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Hesitant fuzzy soft sets

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Abstract – A new hybrid structure- hesitant fuzzy soft set, involving soft set is introduced. Soft set is relatively a new approach initiated by Molodtsov in 1999 to deal impreciseness and uncertainty. Hesitant fuzzy set is a generalization of fuzzy set whose membership is a subset of $[0,1]$. This paper is an endeavor to establish a link between soft sets and hesitant fuzzy sets. Basic operation such as intersection, union, compliment is defined and De Morgan's law is also proved. It also discusses its use in decision making problem

Keywords –
*Soft sets, Hesitant fuzzy sets,
Hesitant fuzzy soft sets,
De Morgan's laws.*

1. Introduction

The theory of soft set was introduced by Molodtsov in 1999[1]. It is completely a new approach for modeling vagueness and uncertainty. The traditional soft set is a mapping from parameter to the crisp subset of universe. In [2] Maji et al discussed theoretical aspect of soft sets and they introduced several operations on soft sets. Soft set theory has proven useful in many different fields such as decision making [3-8], data analysis [9], forecasting [10] and simulation [11]. Zadeh [12] introduce the concept of fuzzy sets as new mathematical tool for uncertainty and it made its own place in decision making problems. Later to make decision making evaluation more effective so many generalization are defined such as Intuitionistic fuzzy sets [13], type 2 fuzzy sets [14], interval valued fuzzy sets [15]. However, when defining the membership degree of an element to a set, the difficulty of establishing the membership degree is not because we have a margin of error (as in Intuitionistic fuzzy set or interval-valued fuzzy set), or some possibility distribution (as in type-2 fuzzy set) on the possible values, but because we have a set of possible values. To deal with

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such cases, Torra and Narukawa [16] and Torra [17] introduced another generalization of fuzzy set, hesitant fuzzy set, allowing the membership degree having a set of possible values.

They have established a relationship between their envelope and Intuitionistic fuzzy sets and have shown that although they can be represented as fuzzy multisets, the interpretation is different and their operations cannot be applied.

Hybrid structures involving soft sets such as fuzzy soft set, soft fuzzy sets, rough soft sets, and soft rough sets are introduced by several researchers. In Yang et. al [18] the standard soft set theory is expanded to a fuzzy one in which the fuzzy character of parameters in real world is taken into consideration. Feng et.al. [19] have investigated the problem of combining soft sets with fuzzy sets and rough sets. Ali [20] discussed the concept of an approximation space associated with each parameter in a soft set and an approximation space associated with the soft set is defined. Also, based on a novel granulation structures called soft approximation spaces, Feng [21] introduced soft rough approximations and soft rough sets. Jiang et al. [22] combined the interval-valued Intuitionistic fuzzy sets and soft sets, from which a new soft set model, i.e., interval-valued Intuitionistic fuzzy soft set theory, was obtained.

The algebraic nature of soft set is studied by Aktas and Cagman [23] who initiated soft groups, F. Feng [24] defined soft semirings. Sun [25] introduced a basic version of soft module theory. Studies on topological structure on soft set is also going fast by researchers in this field. There is two version of topology defined on soft sets by Shabir and Naz [26], and Cagman et. al. [27]. The concept of fuzzy soft topology is introduced are studied by Tanay and Kandemir [28]. The concepts of soft set relations, the Cartesian product of the soft sets and soft set functions are defined by Babitha and Sunil [29]. As a continuation of their work, Yang and Guo [30] defined on soft set theory the notions of anti-reflexive kernel, symmetric kernel, reflexive closure, and symmetric closure of a soft set relation. Moreover transitive closure of soft set relation and ordering on soft set is defined by Babitha and Sunil [31]

In soft set theory membership is decided by adequate parameters, hesitant fuzzy set employ all possible values for the membership of an element. Although these two theories are quite distinct yet deal with uncertainty joint application of these theories may result in a fruitful way. To get into this direction this paper introduce the concept of hesitant fuzzy soft sets as new hybrid model to handle uncertainties. The rest of this paper is organized as follows.

The second section presents some fundamental concepts in soft set and hesitant fuzzy sets. Section 3 devoted for detailed study on hesitant fuzzy sets and proves the De Morgan's law. It introduce the concept of hesitant fuzzy soft sets and gives basic operation such as union, intersection, compliment. De Morgan's law in hesitant fuzzy soft case is also proved. In section 4 discuss about the relationship between multi group decision making problems and hesitant fuzzy soft set a. It show that how HFS set can be useful in decision making problem by proposing an algorithm for it. Finally, conclusions are presented in the last section.

2. Preliminaries and basic definition

In the current section we recollect the basic definitions and notations as introduced by Molodtsov [1] and Maji et.al. [2]

Definition 2.1 [1] Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U and $A \subset E$. A pair (F, A) is called a soft set over U , where F is a mapping given by $F: A \rightarrow P(U)$.

