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Lattice for Rough Intervals

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Abstract This paper deals with the rough interval approach on lattice theory. In the interval-set model, a pair of sets is referred to as the lower and upper bounds which define a family of sets. A significant difference between these concepts lies in the definition and interpretation of their extended set-theoretic operators. The operators in the rough-set model are not truth-functional, while the operators in the interval-set model are truth-functional. We have showed that the collection of all rough intervals in an approximation space forms a distributive lattice. Some important results are also proved. Finally, an example is considered to illustrated the paper.

Keywords *Rough Set, Interval Set, Equivalence Class, Knowledge Representation, Lattice.*

1 Introduction

The rough-set and interval-set models are two related but distinct extensions of set theory for modelling vagueness. In the rough-set model, a given set is represented by a pair of ordinary sets called the lower and upper approximations [7]. The approximation space is constructed based on an equivalence relation defined by a set of attributes [9]. There are two views for the interpretation of the rough-set model. Under one view the approximation space can be understood in terms of two additional set theoretic operators [3]. In which we assign for each subset of the universe a lower approximation

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and an upper approximation. We regard this interpretation to be the operator-oriented view. The other interpretation is a set-oriented view which considers a rough set as the family of sets having the same lower and upper approximations [1]. The rough-set model is useful in the study of information system, classification and machine learning [8]. In the interval-set model, it is assumed that the available information is insufficient to define a set precisely. Instead, a pair of sets referred to as the lower and upper bounds is used to define the range of the unknown set. In other words, any member of the family of sets bounded by the lower and upper bounds can, in fact be in the set. Orlowska [6] proposed a logic for reasoning about concepts using the notion of rough sets, which is essentially the modal logic system S5 with the modal operators interpreted using the lower and upper approximations [4]. A similar approach was also adopted by Chakraborty and Banerjee [2]. The semantics of these logic systems have been investigated by many authors. Recently, Yao and Li [10] examined the relationship between the interval-set model and Kleene three valued logic. It has been shown that the interval-set model provides the possible-worlds semantics for Kleene three-valued logic.
Based on the above studies, this paper provides a comparison of rough-set and interval-set models with emphasis on uncertain reasoning. The discussion will focus in our paper on the interpretation of these models and their connections and differences. The main objective of such a comparative study in our paper is to show that these two models provide different and complementary extensions of set theory, although both use a pair of sets in their formulations.

2 Rough Sets and Interval Sets

In the rough set model, there is an equivalence relation E defined on U, namely, E is reflexive, symmetric and transitive. This relation partitions U into disjoint subsets $U/E = \{E_1, E_2, \ldots, E_n\}$, where $E_i, i \in (Index\ set)$, is an equivalence class of E. The family U/E is called a quotient set and the pair (U, E) is called an approximation space. The empty set and the elements of U/E are called the elementary or atomic set. The union of one or more elementary sets is called a composed set. The family of all composed sets is denoted by $Com(U, E)$. The elementary sets are the building blocks for the construction of rough sets. In the approximation space (U, E), given an arbitrary set $X \subseteq U$, one may represent X by a pair of lower and upper approximations: $X \downarrow = \{x \in U : x/E \subseteq X\}$ and $X \uparrow = \{x \in U : x/E \cap X \neq \emptyset\}$, where x/E denote the equivalence class containing x. The lower approximation $X \downarrow$ is the union of all the elementary sets which are subsets of X, and the upper approximation $X \uparrow$ is the union of all the elementary sets which have a non-empty intersection with X. In fact, $X \downarrow$ is the greatest composed set contained by X, while $X \uparrow$ is the least composed set containing X. The pair $(X \downarrow, X \uparrow)$ is called the rough set of X. Mathematically rough interval set are defined as follows. Let U be a finite set, called the universe or the reference set, and $P(U)$ be its power set. A subset of $P(U)$ of the form $\mathbf{X} = [X \downarrow, X \uparrow] = \{X \in$

