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Soft Lattices

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Abstract

Soft set theory was introduced by Molodtsov in 1999 as a general mathematical tool for dealing with problems that contain uncertainity. In this paper, we define concept of a soft lattice, soft sublattice, complete soft lattice, modular soft lattice, distributive soft lattice, soft chain and study their related properties.

Keywords: *Soft sets, soft sublattices, complete soft lattices, modular soft lattices, distributive soft lattices, soft chain.*

1 Introduction

Soft set theory [31] was firstly introduced by Molodtsov in 1999 as a general mathematical tool for dealing with uncertainty. The operations of soft sets are defined by Maji et al.[30] and redefined by Cagman and Enginoglu[6]. Recently, the properties and applications on the soft set theory have been studied increasingly [2, 9, 17, 34, 38]. The algebraic structure of soft set theory has also been studied in more detail [1, 4, 11, 18, 19, 21, 22, 23, 24, 25],

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and many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets [4, 8, 12, 28, 35, 37].

The soft lattice structures are constructed by Nagarajan and Meenambigai [32] and Li [27] over a soft set. In this paper, different than Li [27] and Nagarajan and Meenambigai [32], we define soft lattices over a collection of soft sets by using Cagman and Enginoglu's [6] operations of the soft sets. We also give an algebraical and a set-theorical definition of soft lattices and we prove that algebrical and set-theorical definitions are equivalent. In addition, we introduce complete soft lattice, soft sublattice, soft chain, distributive soft lattice, modular soft lattice and discuss their related properties.

2 Soft set theory

In this section, for subsequent discussions, we have presented the basic definitions and results of soft set theory which are taken from earlier studies [6, 30, 31].

Throughout this work, U refers to an initial universe, $P(U)$ is the power set of U, E is a set of parameters and $A \subseteq E$.

Definition 2.1. *A function $f_A : E \to P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$, is called a soft set over U.*

The set of all soft sets over U is denoted by $S(U)$.

Definition 2.2. *Let $f_A \in S(U)$. If $f_A(x) = \emptyset$ for all $x \in E$, then f_A is called an empty soft set, denoted by f_Φ.*

If $f_A(x) = U$ for all $x \in A$, then f_A is called A-universal soft set, denoted by $f_{\tilde{A}}$.

If $A = E$, then the A-universal soft set is called universal soft set denoted by $f_{\tilde{E}}$.

Definition 2.3. *Let $f_A, f_B \in S(U)$. Then, f_A is a soft subset of f_B, denoted by $f_A \tilde{\subseteq} f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.*

f_A and f_B are equal, denoted by $f_A = f_B$, if and only if $f_A(x) = f_B(x)$ for all $x \in E$.

Remark 2.4. *$f_A \tilde{\subseteq} f_B$ does not imply that every element of f_A is an element of f_B. Therefore the definition of classical subset is not valid for the soft subset. For example, let $U = \{u_1, u_2, u_3, u_4\}$ be a universal set of objects and $E = \{x_1, x_2, x_3\}$ be the set of all parameters. If $A = \{x_1\}$ and $B = \{x_1, x_3\}$, and $f_A = \{(x_1, \{u_2, u_4\})\}$, $f_B = \{(x_1, \{u_2, u_3, u_4\}), (x_3, \{u_1, u_5\})\}$, then for all $e \in f_A$, $f_A(x) \subseteq f_B(x)$ is valid. Hence $f_A \tilde{\subseteq} f_B$. It is clear that, $(x_1, f_A(x_1)) \in f_A$ but $(x_1, f_A(x_1)) \notin f_B$.*

Proposition 2.5. *If $f_A, f_B \in S(U)$, then*

1. $f_A \tilde{\subseteq} f_{\tilde{E}}$
2. $f_\Phi \tilde{\subseteq} f_A$
3. $f_A \tilde{\subseteq} f_A$
4. $f_A \tilde{\subseteq} f_B$ *and* $f_B \tilde{\subseteq} f_C \Rightarrow f_A \tilde{\subseteq} f_C$

Definition 2.6. *Let $f_A \in S(U)$. Then, soft complement of f_A is defined by $f_A^{\tilde{c}} = f_{A^{\tilde{c}}}$ such that $f_{A^{\tilde{c}}}(x) = f_A^c(x) = U \setminus f_A(x)$ for all $x \in E$.*

