_{12}(\lambda) \leq \tau^S_{12}(\lambda)$ for all $\lambda \in I^X$. □

Remark 3.17. *Let (X, τ_1, τ_2) be a smooth bts, let $\mu, \eta \in I^X$ and $r \in I_0$, then $C^S_{12}(\mu, r) \vee C^S_{12}(\eta, r) \neq C^S_{12}(\mu \vee \eta, r)$.*

Example 3.18. *Let $X = \{a, b, c\}$. Define $\lambda_1, \lambda_2 \in I^X$ as follows:*

$$\lambda_1 = a_{0.8} \vee b_{0.7} \vee c_1 \quad , \qquad \lambda_2 = a_{0.5} \vee b_{0.9} \vee c_1$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & if\ \lambda = \bar{0}, \bar{1} \\ 0.3 & if\ \lambda = \lambda_1 \\ 0 & o.w \end{cases}, \qquad \tau_2(\lambda) = \begin{cases} 1 & if\ \lambda = \bar{0}, \bar{1} \\ 0.4 & if\ \lambda = \lambda_2 \\ 0 & o.w \end{cases}$$

The associated smooth supra topological space of (X, τ_1, τ_2), is defined as follows: $\tau_{12} : I^X \longrightarrow I$ such that

$$\tau_{12}(\lambda) = \begin{cases} 1 & if\ \lambda = \bar{0}, \bar{1} \\ 0.3 & if\ \lambda = \lambda_1 \\ 0.4 & if\ \lambda = \lambda_2 \\ 0.3 & if\ \lambda = \lambda_1 \vee \lambda_2 \\ 0 & o.w \end{cases}$$

The S-smooth supra topological space (X, τ_{12}^S) defined as follows:

$$\tau_{12}^S(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ 0.3 & \text{if } \lambda = \{a_{r_1} \vee b_{r_2} \vee c_1; 0.8 \leq r_1 \leq 1,\ 0.7 \leq r_2 \leq 0.9\} \\ 0.4 & \text{if } \lambda = \{a_{r_1} \vee b_{r_2} \vee c_1; 0.5 \leq r_1 \leq 1,\ 0.9 \leq r_2 \leq 1\} \\ 0 & o.w \end{cases}$$

Let $r = 0.3$, and $\mu = a_{0.2} \vee b_{0.3} \vee c_0$, $\eta = a_{0.5} \vee b_{0.1} \vee c_0$. Then $C_{12}^S(\mu, 0.3) = \mu$, $C_{12}^S(\eta, 0.3) = \eta$ and $C_{12}^S(\mu \vee \eta, 0.3) = \bar{1}$, therefore we have $C_{12}^S(\mu \vee \eta, 0.3) \neq C_{12}^S(\mu, 0.3) \vee C_{12}^S(\eta, 0.3)$.

4 Fuzzy semi continuous and fuzzy irresolute mappings in smooth bts

In this section we use smooth supra topological spaces (X, τ_{12}) and (Y, τ_{12}^*) which are induced from smooth bitopological spaces (X, τ_1, τ_2) and (Y, τ_1^*, τ_2^*) respectively, to introduce and study fuzzy semi continuous (resp., open, closed) mappings and fuzzy irresolute, fuzzy irresolute open (resp., closed) mappings in smooth bitopological spaces. Throughout this section the supra fuzzy closure operators of (X, τ_{12}) and (Y, τ_{12}^*) are denoted by C_{12} and C_{12}^*, respectively.

Definition 4.1. *A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ is said to be*

1. *FP^*-semi continuous (FP^*S-continuous, for short) iff if $f^{-1}(\mu)$ is r-τ_{12}-fso set in X for all $\mu \in I^Y$, $\tau_{12}^*(\mu) \geq r$.*

2. *FP^*-semi open (FP^*S-open, for short) iff $f(\mu)$ is r-τ_{12}^*-fso set in Y for all $\mu \in I^X$, $\tau_{12}(\mu) \geq r$.*

3. *FP^*-semi closed (FP^*S-closed, for short) iff $f(\mu)$ is r-τ_{12}^*-fsc set in Y for all $\mu \in I^X$, $\tau_{12}(\bar{1} - \mu) \geq r$.*

4. *FP^*-irresolute iff $f^{-1}(\mu)$ is r-τ_{12}-fso set in X for each r-τ_{12}^*-fso set μ in Y.*