$P(U) : X\downarrow \subseteq X\uparrow\}$ is called a closed interval set, where it is assumed $X\downarrow \subseteq X\uparrow$. The set of all closed interval sets is denoted by $Int(P(U))$. Degenerate interval sets of the form $[X, X]$ are equivalent to ordinary sets. An interval set, when interpreted as family of sets of objects, provides an approximate means to represents a partially known concept. Although the extension of a concept is actually a subset of U, a lack of knowledge makes us unable to specify this subset. We can only provide a lower bound $X\downarrow$ and upper bound $X\uparrow$. Any subset X that lies between $X\downarrow$ and $X\uparrow$, namely $X\downarrow \subseteq X\uparrow$ can be actual extension of the concept. The set $BN[X\downarrow, X\uparrow] = X\uparrow - X\downarrow$ is called the boundary of the interval set $[X\downarrow, X\uparrow]$.

3 Operations on Rough Interval

Let $\cap, \cup$ and $\neg$ be the usual set intersection, union and difference defined on $P(U)$, respectively. We define the following binary operations into interval-set operation. For two interval sets $\mathbf{X} = [X\downarrow, X\uparrow]$ and $\mathbf{Y} = [Y\downarrow, Y\uparrow]$

$$X \sqcap Y = \{X \cap Y : X \in \mathbf{X}, Y \in \mathbf{Y}\},$$
$$X \sqcup Y = \{X \cup Y : X \in \mathbf{X}, Y \in \mathbf{Y}\},$$
$$X \setminus Y = \{X \setminus Y : X \in \mathbf{X}, Y \in \mathbf{Y}\}.$$

These operations are referred to as interval-set intersection, union and difference. They are closed on $Int(P(U))$, namely, $X \sqcap Y$, $X \sqcup Y$ and $X \setminus Y$ are interval sets. In fact, these interval sets can be explicitly computed by using the following formulas: $X \sqcap Y = [X\downarrow \cap Y\downarrow,\ X\uparrow \cap Y\uparrow]$, $X \sqcup Y = [X\downarrow \cup Y\downarrow,\ X\uparrow \cup Y\uparrow]$, $X \setminus Y = [X\downarrow - Y\downarrow, X\uparrow - Y\uparrow]$, Similarly, the interval-set complement $\neg[X\downarrow, X\uparrow]$ of $[X\downarrow, X\uparrow]$ is defined as $[U, U] \setminus [X\downarrow, X\uparrow]$. This is equivalent to $[U - X\uparrow, U - X\downarrow] = [X\uparrow^{co}, X\downarrow^{co}]$, where $X^{co} = U - X$ denote the usual set complement operation. Clearly, we have $\neg[\emptyset, \emptyset] = [U, U]$ and $\neg[U, U] = [\emptyset, \emptyset]$. With the above operations $\cap, \cup$ and $\neg$, $IntP(U)$ is a completely distributive lattice [5]. Both the operation $\cap$ and $\cup$ are idempotent, commutative, associative, absorptive and distributive. For interval-set complement, De Morgan's laws and double negation law hold. Moreover, $[U, U]$ and $[\emptyset, \emptyset]$ are the unique identities for interval-set intersection and union. These properties may be regarded as the counterparts of the properties of their corresponding set-theoretic operations. Unlike elementary set theory, for an interval set $X, X \cap \neg X$ is not necessarily equal to $[\emptyset, \emptyset]$, and $X \cup \neg X$ is not necessarily equal to $[U, U]$. In qualitative knowledge representation $X \setminus X$ is not necessarily equal to $[\emptyset, \emptyset]$. Nevertheless, $\emptyset \in X \cap \neg X, U \in X \cup \neg X$, and $\emptyset \in X \setminus Y$. Therefore, $Int(P(U))$ is a completely distributive lattice but not a boolean algebra, whereas $P(U)$ is a boolean algebra [5]. Moreover, degenerate interval sets of the form $[X, X]$ are equivalent to ordinary sets. For degenerate interval sets, the proposed operators $\cap, \cup, \setminus$ and $\neg$ reduce to the usual set-theoretic operators. Thus, interval-set model may be considered as an extension of set theory with extended set-theoretic operators. From the above discussion, it is clear that rough-set and interval-set models are different extensions of set theory. The rough-set model introduces two additional set-theoretic operators based on an equivalence relation on the universe. It forms a system $(P(U), \cap, \cup, ^{co}, \downarrow, \uparrow)$ by adding a pair of unary operators to the Boolean algebra $(P(U), \cap, \cup, ^{co})$. On the other hand, the interval-set model extends the standard set-theoretic operators $\cap, \cup$ and ^{co}. It establishes a completely distributive lattice $(Int(P(U)), \sqcap, \sqcup)$ from the Boolean algebra $(P(U), \cap, \cup, ^{co})$. It should be noted that interval-set operators are truth-functional, namely, the value of a compound formula can be computed from its components. An interval set $[X \downarrow, X \uparrow]$ is also referred to as a flou set [5]. The lower bound $X \downarrow$ defines the sure region, the upper bound $X \uparrow$ defines the maximum region, and the difference $X \uparrow -X \downarrow$ defines the flou region. The sure region corresponds to the positive region of a rough set and the flou region corresponds to the doubtful region. From this point of view, the notion of interval sets is related to the concept of rough sets. An alternative view of rough-set model along the same line of argument as that of the interval set model has been discussed by Bonikowski [1]. It presents a set-oriented interpretation of rough-set model. Given two elements $X \downarrow, X \uparrow \in Com(U, E)$ with $X \downarrow \subseteq X \uparrow$, a rough set is defined as the following family of subsets of U: $< X_1, X_2 >= \{A \in P(U) : A \downarrow = X_1, A \uparrow = X_2\}$. Under this interpretation, one can extend the set-theoretic operators $\cap, \cup$ and co into the corresponding rough-set operators: for two rough sets $< X_1, X_2 >$ and $< Y_1, Y_2 >$,