Definition 2.7. *Let $f_A, f_B \in S(U)$. Then, soft union of f_A and f_B is defined by $f_A \tilde{\cup} f_B = f_{A\tilde{\cup}B}$ such that $f_{A\tilde{\cup}B}(x) = f_A(x) \cup f_B(x)$ for all $x \in E$.*

Soft intersection of f_A and f_B is defined by $f_A \tilde{\cap} f_B = f_{A\tilde{\cap}B}$ such that $f_{A\tilde{\cap}B}(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.

Proposition 2.8. *If $f_A, f_B, f_C \in S(U)$, then*

1. $f_A \tilde{\cup} f_A = f_A$
2. $f_A \tilde{\cup} f_\Phi = f_A$
3. $f_A \tilde{\cup} f_{\tilde{E}} = f_{\tilde{E}}$
4. $f_A \tilde{\cup} f_A^{\tilde{c}} = f_{\tilde{E}}$
5. $f_A \tilde{\cup} f_B = f_B \tilde{\cup} f_A$
6. $(f_A \tilde{\cup} f_B) \tilde{\cup} f_C = f_A \tilde{\cup} (f_B \tilde{\cup} f_C)$

Proposition 2.9. *If $f_A, f_B, f_C \in S(U)$, then*

1. $f_A \tilde{\cap} f_A = f_A$
2. $f_A \tilde{\cap} f_\Phi = f_\Phi$
3. $f_A \tilde{\cap} f_{\tilde{E}} = f_A$
4. $f_A \tilde{\cap} f_A^{\tilde{c}} = f_\Phi$
5. $f_A \tilde{\cap} f_B = f_B \tilde{\cap} f_A$
6. $(f_A \tilde{\cap} f_B) \tilde{\cap} f_C = f_A \tilde{\cap} (f_B \tilde{\cap} f_C)$

Proposition 2.10. *[6] If $f_A, f_B, f_C \in S(U)$, then*

1. $f_A \tilde{\cup} (f_B \tilde{\cap} f_C) = (f_A \tilde{\cup} f_B) \tilde{\cap} (f_A \tilde{\cup} f_C)$
2. $f_A \tilde{\cap} (f_B \tilde{\cup} f_C) = (f_A \tilde{\cap} f_B) \tilde{\cup} (f_A \tilde{\cap} f_C)$

3 Soft Lattices

In this section, the notion of soft lattices is introduced and several related properties and some characterization theorems are investigated.

Definition 3.1. *Let $\mathcal{L} \subseteq S(U)$, and $\curlyvee$ and $\curlywedge$ be two binary operations on $\mathcal{L}$. If the set $\mathcal{L}$ is equipped with two commutative and associative binary operations $\curlyvee$ and $\curlywedge$ which are connected by the absorption law, then algebraic structure $(\mathcal{L}, \curlyvee, \curlywedge)$ is called a soft lattice.*

Theorem 3.2. *Let $(\mathcal{L}, \curlyvee, \curlywedge)$ be a soft lattice and $f_A, f_B \in \mathcal{L}$. Then*

$$f_A \curlywedge f_B = f_A \Leftrightarrow f_A \curlyvee f_B = f_B$$

Proof.

$$\begin{aligned} f_A \curlyvee f_B &= (f_A \curlywedge f_B) \curlyvee f_B \\ &= f_B \curlyvee (f_A \curlywedge f_B) \\ &= f_B \curlyvee (f_B \curlywedge f_A) \\ &= f_B \end{aligned}$$

Conversely,

$$\begin{aligned} f_A \curlywedge f_B &= f_A \curlywedge (f_A \curlyvee f_B) \\ &= f_A \end{aligned}$$

□

Example 3.3. *Let* $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ *and* $\mathcal{L} = \{f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}, f_{A_5}\} \subseteq S(U)$. *Assume that*

$$\begin{aligned} f_{A_1} &= \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_3, u_5\}), (e_3, \{u_4, u_6\})\} \\ f_{A_2} &= \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_3, u_5\})\} \\ f_{A_3} &= \{(e_1, \{u_1, u_3\}), (e_3, \{u_4, u_6\})\} \\ f_{A_4} &= \{(e_1, \{u_1, u_3\})\} \\ f_{A_5} &= \{(e_1, \{u_1\})\} \end{aligned}$$

