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Between $\star$-Closed Sets and $\mathcal{I}_\omega$-Closed Sets

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Abstract – Ravi et al [21] introduced the notion of $\mathcal{I}_{\hat{g}}$-closed sets ($\mathcal{I}_\omega$-closed sets) and further properties of $\mathcal{I}_{\hat{g}}$-closed sets are investigated. In this paper, we introduce the notion of $\mathcal{I}_{m\omega}$-closed sets and obtain the unified characterizations for certain families of subsets between $\star$-closed sets and $\mathcal{I}_\omega$-closed sets in an ideal topological space.

Keywords – *ω-closed set, $\mathcal{I}_\omega$-closed set, $\star$-closed set, m-structure, m-space, $\mathcal{I}_{m\omega}$-closed set.*

1 Introduction

Sheik John [25] (= Veera Kumar [27]) introduced the notion of ω-closed sets (= $\hat{g}$-closed sets). Recently many variations of ω-closed sets [23] were introduced and investigated. They are applied to introduce several low separation axioms. Since then, the further generalizations of ω-closed sets are being introduced and investigated. By combining a topological space (X, τ) and an ideal $\mathcal{I}$ on (X, τ), Dontchev et. al [4] introduced the notion of $\mathcal{I}_g$-closed sets and investigated the properties of $\mathcal{I}_g$-closed sets. By combining an m-space (X, m_x) and an ideal $\mathcal{I}$ on (X, m_x), quite recently Ozbakir and Yildirim [18] have introduced the notion of an ideal minimal spaces. Especially, the notion of m-$\mathcal{I}_g$-closed sets is introduced and investigated.

In this paper we introduce the notion of $\mathcal{I}_{m\omega}$-closed sets. In Sections 4 and 5, we obtain some basic properties and characterizations of $\mathcal{I}_{m\omega}$-closed sets. In the last section, we define several new subsets in ideal topological spaces which lie between $\star$-closed sets and $\mathcal{I}_\omega$-closed sets.

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2 Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by cl(A) and int(A) respectively. A subset A of a space (X, τ) is an α-open [16] (resp. preopen [13]) if A $\subset$ int(cl(int(A))) (resp. A $\subset$ int(cl(A))). The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ. The closure of a subset A in (X, τ^α) is denoted by cl_α(A).

Definition 2.1. *A subset A of a topological space (X, τ) is said to be*

1. *semi-open [11] if A $\subset$ cl(int(A)).*
2. *semi-preopen [5] if A $\subset$ cl(int(cl(A))).*

The complement of semi-open (resp. semi-preopen) set is said to be semi-closed (resp. semi-preclosed).

The family of all semi-open (resp. semi-preopen) sets in X is denoted by SO(X) (resp. SPO(X)).

The semi-closure of A [3] (resp. semi-preclosure of A [5]), denoted by scl(A) (resp. spcl(A)), is defined by

scl(A) = $\cap\{F$: A $\subset$ F and X $-$ F $\in$ SO(X)$\}$,

spcl(A) = $\cap\{F$: A $\subset$ F and X $-$ F $\in$ SPO(X)$\}$.

Definition 2.2. *A subset A of a topological space (X, τ) is said to be*

1. *g-closed [10] if cl(A) $\subset$ U whenever A $\subset$ U and U is open in X.*
2. *ω-closed [25] (or $\hat{g}$-closed [27]) if cl(A) $\subset$ U whenever A $\subset$ U and U is semi-open in X. The complement of ω-closed set is said to be ω-open in X.*
3. **g-closed [29] if cl(A) $\subset$ U whenever A $\subset$ U and U is ω-open in X. The complement of *g-closed set is said to be *g-open in X.*
4. *$^\#$gs-closed [28] if scl(A) $\subset$ U whenever A $\subset$ U and U is *g-open in X. The complement of $^\#$gs-closed set is said to be $^\#$gs-open in X.*
5. *spg-closed [30] if spcl(A) $\subset$ U whenever A $\subset$ U and U is semi-open in X. The complement of spg-closed set is said to be spg-open in X.*
6. *$\tilde{g}$-closed [7] if cl(A) $\subset$ U whenever A $\subset$ U and U is $^\#$gs-open in X.*
7. *$\tilde{g}$s-closed [20] if scl(A) $\subset$ U whenever A $\subset$ U and U is $^\#$gs-open in X. The complement of $\tilde{g}$s-closed set is said to be $\tilde{g}$s-open in X.*
8. *sg-closed [2] if scl(A) $\subset$ U whenever A $\subset$ U and U is semi-open in X. The complement of sg-closed set is said to be sg-open in X.*
9. *gs-closed [1] if scl(A) $\subset$ U whenever A $\subset$ U and U is open in X. The complement of gs-closed set is said to be gs-open in X.*
10. *gsp-closed [5] if spcl(A) $\subset$ U whenever A $\subset$ U and U is open in X. The complement of gsp-closed set is said to be gsp-open in X.*