5. *FP^*-irresolute open iff $f(\mu)$ is r-τ_{12}^*-fso set in Y for each r-τ_{12}-fso set μ in X.*

6. *FP^*-irresolute closed iff $f(\mu)$ is r-τ_{12}^*-fsc set in Y for each r-τ_{12}-fsc set μ in X.*

Proposition 4.2. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then*

1. *f is FP^*S-continuous iff $f : (X, \tau_{12}) \longrightarrow (Y, \tau_{12}^*)$ is fs-continuous.*

2. *f is FP^*S-open (resp., FP^*S-closed) iff $f : (X, \tau_{12}) \longrightarrow (Y, \tau_{12}^*)$ is fs-open (resp., fs-closed).*

3. *f is FP^*-irresolute iff $f : (X, \tau_{12}) \longrightarrow (Y, \tau_{12}^*)$ is f-irresolute.*

4. *f is FP^*-irresolute open (resp., FP^*-irresolute closed) iff $f : (X, \tau_{12}) \longrightarrow (Y, \tau_{12}^*)$ is f-irresolute open (resp., f-irresolute closed).*

Proof. Follows direct from Definition 4.1. □

Theorem 4.3. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then the following are equivalent.*

1. *f is FP^*S-continuous.*
2. *$f^{-1}(\mu)$ is r-τ_{12}-fsc set in X for each $\tau_{12}^*(\bar{1} - \mu) \geq r$.*

Proof. It is clear. □

Theorem 4.4. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ be a mapping.*

1. *If f is FP^*- continuous then it's FP^*S- continuous.*
2. *If f is FP^*- open (resp., closed) then it's FP^*S- open (resp., FP^*S- closed).*

Proof. (1) Let $\mu \in I^Y$ such that $\tau_{12}^*(\mu) \geq r$. Since f is FP^*- continuous then $\tau_{12}(f^{-1}(\mu)) \geq r$, implies $f^{-1}(\mu)$ is r-τ_{12}-open fuzzy set in X and therefore $f^{-1}(\mu)$ is r-τ_{12}-fso set in X. Thus f is FP^*S- continuous.

(2) We prove f is FP^*S- open and the prove of f is FP^*S- closed is similar. Let $\mu \in I^X$ such that $\tau_{12}(\mu) \geq r$, since f is FP^*-open, then $\tau_{12}^*(f(\mu)) \geq r$, implies $f(\mu)$ is r-τ_{12}^*-fso set in Y. Thus f is FP^*S- open. □

The following example show the converse of Theorem 4.4 part(1) is not true.

Example 4.5. *Let $X = \{a, b\}$ and $Y = \{p, q\}$. Define $\lambda_1, \lambda_2 \in I^X$ and $\mu_1, \mu_2 \in I^Y$ as follows:*

$$\lambda_1 = a_{\frac{1}{2}} \vee b_{\frac{2}{3}} \quad , \qquad \lambda_2 = a_{\frac{2}{3}} \vee b_{\frac{1}{2}}$$
$$\mu_1 = p_{\frac{3}{4}} \vee q_{\frac{1}{2}} \quad , \qquad \mu_2 = p_{\frac{1}{2}} \vee q_{\frac{3}{4}}$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ and $\tau_1^*, \tau_2^* : I^Y \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ \frac{1}{2} & \text{if } \lambda = \lambda_1 \\ 0 & o.w \end{cases} , \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ \frac{1}{3} & \text{if } \lambda = \lambda_2 \\ 0 & o.w \end{cases}$$

$$\tau_1^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1} \\ \frac{1}{2} & \text{if } \mu = \mu_1 \\ 0 & o.w \end{cases} , \qquad \tau_2^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1} \\ \frac{1}{3} & \text{if } \mu = \mu_2 \\ 0 & o.w \end{cases}$$

From smooth bts's (X, τ_1, τ_2) and (Y, τ_1^*, τ_2^*) we can induce smooth supra topologies τ_{12} and τ_{12}^* as follows:

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ \frac{1}{2} & \text{if } \lambda = \lambda_1 \\ \frac{1}{3} & \text{if } \lambda = \lambda_2 \\ \frac{1}{3} & \text{if } \lambda = \lambda_1 \vee \lambda_2 \\ 0 & o.w \end{cases} , \qquad \tau_{12}^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1} \\ \frac{1}{2} & \text{if } \mu = \mu_1 \\ \frac{1}{3} & \text{if } \mu = \mu_2 \\ \frac{1}{3} & \text{if } \mu = \mu_1 \vee \mu_2 \\ 0 & o.w \end{cases}$$

consider the mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ defined by $f(a) = q,\ f(b) = p$. Then f is FP^*S- continuous but not FP^*-continuous since there exists $\mu_1 \vee \mu_2$ is $\frac{1}{3}$-τ_{12}^*-open fuzzy set in Y, but $\tau_{12}(f^{-1}(\mu_1 \vee \mu_2)) = \tau_{12}(a_{\frac{3}{4}} \vee b_{\frac{3}{4}}) = 0 \ngeq \frac{1}{3}$.

Also, the converse of Theorem 4.4 part(2) is not true, as the following example show.

Example 4.6. *Let $X = \{a, b\}$ and $Y = \{p, q\}$. Define $\lambda_1, \lambda_2 \in I^X$ and $\mu_1, \mu_2 \in I^Y$ as follows:*

$$\lambda_1 = a_{0.3} \vee b_{0.7} \quad , \quad \lambda_2 = a_{0.7} \vee b_{0.3}$$

$$\mu_1 = p_{0.1} \vee q_{0.3} \quad , \quad \mu_2 = p_{0.3} \vee q_{0.1}$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ and $\tau_1^*, \tau_2^* : I^Y \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ \frac{1}{2} & \text{if } \lambda = \lambda_1 \\ 0 & o.w \end{cases}, \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ \frac{1}{3} & \text{if } \lambda = \lambda_2 \\ 0 & o.w \end{cases}$$

$$\tau_1^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1} \\ \frac{1}{2} & \text{if } \mu = \mu_1 \\ 0 & o.w \end{cases}, \qquad \tau_2^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1} \\ \frac{1}{3} & \text{if } \mu = \mu_2 \\ 0 & o.w \end{cases}$$

From smooth bts's (X, τ_1, τ_2) and (Y, τ_1^*, τ_2^*) we can induce smooth supra topologies τ_{12} and τ_{12}^* as follows:

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ \frac{1}{2} & \text{if } \lambda = \lambda_1 \\ \frac{1}{3} & \text{if } \lambda = \lambda_2 \\ \frac{1}{3} & \text{if } \lambda = \lambda_1 \vee \lambda_2 \\ 0 & o.w \end{cases}, \qquad \tau_{12}^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1} \\ \frac{1}{2} & \text{if } \mu = \mu_1 \\ \frac{1}{3} & \text{if } \mu = \mu_2 \\ \frac{1}{3} & \text{if } \mu = \mu_1 \vee \mu_2 \\ 0 & o.w \end{cases}$$

consider the mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ defined by: $f(a) = q, f(b) = p$. Then f is FP^*S- open but not FP^*-open since there exists λ_1 is $\frac{1}{2}$-τ_{12}-open fuzzy set in X, but $\tau_{12}^*(f(\lambda_1)) = \tau_{12}^*(p_{0.7} \vee q_{0.3}) = 0 < \frac{1}{2}$.
Also, in the same example we can show f is FP^*S- closed but not FP^*-closed. Since there exists $\bar{1} - \lambda_1$ is $\frac{1}{2}$-τ_{12}-closed fuzzy set in X, but $\tau_{12}^*(f(\bar{1} - \lambda_1))$ is not $\frac{1}{2}$-closed fuzzy set in Y since $\tau_{12}^*(\bar{1} - f(\bar{1} - \lambda_1)) = \tau_{12}^*(f(\lambda_1)) = 0 < \frac{1}{2}$.

Theorem 4.7. *If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ is FP^*- irresolute then it is FP^*S-continuous.*

Proof. Let $\mu \in I^Y$ such that $\tau_{12}^*(\mu) \geq r$, implies μ is r- τ_{12}^*-fso set in Y and since f is FP^*- irresolute, then $f^{-1}(\mu)$ is r- τ_{12}-fso set in X. Thus f is FP^*S- continuous. □

The converse of Theorem 4.7 is not true as the following example show.