Then $(\mathcal{L}, \tilde{\cup}, \tilde{\cap})$ *is a soft lattice. Tables of the operations are as follows, respectively;*

$\tilde{\cup}$	f_{A_1}	f_{A_2}	f_{A_3}	f_{A_4}	f_{A_5}
f_{A_1}	f_{A_1}	f_{A_1}	f_{A_1}	f_{A_1}	f_{A_1}
f_{A_2}	f_{A_1}	f_{A_2}	f_{A_1}	f_{A_2}	f_{A_2}
f_{A_3}	f_{A_1}	f_{A_1}	f_{A_3}	f_{A_3}	f_{A_3}
f_{A_4}	f_{A_1}	f_{A_2}	f_{A_3}	f_{A_4}	f_{A_4}
f_{A_5}	f_{A_1}	f_{A_2}	f_{A_3}	f_{A_4}	f_{A_5}

and

$\tilde{\cap}$	f_{A_1}	f_{A_2}	f_{A_3}	f_{A_4}	f_{A_5}
f_{A_1}	f_{A_1}	f_{A_2}	f_{A_3}	f_{A_4}	f_{A_5}
f_{A_2}	f_{A_2}	f_{A_2}	f_{A_4}	f_{A_4}	f_{A_5}
f_{A_3}	f_{A_3}	f_{A_4}	f_{A_3}	f_{A_4}	f_{A_5}
f_{A_4}	f_{A_4}	f_{A_4}	f_{A_4}	f_{A_4}	f_{A_5}
f_{A_5}	f_{A_5}	f_{A_5}	f_{A_5}	f_{A_5}	f_{A_5}

The Hasse Diagram of it appears in Figure 1.

Theorem 3.4. $(\mathcal{L}, \curlyvee, \curlywedge)$ *be a soft lattice and* $f_A, f_B \in \mathcal{L}$. *Then a relation* $\preceq$ *that is defined by*

$$f_A \preceq f_B \Leftrightarrow f_A \curlywedge f_B = f_A \textit{ or } f_A \curlyvee f_B = f_B$$

is an ordering relation on $\mathcal{L}$.

Proof. 1. $\preceq$ is reflexive. $f_A \preceq f_A \Leftrightarrow f_A \curlywedge f_A = f_A$.

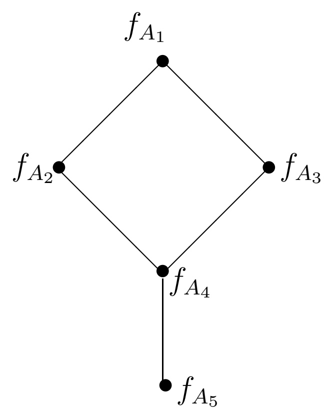

Figure 1: A soft lattice structure

2. $\preceq$ is antisymmetric. Let be $f_A \preceq f_B$ and $f_B \preceq f_A$. Then from hypotesis,

$$\begin{aligned} f_A &= f_A \curlywedge f_B \\ &= f_B \curlywedge f_A \\ &= f_B \end{aligned}$$

3. $\preceq$ is transitive. Let be $f_A \preceq f_B$ and $f_B \preceq f_C$. Then

$$\begin{aligned} f_A \curlywedge f_C &= (f_A \curlywedge f_B) \curlywedge f_C \\ &= f_A \curlywedge (f_B \curlywedge f_C) \\ &= f_A \curlywedge f_B \\ &= f_A \end{aligned}$$

from hypothesis $f_A \preceq f_C$.

□

Theorem 3.5. *Let $(\mathcal{L}, \curlyvee, \curlywedge)$ be a soft lattice and $f_A, f_B \in \mathcal{L}$. Then,*

1. $f_A \curlywedge f_B \preceq f_A$ and $f_A \curlywedge f_B \preceq f_B$

2. $f_A \preceq f_A \curlyvee f_B$ and $f_B \preceq f_A \curlyvee f_B$

Proof. 1. By Definition 3.1,

$$(f_A \curlywedge f_B) \curlyvee f_A = f_A \curlyvee (f_A \curlywedge f_B) = f_A$$

from Theorem 3.4. We get $f_A \curlywedge f_B \preceq f_A$. It can be show that $f_A \curlywedge f_B \preceq f_B$.