$< X_1, X_2 > \otimes < Y_1, Y_2 >= \{A \in P(U) : A \downarrow = X_1 \cap Y_1, A \uparrow = X_2 \cap Y_2\}$,

$< X_1, X_2 > \oplus < Y_1, Y_2 >= \{A \in P(U) : A \downarrow = X_1 \cup Y_1, A \uparrow = X_2 \cup Y_2\}$,

$\ominus < X_1, X_2 >= \{A \in P(U) : A \downarrow = X_2^{co}, A \uparrow = X_1^{co}\}$.

The symbols $\otimes, \oplus$ and $\ominus$ represent the rough-set intersection, union and complement. Unlike the interval-set operators, these rough-set operators are not truth-functional. Let $\Re$ denote the set of all rough sets in the approximation space (U, E). The extended system $(\Re, \otimes, \oplus)$ is a completely distributive lattice [5].

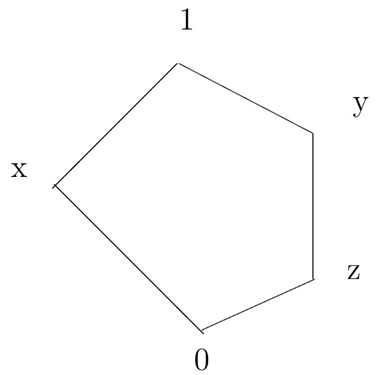

Figure 1: A non-distributive lattice

Consider a non-distributive lattice given in Figure 1 on $L = \{0, x, y, z, 1\}$. For two intervals $[x, 1]$ and $[y, 1]$, we have: $[x, 1] \otimes [y, 1] = \{0, x, y, 1\}$, which is not an interval. Similarly, for two intervals $[0, x]$ and $[0, z]$, we have: $[0, x] \oplus [0, z] = \{0, x, z, 1\}$, which is also not an interval. Operations $\otimes$ and $\oplus$ on L we have $(x \preceq x', y \preceq y') \Longrightarrow x \otimes y \preceq x' \otimes y', (x \preceq x', y \preceq y') \Longrightarrow x \oplus y \preceq x' \oplus y'$. It is expected that a simple computation method can be used if extended operations are indeed closed on $I(L)$, the set of all intervals formed from I. As shown by the following theorem, a sufficient condition for these operations to be closed is that the lattice L is distributive. In addition, the extended operations can be easily computed by considering only ending points of intervals.