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ imply $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$ [9]. Given a topological space (X, τ) with ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^\star$: $\wp(X) \to \wp(X)$, called a local function [9] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subset X$, $A^\star(\mathcal{I}, \tau) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. We will make use of the basic facts about the local function ([8], Theorem 2.3) without mentioning it explicitly. In particular, the local function is monotonic. A Kuratowski closure operator $cl^\star(.)$ for a topology $\tau^\star(\mathcal{I}, \tau)$, called the $\star$-topology and finer than τ, is defined by $cl^\star(A) = A \cup A^\star(\mathcal{I}, \tau)$ [26]. Clearly, if $\mathcal{I} = \{\emptyset\}$, then $cl^\star(A) = cl(A)$ for every subset A of X. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I}, \tau)$ and $\tau^\star$ for $\tau^\star(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal space or an ideal topological space. A subset A of an ideal space (X, τ, $\mathcal{I}$) is $\star$-closed [8] if $A^\star \subset A$. The interior of a subset A in (X, $\tau^\star(\mathcal{I})$) is denoted by $int^\star(A)$. A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is $\mathcal{I}_g$-closed [15] if $A^\star \subset U$ whenever U is open and $A \subset U$.

Lemma 2.3. *[8] Let (X, τ, $\mathcal{I}$) be an ideal topological space and A, B subsets of X. Then the following properties hold:*

1. $A \subset B \Rightarrow A^\star \subset B^\star$,
2. $A^\star = cl(A^\star) \subset cl(A)$,
3. $(A^\star)^\star \subset A^\star$,
4. $(A \cup B)^\star = A^\star \cup B^\star$,
5. $(A \cap B)^\star \subset A^\star \cap B^\star$.

Definition 2.4. *A subset A of an ideal space (X, τ, $\mathcal{I}$) is $\mathcal{I}_\omega$-closed (or $\mathcal{I}_{\hat{g}}$-closed) [21] if $A^\star \subset U$ whenever U is semi-open and $A \subset U$. A subset A of an ideal space (X, τ, $\mathcal{I}$) is said to be $\mathcal{I}_\omega$-open [21](resp. $\mathcal{I}_g$-open [15]) if $X - A$ is $\mathcal{I}_\omega$-closed (resp. $\mathcal{I}_g$-closed).*

Lemma 2.5. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. Then every ω-closed set is an $\mathcal{I}_\omega$-closed but not conversely ([21], Theorem 2.13).*

3 m-structures

Definition 3.1. *[19] A subfamily $m_x \subset \wp(X)$ is said to be a minimal structure (briefly, m-structure) on X if $\emptyset$, $X \in m_x$. The pair (X, m_x) is called a minimal space (briefly m-space). Each member of m_x is said to be m-open and the complement of an m-open set is said to be m-closed.*

Notice that (X, m_x, $\mathcal{I}$) is called an ideal m-space.

Remark 3.2. *Let (X, τ) be a topological space. Then $m_x = \tau$, SO(X) and SPO(X) are minimal structures on X.*

Definition 3.3. *Let (X, m_x) be an m-space. For a subset A of X, the m-closure of A and the m-interior of A are defined in [12] as follows:*

1. $m\text{-}cl(A) = \cap\{F : A \subset F,\ F^c \in m_x\}$,
2. $m\text{-}int(A) = \cup\{U : U \subset A,\ U \in m_x\}$.

4 $\mathcal{I}_{m\omega}$-closed sets

In this section, let (X, τ, $\mathcal{I}$) be an ideal topological space and m_x an m-structure on X. We obtain several basic properties of $\mathcal{I}_{m\omega}$-closed sets.