□

The proof *2* can be made similarly way.

Theorem 3.6. *Let* $(\mathcal{L}, \curlyvee, \curlywedge)$ *be a soft lattice and* $f_A, f_B, f_C, f_D \in \mathcal{L}$. *Then*

$$f_A \preceq f_B \text{ and } f_C \preceq f_D \Rightarrow f_A \curlywedge f_C \preceq f_B \curlywedge f_D$$

Proof. From hypothesis and Theorem 3.4, $f_A \curlywedge f_B = f_A$ and $f_C \curlywedge f_D = f_C$

$$\begin{aligned}(f_A \curlywedge f_C) \curlywedge (f_B \curlywedge f_D) &= [(f_A \curlywedge f_C) \curlywedge f_B] \curlywedge f_D \\ &= [f_A \curlywedge (f_C \curlywedge f_B)] \curlywedge f_D \\ &= [f_A \curlywedge (f_B \curlywedge f_C)] \curlywedge f_D \\ &= [(f_A \curlywedge f_B) \curlywedge f_C] \curlywedge f_D \\ &= (f_A \curlywedge f_B) \curlywedge (f_C \curlywedge f_D) \\ &= f_A \curlywedge f_C\end{aligned}$$

Then, from Theorem 3.4, $f_A \curlywedge f_C \preceq f_B \curlywedge f_D$. □

Theorem 3.7. *Let* $(\mathcal{L}, \curlyvee, \curlywedge)$ *be a soft lattice and* $f_A, f_B, f_C, f_D \in \mathcal{L}$. *Then,*

$$f_B \preceq f_A \text{ and } f_D \preceq f_C \Rightarrow f_B \curlyvee f_D \preceq f_A \curlyvee f_C$$

Proof. Proof is made similarly to Theorem 3.6. □

Example 3.8. *From Example 3.3.* $f_{A_2} \tilde{\subseteq} f_{A_1}$ *and* $f_{A_4} \tilde{\subseteq} f_{A_3}$. *Then* $f_{A_2} \tilde{\cap} f_{A_4} \tilde{\subseteq} f_{A_1} \tilde{\cap} f_{A_3}$.

Lemma 3.9. *Let* $(\mathcal{L}, \curlyvee, \curlywedge)$ *be a soft lattice and* $f_A, f_B \in \mathcal{L}$. *Then,* $f_A \curlyvee f_B$ *and* $f_A \curlywedge f_B$ *are the least upper and the greatest lower bound of* f_A *and* f_B*, respectively.*

Proof. From Theorem 3.5, $f_A \curlywedge f_B$ and $f_A \curlyvee f_B$ are a lower bound and an upper bound of f_A and f_B, respectively. Assume that, $f_A \curlywedge f_B$ is not a greatest lower bound of f_A and f_B. Then, $f_C \in \mathcal{L}$ is exist, such that $f_A \curlywedge f_B \preceq f_C \preceq f_A$ and $f_A \curlywedge f_B \preceq f_C \preceq f_B$. Hence, by Theorem 3.6, $f_C \curlywedge f_C \preceq f_A \curlywedge f_B$. Thus $f_C \preceq f_A \curlywedge f_B$. That is $f_C = f_A \curlywedge f_B$. This is a contradiction.

For $f_A \curlyvee f_B$ the proof can be made similarly. □

Theorem 3.10. *A soft lattice is a poset.*

Proof. The proof is obviously, from Lemma 3.9. □

Theorem 3.11. *Let* $\mathcal{L} \subseteq S(U)$. *Then, an algebraic structure* $(\mathcal{L}, \curlyvee, \curlywedge, \preceq)$ *is a soft lattice.*

Proof. For all f_A, f_B and $f_C \in \mathcal{L}$,

1. From Lemma 3.9,

   $$f_A \curlywedge f_B \preceq f_A \text{ and } f_A \curlywedge f_B \preceq f_B$$

   from Theorem 3.6

   $$f_A \curlywedge f_B \preceq f_B \curlywedge f_A$$

   Similarly,

   $$f_B \curlywedge f_A \preceq f_A \curlywedge f_B$$

   Then, $f_A \curlywedge f_B = f_B \curlywedge f_A$. By the same way, the proof of $f_A \curlyvee f_B = f_B \curlyvee f_A$ can be made.