Example 4.8. *Let $X = \{a, b\}$ and $Y = \{p, q\}$. Define $\lambda_1, \lambda_2 \in I^X$ and $\mu_1, \mu_2 \in I^Y$ as follows:*

$$\lambda_1 = a_{0.1} \vee b_{0.2} \quad , \quad \lambda_2 = a_{0.2} \vee b_{0.1}$$

$$\mu_1 = p_{0.6} \vee q_{0.7} \quad , \quad \mu_2 = p_{0.7} \vee q_{0.6}$$

We define smooth topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ *and* $\tau_1^*, \tau_2^* : I^Y \longrightarrow I$ *as follows:*

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ \frac{2}{3} & \text{if } \lambda = \lambda_1 \\ 0 & o.w \end{cases}, \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ \frac{1}{4} & \text{if } \lambda = \lambda_2 \\ 0 & o.w \end{cases}$$

$$\tau_1^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1} \\ \frac{2}{3} & \text{if } \mu = \mu_1 \\ 0 & o.w \end{cases}, \qquad \tau_2^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1} \\ \frac{1}{4} & \text{if } \mu = \mu_2 \\ 0 & o.w \end{cases}$$

From smooth bts's (X, τ_1, τ_2) *and* (Y, τ_1^*, τ_2^*) *we can induce smooth supra topologies* τ_{12} *and* τ_{12}^* *as follows:*

$$\tau_{12}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0}, \bar{1} \\ \frac{2}{3} & \text{if } \lambda = \lambda_1 \\ \frac{1}{4} & \text{if } \lambda = \lambda_2 \\ \frac{1}{4} & \text{if } \lambda = \lambda_1 \vee \lambda_2 \\ 0 & o.w \end{cases}, \qquad \tau_{12}^*(\mu) = \begin{cases} 1 & \text{if } \mu = \bar{0}, \bar{1} \\ \frac{2}{3} & \text{if } \mu = \mu_1 \\ \frac{1}{4} & \text{if } \mu = \mu_2 \\ \frac{1}{4} & \text{if } \mu = \mu_1 \vee \mu_2 \\ 0 & o.w \end{cases}$$

consider the mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ *defined by:* $f(a) = q, f(b) = p$. *Then* f *is* FP^*S*- continuous but not is* FP^**- irresolute, since there exists* $\mu \in I^Y$ *such that* $\mu = p_{0.9} \vee q_{0.9}$ *and* μ *is* $\frac{2}{3}$*-*τ_{12}^**-fso set but* $f^{-1}(\mu) = a_{0.9} \vee b_{0.9}$ *is not* $\frac{2}{3}$*-*τ_{12}*-fso set.*

Theorem 4.9. *Let* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ *be a mapping. Then for each* $r \in I_0$*, the following statements are equivalent.*

1. f *is* FP^*S*-continuous.*
2. $f(C_{12}^S(\lambda, r)) \leq C_{12}^*(f(\lambda), r)$*, for each* $\lambda \in I^X$
3. $C_{12}^S(f^{-1}(\mu), r) \leq f^{-1}(C_{12}^*(\mu, r))$*, for each* $\mu \in I^Y$.

Proof. (1) $\Longrightarrow$ (2) Suppose $f(C_{12}^S)(\lambda, r)) > C_{12}^*(f(\lambda), r)$. So there exists $y \in Y$ *and* $t \in I_0$ such that

$$f(C_{12}^S(\lambda, r))(y) > t > C_{12}^*(f(\lambda), r)(y). \tag{5}$$

If $f^{-1}(y) = \emptyset$, then $f(C_{12}^S(\lambda, r))(y) = 0$, which is a contradiction with (5), hence there exists $x \in f^{-1}(y)$ such that

$$f(C_{12}^S(\lambda, r))(y) > C_{12}^S(\lambda, r)(x) > t > C_{12}^*(f(\lambda), r)(y). \tag{6}$$