Definition 4.1. *Let (X, τ) be a topological space and m_x an m-structure on X. A subset A of X is said to be m-semiopen [14] if A $\subset$ m-cl(m-int(A)).*

The family of all m-semiopen sets in X is denoted by m-SO(X).

The complement of m-semiopen set is said to be m-semiclosed.

Definition 4.2. *[24] Let (X, τ) be a topological space and m_x an m-structure on X. A subset A of X is said to be*

1. *m-ω-closed if cl(A) $\subset$ U whenever A $\subset$ U and U is m-semiopen,*
2. *m-ω-open if its complement is m-ω-closed.*

Definition 4.3. *[14] Let (X, m_x) be an m-space. For a subset A of X, the m-semi-closure of A and the m-semi-interior of A, denoted by m-scl(A) and m-sint(A), respectively are defined as follows:*

1. *m-scl(A) = $\cap\{F : A \subset F,$ F is m-semiclosed in X$\}$,*
2. *m-sint(A) = $\cup\{U : U \subset A,$ U is m-semiopen in X$\}$.*

Remark 4.4. *Let (X, τ) be a topological space and A a subset of X. If m-SO(X) = SO(X) (resp. τ) and A is m-ω-closed, then A is ω-closed (g-closed).*

Definition 4.5. *A subset A of an ideal m-space (X, m_x, $\mathcal{I}$) is said to be*

1. *$\mathcal{I}_{m\omega}$-closed if $A^\star \subset U$ whenever A $\subset$ U and U is m-semiopen.*
2. *$\mathcal{I}_{m\omega}$-open if X $-$ A is $\mathcal{I}_{m\omega}$-closed.*

Remark 4.6. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A a subset of X. If m-SO(X) = SO(X) (resp. τ) and A is $\mathcal{I}_{m\omega}$-closed, then A is $\mathcal{I}_\omega$-closed (resp. $\mathcal{I}_g$-closed).*

Proposition 4.7. *Every m-ω-closed set is $\mathcal{I}_{m\omega}$-closed but not conversely.*

Proof. Let A be an m-ω-closed, then cl(A) $\subset$ U whenever A $\subset$ U and U is m-semiopen. By Lemma 2.3, $A^\star \subset$ cl(A). Hence A is $\mathcal{I}_{m\omega}$-closed.

Example 4.8. *Let X = $\{a, b, c\}$, $m_x = \{\emptyset, X, \{c\}\}$, $\tau = \{\emptyset, X, \{c\}, \{b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then m-semiopen sets are $\emptyset$, X, $\{c\}$, $\{a, c\}$ and $\{b, c\}$; semi-open sets are $\emptyset$, X, $\{c\}$, $\{a, c\}$ and $\{b, c\}$; m-ω-closed sets are $\emptyset$, X, $\{a\}$ and $\{a, b\}$; and $\mathcal{I}_{m\omega}$-closed sets are $\emptyset$, X, $\{a\}$, $\{b\}$ and $\{a, b\}$. It is clear that $\{b\}$ is $\mathcal{I}_{m\omega}$-closed set but it is not m-ω-closed.*

Proposition 4.9. *Let SO(X) $\subset$ m-SO(X). Then every $\mathcal{I}_{m\omega}$-closed set is $\mathcal{I}_\omega$-closed but not conversely.*

Proof. Suppose that A is an $\mathcal{I}_{m\omega}$-closed set. Let A $\subset$ U and U $\in$ SO(X). Since SO(X) $\subset$ m-SO(X), $A^\star \subset$ U and hence A is $\mathcal{I}_\omega$-closed.

Example 4.10. *Let $X = \{a, b, c\}$, $m_x = \{\emptyset, X, \{c\}\}$, $\tau = \{\emptyset, X, \{b\}, \{a, c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then ω-closed sets are the power set of X; m-ω-closed sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{a, b\}$ and $\{a, c\}$; $\mathcal{I}_{m\omega}$-closed sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{a, b\}$ and $\{a, c\}$; and $\mathcal{I}_\omega$-closed sets are the power set of X. It is clear that $\{c\}$ is $\mathcal{I}_\omega$-closed set but it is not $\mathcal{I}_{m\omega}$-closed.*