2. From Theorem 3.5,

$$(f_A \curlywedge f_B) \curlywedge f_C \preceq f_A \curlywedge f_B \preceq f_B \text{ and } (f_A \curlywedge f_B) \curlywedge f_C \preceq f_C$$

   from Theorem 3.6,

$$(f_A \curlywedge f_B) \curlywedge f_C \preceq f_B \curlywedge f_C \tag{1}$$

   Also

$$(f_A \curlywedge f_B) \curlywedge f_C \preceq f_A \curlywedge f_B \preceq f_A \tag{2}$$

   from (1) and (2)

$$(f_A \curlywedge f_B) \curlywedge f_C \preceq f_A \curlywedge (f_B \curlywedge f_C).$$

   Similarly,

$$f_A \curlywedge (f_B \curlywedge f_C) \preceq (f_A \curlywedge f_B) \curlywedge f_C$$

   Then,

$$(f_A \curlywedge f_B) \curlywedge f_C = f_A \curlywedge (f_B \curlywedge f_C)$$

   By the same way, the proof of $f_A \curlyvee (f_B \curlyvee f_C) = (f_A \curlyvee f_B) \curlyvee f_C$ can be made.

3. From Theorem 3.5,

$$f_A \preceq (f_A \curlyvee f_B) \text{ and } f_A \preceq f_A,$$

   and from Theorem 3.6,

$$f_A \preceq (f_A \curlyvee f_B) \curlywedge f_A$$

   Similarly,

$$(f_A \curlyvee f_B) \curlywedge f_A \preceq f_A.$$

   Then, $f_A \curlywedge (f_A \curlyvee f_B) = f_A$. By the same way, the proof of $f_A \curlyvee (f_A \curlywedge f_B) = f_A$ can be made.

□

Note 3.12. *According to this theorem, a soft lattice $(\mathcal{L}, \curlyvee, \curlywedge)$ has the same character with $(\mathcal{L}, \curlyvee, \curlywedge, \preceq)$. Therefore, we shall identify any soft lattice $(\mathcal{L}, \curlyvee, \curlywedge)$ with $(\mathcal{L}, \curlyvee, \curlywedge, \preceq)$ and use these two concepts as interchangeable.*

Lemma 3.13. *Let $\mathcal{L} \subseteq S(U)$. Then, soft inclusion relation $\tilde{\subseteq}$ that is defined by*

$$f_A \tilde{\subseteq} f_B \Leftrightarrow f_A \tilde{\cup} f_B = f_B \text{ or } f_A \tilde{\cap} f_B = f_A$$

is an ordering relation on $\mathcal{L}$.

Proof. For all f_A, f_B and $f_C \in \mathcal{L}$,

1. $\tilde{\subseteq}$ is reflexive. $f_A \tilde{\subseteq} f_A$
2. $\tilde{\subseteq}$ is antisymetric. $f_A \tilde{\subseteq} f_B$ and $f_B \tilde{\subseteq} f_A \Leftrightarrow f_A = f_B$
3. $\tilde{\subseteq}$ is transitive. $f_A \tilde{\subseteq} f_B$ and $f_B \tilde{\subseteq} f_C \Rightarrow f_A \tilde{\subseteq} f_C$

□

Corollary 3.14. *Let* $(\mathcal{L}, \tilde{\cup}, \tilde{\cap}, \tilde{\subseteq})$ *is a soft lattice.*

Definition 3.15. *Let* $(\mathcal{L}, \curlyvee, \curlywedge, \preceq)$ *be a soft lattice and* $f_A \in \mathcal{L}$.
If $f_A \preceq f_B$ *for all* $f_B \in \mathcal{L}$*, then* f_A *is called the minimum element of* $\mathcal{L}$.
If $f_B \preceq f_A$ *for all* $f_B \in \mathcal{L}$*, then* f_A *is called the maximum element of* $\mathcal{L}$.