4 Rough Interval Concept

Rough intervals provide a framework for the interpretation of crisp intervals based on indiscernibility. Indiscernibility is a property defined inside of the Rough set theory, which allows the extraction of precise information from vague concepts, by comparing these vague concepts with the available process knowledge. The required process knowledge may be provided by sensors in the technical process and technical specifications, upon which it can be decided, when a physical (measured) value clearly belongs to the represented concept, and when this membership is uncertain. Rough intervals were introduced because rough sets were conceived only to handle discrete objects they cannot be used to represent continuous values such as temperature, flow etc. A rough interval is a particular case of a rough set. They fulfill all the rough set properties and core concepts, including the upper and lower approximation definitions. Inside of the upper approximation interval, the variable could take the represented qualitative value (a vague concept in rough sets), or what is the same, it is clear that outside this interval, the variable cannot take it. The second element of a rough set, the lower approximation, can be also redefined on this basis: In the lower approximation interval, it is sure that the variable takes the represented qualitative value. The rough interval concept also satisfies the mathematical definition of rough set with upper and lower approximation.

5 Order for Two Interval Rough Sets

Let U be a finite non-empty set called the universe. If we denote by $P(U)$ the power set of U, then $(P(U), \subseteq)$ will be a lattice in which meet and join operators are the classical set intersection $\cap$ and the classical set union $\cup$ respectively. The order of the lattice is the classical set inclusion, and the classical set complement is an order reversing involution. Now we can distinguish two distinct orders for interval rough sets:
1) Inclusion ordering of interval rough sets:
$[X \downarrow, X \uparrow] \subseteq_I [Y \downarrow, Y \uparrow] \Leftrightarrow X \downarrow \subseteq Y \downarrow, X \uparrow \subseteq Y \uparrow$
2) Knowledge ordering of interval rough sets:
$[X \downarrow, X \uparrow] \subseteq_K [Y \downarrow, Y \uparrow] \Leftrightarrow X \downarrow \subseteq Y \downarrow, X \uparrow \subseteq Y \uparrow$. The first one $\subseteq_I$ is an extension of the classical set inclusion.
$(P(U) \times (P(U), \subseteq_I)$ is a lattice whose meet and join operators are the interval set intersection and the union respectively, and the set complement is an order reversing involution of this lattice. The second ordering, $\subseteq_K$ is a new one and can be understood as an order of information that each interval set exhibits. $(P(U) \times (P(U), \subseteq_K)$ will be a lattice in which $\sqcap, \sqcup$ are its meet and join operators, respectively. The interval rough

set complement ^{co} is an order-preserving operator of the lattice. So, with interval rough set algebra, the lattice of classical sets has been generalized to bilattice $((P(U) \times (P(U), \subseteq_I, \subseteq_K, ^{co})$.

Theorem 5.1. *Suppose L is a distributive lattice. Then,*
$[X\downarrow, X\uparrow] \otimes [Y\downarrow, Y\uparrow] = [X\downarrow \otimes Y\downarrow, X\uparrow \otimes Y\uparrow]$ *and*
$[X\downarrow, X\uparrow] \oplus [Y\downarrow, Y\uparrow] = [X\downarrow \oplus Y\downarrow, X\uparrow \oplus Y\uparrow]$
Moreover, $I(L)$, the set of all intervals formed from I with operations $\otimes$ and $\oplus$ forms a distributive lattice.