Remark 4.11. *Let SO(X) $\subset$ m-SO(X). Then we have the following implications for the subsets stated above.*

$$\begin{array}{ccccc} \text{closed} & \longrightarrow & \text{m-}\omega\text{-closed} & \longrightarrow & \omega\text{-closed} \\ \downarrow & & \downarrow & & \downarrow \\ \star\text{-closed} & \longrightarrow & \mathcal{I}_{m\omega}\text{-closed} & \longrightarrow & \mathcal{I}_\omega\text{-closed} \end{array}$$

The implications in the first line are known in [22]. The three vertical implications follow from Lemma 2.3(2), Proposition 4.7 and Lemma 2.5. It is obvious that every $\star$-closed is $\mathcal{I}_{m\omega}$-closed and by Proposition 4.9, every $\mathcal{I}_{m\omega}$-closed set is $\mathcal{I}_\omega$-closed.

Lemma 4.12. *[6] Let $\{A_\lambda : \lambda \in \wedge \}$ is a locally finite family of sets in $(X, \tau, \mathcal{I})$. Then $\cup_{\lambda\in\wedge} A_\lambda^\star = (\cup_{\lambda\in\wedge} A_\lambda)^\star$.*

Proposition 4.13. *If $\{A_\lambda : \lambda \in \wedge \}$ is a locally finite family of sets in $(X, \tau, \mathcal{I})$ and A_λ is $\mathcal{I}_{m\omega}$-closed for each $\lambda \in \wedge$, then $(\cup_{\lambda\in\wedge} A_\lambda)$ is $\mathcal{I}_{m\omega}$-closed.*

Proof. Let $(\cup_{\lambda\in\wedge} A_\lambda) \subset U$ where U is m-semiopen. Then $A_\lambda \subset U$ for each $\lambda \in \wedge$. Since A_λ is $\mathcal{I}_{m\omega}$-closed for each $\lambda \in \wedge$, we have $A_\lambda^\star \subset U$ and hence $\cup_{\lambda\in\wedge} A_\lambda^\star \subset U$. By Lemma 4.12, $(\cup_{\lambda\in\wedge} A_\lambda)^\star \subset U$. Hence $(\cup_{\lambda\in\wedge} A_\lambda)$ is $\mathcal{I}_{m\omega}$-closed.

Corollary 4.14. *If A and B are $\mathcal{I}_{m\omega}$-closed sets in $(X, \tau, \mathcal{I})$, then $A\cup B$ is $\mathcal{I}_{m\omega}$-closed.*

Proof. Let $A\cup B \subset U$ where U is m-semiopen. Then $A \subset U$ and $B \subset U$. Since A and B are $\mathcal{I}_{m\omega}$-closed, then $A^\star \subset U$ and $B^\star \subset U$ and so $A^\star \cup B^\star \subset U$. By Lemma 2.3, $A^\star\cup B^\star = (A\cup B)^\star$. Hence $A\cup B$ is $\mathcal{I}_{m\omega}$-closed.

Definition 4.15. *An m-structure m_x on a nonempty set X is said to have property $\mathcal{B}$ [19] if the union of any family of subsets belonging to m_x belongs to m_x.*

Example 4.16. *Let $X = \{a, b, c, d\}$, $m_x = \{\emptyset, X, \{a, b\}, \{a, c\}, \{b, d\}\}$ and $\tau = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}, \{a, b, d\}\}$. Then m-$\omega$-open sets $\emptyset$, X, $\{a\}$, $\{b\}$, $\{d\}$, $\{a, b\}$, $\{a, d\}$ and $\{a, b, d\}$. It is shown that this collection does not have property $\mathcal{B}$.*

Remark 4.17. *Let (X, τ) be a topological space. Then the families SO(X) and τ^α are all m-structure with property $\mathcal{B}$.*

Proposition 4.18. *Let SO(X) $\subset$ m-SO(X) and m-SO(X) have property $\mathcal{B}$. If A is $\mathcal{I}_{m\omega}$-closed in $(X, \tau, \mathcal{I})$ and B is closed in (X, τ), then $A\cap B$ is $\mathcal{I}_{m\omega}$-closed.*

Proof. Let $A\cap B \subset U$ where U is m-semiopen. Then we have $A \subset U\cup(X-B)$. Since $\tau \subset$ SO(X) $\subset$ m-SO(X) and so $U\cup(X-B)$ is m-semiopen. Since A is $\mathcal{I}_{m\omega}$-closed, then $A^\star \subset U\cup(X-B)$ and hence $A^\star\cap B \subset U\cap B \subset U$. By Lemma 2.3. $(A\cap B)^\star \subset A^\star\cap B^\star$. Since $\tau \subset \tau^\star$, B is $\star$-closed and $B^\star \subset B$. Therefore, we obtain $(A\cap B)^\star \subset A^\star\cap B^\star \subset A^\star\cap B \subset U$. This shows that $A\cap B$ is $\mathcal{I}_{m\omega}$-closed.