Definition 2.2 [2] For two soft sets (F, A) and (G, B) over a common universe U , we say that (F, A) is a soft subset of (G, B) if

- (i) $A \subset B$, and
- (ii) $\forall \varepsilon \in A, F(\varepsilon)$ and $G(\varepsilon)$ are identical approximations.

We write $(F, A) \tilde{\subset} (G, B)$.

(F, A) is said to be a soft super set of (G, B) , if (G, B) is a soft subset of (F, A) . We denote it by $(F, A) \tilde{\supset} (G, B)$.

Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A)

Definition 2.3 [2] Union of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C) , where $C = A \cup B$, and $\forall e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B \\ G(e), & \text{if } e \in B - A \\ F(e) \cup G(e), & \text{if } e \in A \cap B \end{cases}$$

We write $(F, A) \tilde{\cup} (G, B) = (H, C)$.

Definition 2.4 [2] Intersection of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C) , where $C = A \cap B$, and $\forall e \in C, H(e) = F(e) \cap G(e)$.

We write $(F, A) \tilde{\cap} (G, B) = (H, C)$.

Definition 2.5 [2] If (F, A) and (G, B) over a common universe U are soft sets. Then (F, A) AND (G, B) denoted by $(F, A) \wedge (G, B)$ is defined as $(F, A) \wedge (G, B) = (H, A \times B)$ where

$$H(a, b) = F(a) \cap G(b)$$

Definition 2.6 [2] If (F, A) and (G, B) over a common universe U are soft sets then (F, A) OR (G, B) denoted by $(F, A) \vee (G, B)$ is defined as $(F, A) \vee (G, B) = (H, A \times B)$ where

$$H(a, b) = F(a) \cup G(b)$$

3. Hesitant fuzzy sets and hesitant fuzzy soft sets

Hesitant fuzzy sets

In this section the definition of hesitant fuzzy sets and some basic operations are given. Moreover the De Morgan's law is proved.

Definition 3.1 [17] Given a fixed set X, then a hesitant fuzzy set (shortly HFS) in X is in terms of a function that when applied to X return a subset of [0, 1]

Definition 3.2 [17] Given an hesitant fuzzy set h, we define below it lower and upper bound as

$$\begin{aligned} \text{lower bound } h^-(x) &= \min h(x) \\ \text{upper bound } h^+(x) &= \max h(x) \end{aligned}$$

Definition 3.3[17] Given a hesitant fuzzy set represented by its membership function h we define its compliment as follows

$$h^c(x) = \bigcup_{\gamma \in h(x)} \{1 - \gamma\}$$

Definition 3.4 [17] Given two hesitant fuzzy sets represented by their membership functions h_1 and h_2 , we define their union represented by $h_1 \cup h_2$ as

$$h_1 \cup h_2(x) = \{h \in (h_1(x) \cup h_2(x)) / h \geq \max(h_1^-, h_2^-)\}$$

Definition 3.5 [17] Given two hesitant fuzzy sets represented by their membership functions h_1 and h_2 , we define their intersection represented by $h_1 \cap h_2$ as

$$h_1 \cap h_2(x) = \left\{ h \in \frac{h_1(x) \cup h_2(x)}{h} \leq \min(h_1^+, h_2^+) \right\}$$

Proposition 3.6 Let h_A be hesitant fuzzy set defined on A. Then we have

1. $h_A^- + h_{A_c}^+ = 1$
2. $h_A^+ + h_{A_c}^- = 1$

Proof: The proof is obvious from the definition of h_A^- , h_{A_c} , h_A^+ and using the fact that for every

$$x \in [0, 1], \max\{x : x \in [0, 1]\} = 1 - \min\{1 - x : x \in [0, 1]\}$$

De Morgan' s laws in Hesitant fuzzy sets

Proposition 3.7 Let A and B be two hesitant fuzzy sets with membership functions h_A and h_B . Then

1. $(h_A \cup h_B)^c = h_A^c \cap h_B^c$
2. $(h_A \cap h_B)^c = h_A^c \cup h_B^c$

Proof: Let $\gamma \in (h_A \cup h_B)^c(x)$ for some $x \in X$. Then

$$1-\gamma \in h_A \cup h_B \text{ such that } 1-\gamma \geq \max\{h_A^-, h_B^-\}$$