Proof : The inclusion $[X\downarrow, X\uparrow] \otimes [Y\downarrow, Y\uparrow] \subseteq [X\downarrow \otimes Y\downarrow, X\uparrow \otimes Y\uparrow]$ follows trivially from the properties of lattice, namely, $X\downarrow \preceq A \preceq X\uparrow$ and $Y\downarrow \preceq B \preceq Y\uparrow$ imply $X\downarrow \otimes Y\downarrow \preceq A \otimes B \preceq X\uparrow \otimes Y\uparrow$. Now suppose $C \in [X\downarrow \otimes Y\downarrow, X\uparrow \otimes Y\uparrow]$. We only need to show there exists a pair $A \in [X\downarrow, X\uparrow]$ and $B \in [Y\downarrow, Y\uparrow]$ such that $X\downarrow \otimes Y\downarrow = C$. Let $A = (X\downarrow \oplus C) \otimes X\uparrow$ and $B = (Y\downarrow \oplus C) \otimes Y\uparrow$. It is easily verified that $A \in [X\downarrow, X\uparrow]$, $\quad B \in [Y\downarrow, Y\uparrow]$ It follows,

$$\begin{aligned}
A \otimes B &= ((X\downarrow \oplus C) \otimes X\uparrow) \otimes ((Y\downarrow \oplus C) \otimes Y\uparrow) \\
&= ((X\downarrow \oplus C) \otimes (Y\downarrow \oplus C)) \otimes (X\uparrow \otimes Y\uparrow) \\
&= (X\downarrow) \otimes X\downarrow) \oplus C) \otimes (X\uparrow \otimes Y\uparrow) \\
&= C \otimes (X\uparrow \otimes Y\uparrow) \\
&= C.
\end{aligned}$$

Therefore, $[X\downarrow, X\uparrow] \otimes [Y\downarrow, Y\uparrow] = [X\downarrow \otimes Y\downarrow, X\uparrow \otimes Y\uparrow]]$. Similarly, we can show that the operation $\oplus$ is also closed. It can be easily checked that if $(\Re, \otimes, \oplus)$ is a distributive lattice, then $I(\Re)$ is a distributive lattice, where $I(\Re)$ denote the set of all intervals formed from $\Re$ and $\Re$ is a lattice with operations $\otimes$ and $\oplus$. In particular, the order relation on intervals is given by $[X\downarrow, X\uparrow] \preceq [Y\downarrow, Y\uparrow]$ if and only if $X\downarrow \preceq X\uparrow$ and $Y\downarrow \preceq Y\uparrow$.
To differentiate it from the original lattice $\Re$, we call $I(\Re)$ an interval lattice. If $\Re$ is a Boolean lattice, one may extend the complement operation $\ominus$ as follows:

$$\begin{aligned}
\ominus[X\downarrow, X\uparrow] &= \{\ominus A : A \in [X\downarrow, X\uparrow]\} \\
&= [\ominus X\downarrow, \ominus X\uparrow]
\end{aligned}$$

Therefore we can say that $I(\Re)$ is not a Boolean lattice but a complete distributive lattice.
Example-1: Suppose that we have a number of jurcy with colored blue or red, numbered information about the color of different jurcy each making a interval rough set, for the concept of being red, on the basis of their observation as follows:

Agent 1: Red jurcy: 1,2,3
Blue jurcy: 6,7,8,9
therefore, $[A_1\downarrow, A_1\uparrow] = (\{1, 2, 3\}, \{6, 7, 8, 9\})$

Agent 2: Red jurcy: 1,2,3,4,6
Blue jurcy:5,6,7,8,9

therefore, $[A_2\downarrow, A_2\uparrow] = (\{1,2,3,4,6\},\{5,6,7,8,9\})$

Agent 3: Red jurcy: 1,2,10
Blue jurcy: 5,6,7,8

therefore, $[A_3\downarrow, A_3\uparrow] = (\{1,2,10\},\{5,6,7,8,\})$

According to the definition, $A_1\downarrow\subseteq A_2\uparrow$, $A_1\downarrow\subseteq A_2\uparrow\Longrightarrow (A_1\downarrow, A_1\uparrow)\subseteq_K (A_2\downarrow, A_2\uparrow)$ therefore, $[A_1\downarrow, A_1\uparrow]\otimes[A_3\downarrow, A_3\uparrow] = [\{1,2\},\{6,7,8\}]$
$[A_1\downarrow, A_1\uparrow]\oplus[A_3\downarrow, A_3\uparrow] = [\{1,2,3,10\},\{5,6,7,8,9\}]$ Some relevant comments are as follows:

1. As far as Agent 2 is concerned, jurcy No. 6 presents a contradicting behavior (it is both red and blue) and there is no information for jurcy No. 10.
2. Since $[A_1\downarrow, A_1\uparrow]\subseteq_K [A_2\downarrow, A_2\uparrow], [A_2\downarrow, A_2\uparrow]$ exhibits more knowledge (information) about different jurcies. In other hand, Agent 2 has more knowledge about red jurcy than Agent 1. Note that $[A_1\downarrow, A_1\uparrow]$ and $[A_3\downarrow, A_3\uparrow]$ are incomparable due to the order of knowledge.
3. If one is going to make deductions on the basis of the knowledge of Agents 1 and 3, he or she can choose either to accept the information confirmed by both Agents, or to accept all proposed information. The former approach insists on a consensus and will form the interval rough set $[A\downarrow, A\uparrow]$. Here, we only gave an example for the knowledge order, which can be discussed when one is dealing with several agents. Classical set algebra can be considered as a special case when there is only one agent present. The order of inclusion can be clarified if one thinks of a single agent logically comparing two different concepts.

6 Conclusion

In this paper, we have provided a comparative study of rough set and interval set models within the context of uncertain reasoning. The rough set model is a generalization of set theory in which a pair of new set-theoretic operators are introduced. Interval sets provide a new means for representing partially known concepts for approximating undefinable concepts or complex concepts. Interval sets are closely related to rough set which we declared as interval structure rough set or interval rough set, is discussed in this paper. Also based on an inclusion ordering and a knowledge ordering, we have discussed two types of order for rough interval set.

References

[1] Z. Bonikowski, A certain conception of the calculus of rough sets, *Notre Dame Journal of Formal Logic,* 33: 412-421, 1992.

[2] M. Chakraborty, M. Banerjee, Logic and algebra of rough sets. In: Rough Sets, Fuzzy Sets and Knowledge Discovery, *W.P. Ziarko, Fei., London, Springer-Verlag,* 196-207, 1994.

[3] T.Y. Lin, Q. Liu, Rough Approximate Operators.In: Rough Sets, Fuzzy Sets and Knowledge Discovery, *W.P. Ziarko, Ed., London, Springer-Verlag,* 256-260, 1994.

[4] A. Nakamura, On a logic of information for reasoning about knowledge. In: Rough Sets, Fuzzy Sets and Knowledge Discovery, *W.P. Ziarko,Ed.,London,Springer-Verlag,* 186-195, 1994.

[5] C.V. Negoitca, D. A. Ralescu, Applications of Fuzzy Sets to Systems Analysis, *Basel: Birkh ¨auser Verlag,* 1975.

[6] E. Orlowska, Rough set semantics for non-classical logics. In: Rough Sets, Fuzzy Sets and Knowledge Discovery, *W.P. Ziarko, Fei., London, Springer-Verlag,* 143-148, 1994.

[7] Z. Pawlak, Rough Sets, *International Journal of Computer and Information Sciences,* 11: 341-356, 1982.

[8] Z. Pawlak, Rough sets: a new approach to vagueness. In: L.A. Zadeh and J. Kacprzyk, Eds, *Fuzzy Logic for the Management of Uncertainty, New York, John Wiley and Sons,* 105-118, 1992.

[9] D. Rana, S.K. Roy, Rough Set Approach on Lattice, Journal of Uncertain Systems, 5(1): 72-80, 2011.

[10] Y.Y. Yao, X. Li, Uncertain Reasoning With Interval-Set Algebra. *In: Rough Sets, Fuzzy Sets and Knowledge Discovery, W. P. Ziarko. Ed., London, Springer-Verlag,* 178-185, 1994.

[11] Y.Y. Yao, Probabilistic rough set approximations, International Journal of Approximation Reasoning 49:, 255-271, 2008.

[12] Y.Y. Yao, Interval Sets and Interval-Set Algebras, Proceedings of the 8th IEEE International Conference on Cognitive Informatics, 307-314, 2009.