Proposition 4.19. *If A is $\mathcal{I}_{m\omega}$-closed and $A \subset B \subset cl^\star(A)$, then B is $\mathcal{I}_{m\omega}$-closed.*

Proof. Let B ⊂ U where U is m-semiopen. Then A ⊂ U and A is $\mathcal{I}_{m\omega}$-closed. Therefore $A^\star \subset U$ and $B^\star \subset cl^\star(B) \subset cl^\star(A) = A \cup A^\star \subset U$. Hence B is $\mathcal{I}_{m\omega}$-closed.

Proposition 4.20. *A subset A of X is $\mathcal{I}_{m\omega}$-open if and only if $F \subset int^\star(A)$ whenever $F \subset A$ and F is m-semiclosed.*

Proof. Suppose that A is $\mathcal{I}_{m\omega}$-open. Let F ⊂ A and F be m-semiclosed. Then X − A ⊂ X − F and X − F is m-semiopen. Since X − A is $\mathcal{I}_{m\omega}$-closed, then $(X - A)^\star \subset X - F$ and $X - int^\star(A) = cl^\star(X - A) = (X - A) \cup (X - A)^\star \subset X - F$ and hence $F \subset int^\star(A)$.

Conversely, let X − A ⊂ G where G is m-semiopen. Then X − G ⊂ A and X − G is m-semiclosed. By hypothesis, we have $X - G \subset int^\star(A)$ and hence $(X - A)^\star \subset cl^\star(X - A) = X - int^\star(A) \subset G$. Therefore, X − A is $\mathcal{I}_{m\omega}$-closed and A is $\mathcal{I}_{m\omega}$-open.

Corollary 4.21. *Let SO(X) ⊂ m-SO(X) and m-SO(X) have property $\mathcal{B}$. Then the following properties hold.*

1. *Every ⋆-open set is $\mathcal{I}_{m\omega}$-open and every $\mathcal{I}_{m\omega}$-open set is $\mathcal{I}_\omega$-open,*
2. *If A and B are $\mathcal{I}_{m\omega}$-open, then A∩B is $\mathcal{I}_{m\omega}$-open,*
3. *If A is $\mathcal{I}_{m\omega}$-open and B is open in (X, τ), then A∪B is $\mathcal{I}_{m\omega}$-open,*
4. *If A is $\mathcal{I}_{m\omega}$-open and $int^\star(A) \subset B \subset A$, then B is $\mathcal{I}_{m\omega}$-open.*

Proof. This follows from Remark 4.11, Propositions 4.18 and 4.19 and Corollary 4.14.

Lemma 4.22. *[23] Let (X, m_x) be an m-space and A a subset of X. Then x ∈ m-scl(A) if and only if U∩A ≠ ∅ for every U ∈ m-SO(X) containing x.*

Lemma 4.23. *[23] Let X be a nonempty set, m_x an m-structure on X and m-SO(X) have property $\mathcal{B}$. For a subset A of X, the following properties hold:*

1. *A ∈ m-SO(X) if and only if m-sint(A) = A,*
2. *A is m-semiclosed if and only if m-scl(A) = A,*
3. *m-sint(A) ∈ m-SO(X) and m-scl(A) is m-semiclosed.*

5 Characterizations of $\mathcal{I}_{m\omega}$-closed sets

In this section, let (X, τ, $\mathcal{I}$) be an ideal topological space and m_x an m-structure on X. We obtain several characterizations of $\mathcal{I}_{m\omega}$-closed sets.