Then

$$1-\gamma \geq h_A^- \text{ and } 1-\gamma \geq h_B^-$$

Then we have

$$\gamma \leq 1 - h_A^- \text{ and } \gamma \leq 1 - h_B^-$$

By proposition 3.8, we have

$$\gamma \leq h_{A^c}^+ \text{ and } \gamma \leq h_{B^c}^+$$

Then

$$\gamma \leq \min\{h_{A^c}^+, h_{B^c}^+\}.$$

Hence

$$\gamma \in h_A^c \cap h_B^c$$

Then

$$(h_A \cup h_B)^c(x) \subseteq (h_A^c \cap h_B^c)(x) \text{ for every } x \in X$$

For the converse part, let

$$\gamma \in (h_A^c \cap h_B^c)(x)$$

Then by definition

$$\gamma \in (h_A^c \cup h_B^c)$$

such that

$$\gamma \leq \min\{h_{A^c}^+, h_{B^c}^+\}.$$

Then

$$\gamma \leq h_{A^c}^+ \text{ and } \gamma \leq h_{B^c}^+.$$

Then we have

$$1-\gamma \geq h_A^- \text{ and } 1-\gamma \geq h_B^- \text{ and } 1-\gamma \geq \max\{h_A^-, h_B^-\}$$

This imply that

$$1-\gamma \in (h_A \cup h_B)(x)$$

Thus we have

$$\gamma \in (h_A \cup h_B)^c(x)$$

and hence

$$(h_A^c \cap h_B^c)(x) \subseteq (h_A \cup h_B)^c(x) \text{ for every } x \in X.$$

Thus we have

$$(h_A \cup h_B)^c = h_A^c \cap h_B^c$$

2. Similarly we can prove for the second case.

Definition 3.8 Let h_1, h_2 be two hesitant fuzzy sets. Then h_1 is hesitant fuzzy subset of h_2 if

$$h_1(x) \subset h_2(x) \text{ for every } x \text{ in } X$$

Hesitant fuzzy soft sets

Let U be a universal set and E be set of parameters. Let $HF(U)$ denotes the set of all hesitant fuzzy sets defined over U .

Definition 3.9 A pair (F,E) is a hesitant fuzzy soft sets if $F(e) \in HF(U)$ for every e in E .

Example 3.10 let U be set of participants performing dance programme. $U = \{c_1, c_2, c_3, c_4\}$. Let $A = \{\text{confident, creative, timing}\}$. Then hesitant fuzzy soft sets (F,A) defined as below gives the evaluation of the performance of candidates by three judges .

$$\begin{aligned} F(\text{confident}) &= \{c_1 = \{0.7,0.6,0.8\}, c_2 = \{0.4,0.5,0.7\}, c_3 = \{0.8,0.9,0.9\}, c_4 = \{0.8,0.9,0.8\}\} \\ F(\text{creative}) &= \{c_1 = \{0.5,0.6,0.6\}, c_2 = \{0.6,0.7,0.55\}, c_3 = \{0.8,0.9,0.82\}, c_4 = \{1,0.9,0.8\}\} \\ F(\text{timing}) &= \{c_1 = \{0.8,0.7,0.9\}, c_2 = \{0.6,0.8,0.45\}, c_3 = \{0.78,0.9,0.76\}, c_4 = \{0.8,0.65,0.8\}\} \end{aligned}$$

Definition 3.11 For two hesitant fuzzy soft sets (F,A) and (G,B) over a common universe U we say that (F,A) is hesitant fuzzy soft subset of (G,B) if

- (i) $A \subseteq B$
- (ii) $F(a)$ is sub hesitant fuzzy sub set of $G(a)$ for every a in A .

Definition 3.12 Let (F,A) be hesitant fuzzy soft set. Then the compliment of (F,A) is denoted by $(F,A)^c$ is defined by $(F,A)^c = (F^c,A)$ where $F^c(a)$ is the compliment of the hesitant fuzzy set $F(a)$

Proposition 3.13 Let (F,A) and (G,B) be two hesitant fuzzy soft sets over U . Then \cup, \cap, \wedge, \vee of (F,A) and (G,B) are hesitant fuzzy soft sets.

Propositio 3.14 Let (F,A) and (G,B) be two hesitant fuzzy soft sets over U . Then

- (i) $[(F,A) \cup (G,B)]^c = (F,A)^c \cap (G,B)^c$
- (ii) $[(F,A) \cap (G,B)]^c = (F,A)^c \cup (G,B)^c$

Proof: Let $(F,A) \cup (G,B) = (H,A \cup B)$ where $H(c) = F(c) \cup G(c)$ for every $c \in C$. Then

$$\begin{aligned} H^c(c) &= (F(c) \cup G(c))^c \\ &= F^c(c) \cap G^c(c) \end{aligned}$$

using Demorgan’s law for hesitant fuzzy sets.

Similarly we can prove (ii) also.