Since $C_{12}^*(f(\lambda), r)(y) < t$, then $y_t \notin C_{12}^*(f(\lambda), r)$. This yields, that there exists $\mu \in I^Y$ such that $\tau_{12}^*(\bar{1} - \mu) \geq r$, $f(\lambda) \leq \mu$ and $y_t \notin \mu$ (i.e., $\mu(y) < t$). Therefore, $C_{12}^*(f(\lambda), r)(y) < \mu(y) < t$ which means $C_{12}^*(f(\lambda), r)(f(x)) < \mu(f(x)) < t$. Moreover, $f(\lambda) \leq \mu$ (i.e., $\lambda \leq f^{-1}(\mu)$). This yields, $\bar{1} - f^{-1}(\mu) \leq \bar{1} - \lambda$, i.e., $f^{-1}(\bar{1} - \mu) \leq \bar{1} - \lambda$. From the fact that f is FP^*S-continuous, we have $f^{-1}(\bar{1} - \mu)$ is r-τ_{12}-fso in X. Thus, $I_{12}^S(f^{-1}(\bar{1} - \mu), r) \leq I_{12}^S(\bar{1} - \lambda, r)$ which yields $f^{-1}(\bar{1} - \mu) \leq I_{12}^S(\bar{1} - \lambda, r)$, i.e., $\bar{1} - I_{12}^S(\bar{1} - \lambda, r) \leq \bar{1} - f^{-1}(\bar{1} - \mu)$. Hence, $C_{12}^S(\lambda, r) \leq f^{-1}(\mu)$, and consequently,

$C_{12}^S(\lambda, r)(x) \leq f^{-1}(\mu)(x) = \mu(f(x)) < t$. This is a contradiction with (6). Thus $f(C_{12}^S(\lambda, r)) \leq C_{12}^*(f(\lambda), r)$.

(2) $\Longrightarrow$ (3), by letting $\lambda = f^{-1}(\mu)$ in (2), we get

$$f(C_{12}^S(f^{-1}(\mu), r)) \leq C_{12}^*(f(f^{-1}(\mu), r) \leq C_{12}^*(\mu, r).$$

Consequently, $f^{-1}(f(C_{12}^S(f^{-1}(\mu), r))) \leq f^{-1}(C_{12}^*(\mu, r))$ which yields $C_{12}^S\ (f^{-1}(\mu), r) \leq f^{-1}(C_{12}^*(\mu, r))$.

(3) $\Longrightarrow$ (1), let $\mu \in I^Y$ such that $\tau_{12}^*(\mu) \geq r$. Then, $C_{12}^{*S}(\bar{1} - \mu, r) = \bar{1} - \mu$. But $C_{12}^S(f^{-1}(\bar{1} - \mu), r) \leq f^{-1}(C_{12}^*(\bar{1} - \mu, r)) = f^{-1}(\bar{1} - \mu)$. This implies, $C_{12}^S(f^{-1}(\bar{1} - \mu), r) = f^{-1}(\bar{1} - \mu) = \bar{1} - f^{-1}(\mu)$ which gives $\bar{1} - f^{-1}(\mu)$ is r-τ_{12}-fsc in X. So, $f^{-1}(\mu)$ is r-τ_{12}-fso in X. Hence, f is FP^*S-continuous. □

Theorem 4.10. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then for each $r \in I_0$, the following statements are equivalent.*

1. *f is FP^*S-open.*
2. *$f(I_{12}(\lambda, r)) \leq I_{12}^{*S}(f(\lambda), r)$, $\lambda \in I^X$.*
3. *$I_{12}(f^{-1}(\mu), r) \leq f^{-1}(I_{12}^{*S}(\mu, r))$, $\mu \in I^Y$.*

Proof. (1) $\Longrightarrow$ (2), let $\lambda \in I^X$. We know that $I_{12}(\lambda, r) \leq \lambda$, then $f(I_{12}(\lambda, r)) \leq f(\lambda)$. Hence, $I_{12}^{*S}(f(I_{12}(\lambda, r))) \leq I_{12}^{*S}(f(\lambda), r)$. Since f is FP^*S-open, this implies $f(I_{12}(\lambda, r)) \leq I_{12}^{*S}(f(\lambda), r)$ as required.

(2) $\Longrightarrow$ (3), let $\mu \in I^Y$ such that $f^{-1}(\mu) \in I^X$. Set $\lambda = f^{-1}(\mu)$. This yields,

$$f(I_{12}(f^{-1}(\mu), r) \leq I_{12}^{*S}(f(f^{-1}(\mu)), r) \leq I_{12}^{*S}(\mu, r).$$

Consequently, $f^{-1}(f(I_{12}(f^{-1}(\mu), r))) \leq f^{-1}(I_{12}^{*S}(\mu, r))$ which means $I_{12}(f^{-1}(\mu), r) \leq f^{-1}(I_{12}^{*S}(\mu, r))$.