Theorem 5.1. *For a subset A of X, the following properties are equivalent:*

1. *A is $\mathcal{I}_{m\omega}$-closed,*
2. *$cl^\star(A) \subset U$ whenever A ⊂ U and U is m-semiopen,*
3. *$cl^\star(A) \cap F = \emptyset$ whenever A∩F = ∅ and F is m-semiclosed.*

Proof. (1) $\Rightarrow$ (2) Let A $\subset$ U where U is m-semiopen. Then by (1), $A^\star \subset U$ and $cl^\star(A) = A \cup A^\star \subset U$.

(2) $\Rightarrow$ (3) Let $A \cap F = \emptyset$ and F be m-semiclosed. Then $A \subset X - F$ and $X - F$ is m-semiopen. By (2), $cl^\star(A) \subset X - F$. Hence $cl^\star(A) \cap F = \emptyset$.

(3) $\Rightarrow$ (1) Let A $\subset$ U where U is m-semiopen. Then $A \cap (X - U) = \emptyset$ and $X - U$ is m-semiclosed. By (3), $cl^\star(A) \cap (X - U) = \emptyset$ and so $A^\star \subset cl^\star(A) \subset U$. Hence A is $\mathcal{I}_{m\omega}$-closed.

Definition 5.2. *Let (X, τ) be a topological space, m_x an m-structure on X and A a subset of X. The subset $\wedge_{ms}(A)$ is defined as follows:*

$\wedge_{ms}(A) = \cap\{U : A \subset U,\ U \in m\text{-}SO(X)\}$.

Theorem 5.3. *A subset A of X is $\mathcal{I}_{m\omega}$-closed if and only if $cl^\star(A) \subset \wedge_{ms}(A)$.*

Proof. Suppose that A is $\mathcal{I}_{m\omega}$-closed. If $x \notin \wedge_{ms}(A)$, then there exists $U \in$ m-SO(X) such that $A \subset U$ and $x \notin U$. Since A is $\mathcal{I}_{m\omega}$-closed, by Theorem 5.1, $cl^\star(A) \subset U$ and hence $x \notin cl^\star(A)$. Hence we obtain $cl^\star(A) \subset \wedge_{ms}(A)$.

Conversely, suppose that $cl^\star(A) \subset \wedge_{ms}(A)$. Let $A \subset U$ and $U \in$ m-SO(X). Then $cl^\star(A) \subset \wedge_{ms}(A) \subset U$. By Theorem 5.1, A is $\mathcal{I}_{m\omega}$-closed.

Theorem 5.4. *Let SO(X) $\subset$ m-SO(X) and m-SO(X) have property $\mathcal{B}$. For a subset A of X, the following properties are equivalent:*

1. *A is $\mathcal{I}_{m\omega}$-closed,*
2. *$A^\star - A$ contains no nonempty m-semiclosed set,*
3. *$A^\star - A$ is $\mathcal{I}_{m\omega}$-open,*
4. *$A \cup (X - A^\star)$ is $\mathcal{I}_{m\omega}$-closed,*
5. *$cl^\star(A) - A$ contains no nonempty m-semiclosed set,*
6. *m-scl$(\{x\}) \cap A \neq \emptyset$ for each $x \in cl^\star(A)$.*

Proof. (1) $\Rightarrow$ (2) Suppose that A is $\mathcal{I}_{m\omega}$-closed. Let $F \subset A^\star - A$ and F be m-semiclosed. Then $F \subset A^\star$ and $F \not\subset A$. We have $A \subset X - F$ and $X - F$ is m-semiopen. Therefore $A^\star \subset X - F$ and so $F \subset X - A^\star$. Hence $F \subset A^\star \cap (X - A^\star) = \emptyset$.

(2) $\Rightarrow$ (3) Let $F \subset A^\star - A$ and F be m-semiclosed. By (2), we have $F = \emptyset$ and so $F \subset int^\star(A^\star - A)$. By Proposition 4.20, $A^\star - A$ is $\mathcal{I}_{m\omega}$-open.