4. Hesitant fuzzy soft set in decision making problems

In the preceding section we have investigated the application of hesitant fuzzy soft set in group decision making problems. Let U be a universal set consisting of alternatives. Let E be set of criteria. We can represent a group decision making problem using the hesitant fuzzy soft approach in the following way.

Let (F, A) denotes the corresponding hesitant fuzzy soft set in which F_i^j represents the hesitant fuzzy set for the alternative u_j corresponding to the criteria e_i .

Definition 4.1[32] For a hesitant fuzzy element $h(x)$,

$$s(h) = \frac{1}{l(h)} \sum_{\gamma \in h(x)} \gamma$$

is called the score function of $h(x)$ where $l(h)$ denotes number of values in $h(x)$.

Definition 4.3 Let (F, A) denotes hesitant fuzzy soft set, Then the fuzzy soft set (F_S, A) in which each entries in the fuzzy set $F_S(e)$ is the score function of the respective entries in the hesitant fuzzy set $F(e)$ is called as score matrix.

Definition 4.4 The table obtained by calculating the average of $F_S(e_i)$ for each u_j is called as decision table. This table determines the optimal outcome for the decision making problem. Now we will propose an algorithm which show that by considering score matrix, a hesitant fuzzy soft based decision making problem can be reduced into much simpler treatment of fuzzy soft set.

Algorithm

1. Input the hesitant fuzzy soft set (F, A)
2. Obtain the score matrix (F_S, A) corresponds to (F, A)
3. Calculate the average of $F_S(e_i)$ for each u_j and let it be denoted as a_j .
This is the decision table
4. Select the optimal alternative u_k if $a_k = \max_j a_j$
5. If k has more than one value then any one of u_k may be chosen.

Remark 4.5 In decision making problems further representational capability can be added by associating with each parameter e_i a value $w_i \in [0, 1]$ called its weight. In the case of multi-criteria decision making, these weights can be used to represent the different importance of the concerned criteria. In this case there is a small change in the above algorithm. In step 3 instead of average we take weighted average

$$\frac{\sum_{i=1}^n F_S(e_i) w_i}{n}$$

and follows the next step.

Application 4.6 (Job allocation problem)

Let us consider decision-making problem of allocating a particular job to the best possible person who fulfills the requirements of the job. Selection is done by the interview board consisting of three members. Let $U = \{u_1, u_2, u_3, u_4\}$ be crisp set of four persons for the job. Let $A = \{\text{enterprising, confident, willing to take risks, hardworking}\}$ be the set of parameters which represents the criteria for the problem. Let A can be represented as $A = \{e_1, e_2, e_3, e_4\}$ The problem is the selection of best person who satisfy the criteria to utmost extent.

All the available information on these candidates can be characterized by hesitant fuzzy soft set (F, A) . The tabular representation of . hesitant fuzzy soft set (F, A) is shown in Table 1. In Table 1, we can see that the evaluation for an alternative to satisfy a criterion is represented by hesitant fuzzy set representing the grades given by the three interviewers.

	u_1	u_2	u_3	u_4
e_1	{0.6,0.70,0.8}	{0.3,0.40,0.45}	{0.5, 0.6,0.7}	{0.7,0.8,0.8}
e_2	{0.7,0.8,0.85}	{0.6,0.7,0.65}	{0.7,0.8,0.65}	{0.6,0.720,0.8}
e_3	{0.76,0.82,0.65}	{0.76,0.7,0.8}	{0.81,0.66,0.9}	{0.9,0.8,0.9}
e_4	{0.8,0.82,0.88}	{0.74,0.68,0.52}	{0.56,0.7,0.68}	{0.76,0.7,0.8}

Table 1

Then the score matrix (F_s, A) corresponds to (F, A) given in the table 1 is as follows:

	u_1	u_2	u_3	u_4
e_1	0.7	0.3833	0.6	0.766667
e_2	0.7833	0.65	0.7166	0.70667
e_3	0.7433	0.7533	0.79	0.866667
e_4	0.8333	0.6466	0.64667	0.7533

Table 2

The decision table for each person u_j obtained as follows

a_j	Values
a_1	.76497
a_2	.6083
a_3	.688
a_4	.7733

Table 3

From above table its clear that the optimal alternative is the candidate a_4 .

5. Conclusion

Soft set is completely a new approach for modeling vagueness and uncertainty. Hesitant fuzzy set is a generalization of fuzzy set allowing the membership degree having a set of possible values. This paper introduces a new hybrid structure connecting soft sets and hesitant fuzzy sets in more fruitful way. It discusses set theoretic operation such as compliment, union, intersection. More over De Morgan law in hesitant fuzzy soft case is also proved. Finally this concept is used in decision making problem. Although this study is a preliminary proposal concerning the hesitant fuzzy soft set model, we hope it will give rise to a potentially interesting research direction.

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