(3) $\Longrightarrow$ (1), let $\mu \in I^X$ such that $\tau_{12}(\mu) \geq r$. Then $f(\mu) \in I^Y$, and by (3) we have that $I_{12}(f^{-1}(f(\mu), r)) \leq f^{-1}(I_{12}^{*S}(f(\mu), r))$. This yields, $\mu \leq f^{-1}(I_{12}^{*S}(f(\mu), r))$. By taking the image to both side of the last equality we obtain

$$f(\mu) \leq f(f^{-1}(I_{12}^{*S}(f(\mu), r))) \leq I_{12}^{*S}(f(\mu), r).$$

Thus, $f(\mu) \leq I_{12}^{*S}(f(\mu), r)$ which gives $I_{12}^{*S}(f(\mu), r) = f(\mu)$. Hence, $f(\mu)$ is r-τ_{12}^*-fso set in Y, i.e., f is FP^*S-open. □

Theorem 4.11. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then for each $r \in I_0$, the following statements are equivalent.*

1. *f is FP^*S-closed.*
2. *$C_{12}^{*S}(f(\lambda), r) \leq f(C_{12}(\lambda, r))$, $\lambda \in I^X$.*

Proof. (1) $\Longrightarrow$ (2), Let $\lambda \in I^X$ since $\lambda \leq C_{12}(\lambda, r)$, then $f(\lambda) \leq f(C_{12}(\lambda, r))$, such that $f(C_{12}(\lambda, r))$ is r-τ_{12}^*-fsc set, implies $C_{12}^{*S}(f(\lambda), r) \leq C_{12}^{*S}(f(C_{12}(\lambda, r)))$, by(1) $C_{12}^{*S}(f(\lambda), r) \leq f(C_{12}(\lambda, r))$.

(2) $\Longrightarrow$ (1), Let $\lambda \in I^X$ such that $\tau_{12}(\bar{1} - \lambda) \geq r$ Then $f(\lambda) \in I^Y$, and by (2) we have $C_{12}^{*S}(f(\lambda), r) \leq f(C_{12}(\lambda, r)) = f(\lambda)$, since λ is r-τ_{12}-fuzzy closed set, then $C_{12}^{*S}(f(\lambda), r) \leq f(\lambda)$ but clearly $f(\lambda) \leq C_{12}^{*S}(f(\lambda), r)$, then $C_{12}^{*S}(f(\lambda), r) = f(\lambda)$ and consequently $f(\lambda)$ is r-τ_{12}^*-fsc set in Y. □

Theorem 4.12. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then for each $r \in I_0$, the following statements are equivalent.*
1. f is FP^*-irresolute.
2. $f(C_{12}^S(\lambda, r)) \leq C_{12}^{*S}(f(\lambda), r), \lambda \in I^X$.
3. $C_{12}^S(f^{-1}(\mu), r) \leq f^{-1}(C_{12}^{*S}(\mu, r)), \mu \in I^Y$.

Proof. The proof is similar to that of Theorem 4.9. □

Theorem 4.13. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then for each $r \in I_0$, the following statements are equivalent.*
1. f is FP^*-irresolute open.
2. $f(I_{12}^S(\lambda, r)) \leq I_{12}^{*S}(f(\lambda), r)$, $\lambda \in I^X$.
3. $I_{12}^S(f^{-1}(\mu), r) \leq f^{-1}(I_{12}^{*S}(\mu, r))$, $\mu \in I^Y$.

Proof. The proof is similar to that of Theorem 4.10. □

Theorem 4.14. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then for each $r \in I_0$, the following statements are equivalent.*
1. f is FP^*-irresolute closed.
2. $C_{12}^{*S}(f(\lambda), r) \leq f(C_{12}^S(\lambda, r))$, $\lambda \in I^X$.

Proof. The proof is similar to that of Theorem 4.11. □

Theorem 4.15. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then for each $r \in I_0$, the following statements are equivalent.*
1. $f : (X, \tau_{12}^S) \longrightarrow (Y, (\tau_{12}^{*S})$ is F-continuous.
2. $f(C_{12}^S(\lambda, r)) \leq C_{12}^{*S}(f(\lambda), r), \lambda \in I^X$.
3. $C_{12}^S(f^{-1}(\mu), r) \leq f^{-1}(C_{12}^{*S}(\mu, r)), \mu \in I^Y$.

Proof. The proof is similar to that of Theorem 4.9. □

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