(3) $\Rightarrow$ (1) Let $A \subset U$ where U is m-semiopen. Then $X - U \subset X - A \Rightarrow A^\star \cap (X - U) \subset A^\star \cap (X - A) = A^\star - A$. Since $A^\star$ is closed in (X, τ) and hence $A^\star$ is semi-closed in (X, τ). Since every semi-closed set is m-semiclosed and so $A^\star$ is m-semiclosed. Since m-SO(X) has property $\mathcal{B}$, then $A^\star \cap (X - U)$ is m-semiclosed and by (3), $A^\star - A$ is $\mathcal{I}_{m\omega}$-open. Therefore by Proposition 4.20, $A^\star \cap (X - U) \subset int^\star(A^\star - A) = int^\star(A^\star \cap (X - A)) = int^\star(A^\star) \cap int^\star(X - A) = int^\star(A^\star) \cap (X - cl^\star(A)) \subset A^\star \cap (A \cup A^\star)^c = A^\star \cap (A^c \cap (A^\star)^c) = \emptyset$ and hence $A^\star \subset U$. Hence A is $\mathcal{I}_{m\omega}$-closed.

(3) $\Leftrightarrow$ (4) This follows from the fact that $X - (A^\star - A) = X \cap (A^\star \cap A^c)^c = X \cap ((A^\star)^c \cup A) = (X \cap (A^\star)^c) \cup (X \cap A) = A \cup (X - A^\star)$.

(2) $\Leftrightarrow$ (5) This follows from the fact that $cl^\star(A) - A = (A \cup A^\star) - A = (A \cup A^\star) \cap A^c = (A \cap A^c) \cup (A^\star \cap A^c) = A^\star \cap A^c = A^\star - A$.

(1) ⇒ (6) Suppose that A is $\mathcal{I}_{m\omega}$-closed and m-scl({x})∩A = ∅ for some x ∈ cl⋆(A). We know that m-scl({x}) is m-semiclosed. We have A ⊂ X − (m-scl({x})) and X − (m-scl({x})) is m-semiopen. Therefore by Theorem 5.1, cl⋆(A) ⊂ X − (m-scl({x})) ⊂ X − {x}. This contradicts that x ∈ cl⋆(A). Hence m-scl({x})∩A ≠ ∅ for each x ∈ cl⋆(A).

(6) ⇒ (1) Suppose m-scl({x})∩A ≠ ∅ for each x ∈ cl⋆(A). We have to prove that A is $\mathcal{I}_{m\omega}$-closed. Suppose A is not $\mathcal{I}_{m\omega}$-closed. Then by Theorem 5.1, ∅ ≠ cl⋆(A) − U for some m-semiopen set U containing A. There exists x ∈ cl⋆(A) − U. Since x ∉ U, by Lemma 4.22, m-scl({x})∩U = ∅ and hence m-scl({x})∩A ⊂ m-scl({x})∩U = ∅. This shows that m-scl({x})∩A = ∅ for some x ∈ cl⋆(A). This is a contradiction. Hence A is $\mathcal{I}_{m\omega}$-closed.

Corollary 5.5. *Let SO(X) ⊂ m-SO(X) and m-SO(X) have property $\mathcal{B}$. For a subset A of X, the following properties are equivalent:*

1. *A is $\mathcal{I}_{m\omega}$-open,*
2. *A − int⋆(A) contains no nonempty m-semiclosed set,*
3. *m-scl({x})∩(X − A) ≠ ∅ for each x ∈ X − int⋆(A).*

Proof. This follows from Theorem 5.4.

Theorem 5.6. *Let SO(X) ⊂ m-SO(X) and m-SO(X) have property $\mathcal{B}$. A subset A of X is $\mathcal{I}_{m\omega}$-closed if and only if A = F − N where F is ⋆-closed and N contains no nonempty m-semiclosed set.*

Proof. If A is $\mathcal{I}_{m\omega}$-closed, then by Theorem 5.4, N = A⋆ − A contains no nonempty m-semiclosed set. If F = cl⋆(A), then A∪A⋆ = cl⋆(A) = F and by Lemma 2.3, we obtain F⋆ = (A∪A⋆)⋆ = A⋆∪(A⋆)⋆ ⊂A⋆∪A = F. Therefore F is ⋆-closed such that F − N = (A∪A⋆) − (A⋆ − A) = (A∪A⋆)∩(A⋆∩A^c)^c = (A∪A⋆)∩(A∪(A⋆)^c) = A∪(A⋆∩(A⋆)^c) = A.

Conversely, suppose A = F − N where F is ⋆-closed and N contains no nonempty m-semiclosed set. Let U be a m-semiopen set such that A ⊂ U. Then F − N ⊂ U ⇒ F∩(X − U) ⊂ N. Since A⋆ is m-semiclosed, hence A⋆∩(X − U) is m-semiclosed. Since A ⊂ F and F⋆ ⊂ F, then A⋆∩(X − U)⊂ F⋆∩(X − U)⊂F∩(X − U)⊂N. Therefore, A⋆∩(X − U) = ∅ and so A⋆ ⊂ U. Hence A is $\mathcal{I}_{m\omega}$-closed.

6 New forms of closed sets in ideal topological spaces

Definition 6.1. *A subset A of a space (X, τ) is called a $\tilde{g}$-semi-preclosed set (briefly $\tilde{g}$sp-closed set) if spcl(A) ⊂ U whenever A ⊂ U and U is #gs-open in X. The complement of $\tilde{g}$sp-closed set is $\tilde{g}$sp-open in X.*

By SO(X) (resp. #GSO(X), SGO(X), GSO(X), SPO(X), $\tilde{G}$SPO(X), SPGO(X), GSPO(X)) we denote the collection of all semi-open (resp. #gs-open, sg-open, gs-open, semi-preopen, $\tilde{g}$sp-open, spg-open, gsp-open) sets of the topological space (X, τ). These collections are m-structures on X. By the definitions, we obtain the following diagram:

Diagram I

$$\begin{array}{ccccccc} \text{semi-open} & \longrightarrow & \tilde{g}s\text{-open} & \longrightarrow & \text{sg-open} & \longrightarrow & \text{gs-open} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{semi-preopen} & \longrightarrow & \tilde{g}sp\text{-open} & \longrightarrow & \text{spg-open} & \longrightarrow & \text{gsp-open} \end{array}$$

For subsets of an ideal topological space (X, τ, $\mathcal{I}$), we can define new types of closed sets as follows:

Definition 6.2. *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be $\mathcal{I}_\omega$-closed (resp. $\mathcal{I}_{\tilde{g}sg}$-closed, $\mathcal{I}_{sgg}$-closed, $\mathcal{I}_{gsg}$-closed, $\mathcal{I}_{sog}$-closed, $\mathcal{I}_{\tilde{g}sp}$-closed, $\mathcal{I}_{spg}$-closed, $\mathcal{I}_{gsp}$-closed) if $A^\star \subset U$ whenever $A \subset U$ and U is semi-open (resp. $\tilde{g}s$-open, sg-open, gs-open, semi-preopen, $\tilde{g}sp$-open, spg-open, gsp-open) in (X, τ).*

By Diagram I and Definition 6.2, we have the following diagram:

Diagram II

$$\begin{array}{ccccccccc} \mathcal{I}_\omega\text{-closed} & \longleftarrow & \mathcal{I}_{\tilde{g}sg}\text{-closed} & \longleftarrow & \mathcal{I}_{sgg}\text{-closed} & \longleftarrow & \mathcal{I}_{gsg}\text{-closed} & & \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \mathcal{I}_{sog}\text{-closed} & \longleftarrow & \mathcal{I}_{\tilde{g}sp}\text{-closed} & \longleftarrow & \mathcal{I}_{spg}\text{-closed} & \longleftarrow & \mathcal{I}_{gsp}\text{-closed} & \longleftarrow & \star\text{-closed} \end{array}$$

7 Conclusion

Topological ideas are present in almost all areas of today's mathematics. The subject topology consists of different branches, such as point-set topology, algebraic topology and differential topology. Johann Benedict Listing (1802-1882) was the first to use the word topology. In Mathematics, general topology or point-set topology is the branch of topology which studies properties of topological spaces and structures defined on them. General topology provides the most general framework where fundamental concepts of topology such as open/closed sets, continuity, interior/exterior/boundary points and limit points could be defined. General topology, as a distinct branch of mathematics, took shape in the middle of the twentieth century with the publications of the widely used books by Bourbaki and Kelley with numerous applications to Functional Analysis, Theoretical Physics (Quantum Mechanics), Theory of Games, etc. the subject became prestigious.

The notions of sets and functions in topological spaces, ideal topological spaces, minimal spaces and ideal minimal spaces are extensively developed and used in many engineering problems, information systems, particle physics, computational topology and mathematical sciences. By researching generalizations of closed sets in various fields in general topology, some new separation axioms have been founded and they turn out to be useful in the study of digital topology. Therefore, all sets defined in this paper will have many possibilities of applications in digital topology and computer graphics.

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