Definition 3.16. *Let* $(\mathcal{L}, \curlyvee, \curlywedge, \preceq)$ *be a soft lattice. If* $f_B \preceq f_A$ *or* $f_A \preceq f_B$ *for all* $f_A, f_B \in \mathcal{L}$*, then* $\mathcal{L}$ *is called a soft chain.*

Example 3.17. *Let* $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. $\mathcal{L} = \{f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}, f_{A_5}\}$ *and*

$$\begin{aligned}
f_{A_1} &= \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_3, u_5\}), (e_3, \{u_4, u_6\})\} \\
f_{A_2} &= \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_3, u_5\})\} \\
f_{A_3} &= \{(e_1, \{u_1, u_3\}), (e_3, \{u_4, u_6\})\} \\
f_{A_4} &= \{(e_1, \{u_1, u_3\})\} \\
f_{A_5} &= \{(e_1, \{u_1\})\}
\end{aligned}$$

Although, for $\mathcal{S} = \{f_{A_1}, f_{A_3}, f_{A_4}, f_{A_5}\}$, $(\mathcal{S}, \tilde{\cup}, \tilde{\cap}, \tilde{\subseteq})$ *is a soft chain,* $(\mathcal{L}, \tilde{\cup}, \tilde{\cap}, \tilde{\subseteq})$ *is not soft chain because* f_{A_2} *and* f_{A_3} *can not comparable.*

Definition 3.18. *Let* $(\mathcal{L}, \curlyvee, \curlywedge, \preceq)$ *be a soft lattice. If every subsets of* $\mathcal{L}$ *have both a greatest lower bound and a least upper bound, then it is called complete soft lattice.*

Example 3.19. *Let* $U = \{u_1, u_2, u_3\}$ *and* $\mathcal{L} = \{f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}\}$ *such that,*
$f_{A_1} = \{(e_1, \{u_1\})\}$
$f_{A_2} = \{(e_1, \{u_1, u_2\}), (e_2, \{u_3\})\}$
$f_{A_3} = \{(e_1, \{u_1, u_2\}), (e_2, \{u_2, u_3\})\}$,
$f_{A_4} = f_\phi$
Then $(\mathcal{L}, \tilde{\cup}, \tilde{\cap}, \tilde{\subseteq})$ *is a complete soft lattice. Because each finite subset of* $\mathcal{L}$ *has a greatest lower bound and a least upper bound.*

Definition 3.20. $(\mathcal{L}, \curlyvee, \curlywedge, \preceq)$ *be a soft lattice and* $\mathcal{S} \subseteq \mathcal{L}$*. If* $\mathcal{S}$ *is a soft lattice with the operations of* $\mathcal{L}$*, then* $\mathcal{S}$ *is called a soft sublattice of* $\mathcal{L}$.

Theorem 3.21. *Let* $(\mathcal{L}, \curlyvee, \curlywedge, \preceq)$ *be a soft lattice and* $\mathcal{S} \subseteq \mathcal{L}$*. If* $f_A \curlywedge f_B \in \mathcal{S}$ *and* $f_A \curlyvee f_B \in \mathcal{S}$ *for all* $f_A, f_B \in \mathcal{S}$*, then* $\mathcal{S}$ *is a soft sublattice.*

Proof. It is clear from Definition 3.20. □

Corollary 3.22. *Every soft chain is a soft sublattice.*

Corollary 3.23. *Every soft lattice is a soft sublattice of itself.*

Proof. Let $\mathcal{S}$ be a soft chain. Since any two elements of $\mathcal{S}$ is comparable, $f_A \curlywedge f_B \in \mathcal{S}$ and $f_A \curlyvee f_B \in \mathcal{S}$, for all $f_A, f_B \in \mathcal{S}$. Thus $\mathcal{S}$ is a soft sublattice. □

Example 3.24. $\mathcal{S}$*, given in Example 3.17, is a soft sublattice.*

Definition 3.25. *Let* $(\mathcal{L}, \curlyvee, \curlywedge, \preceq)$ *be a soft lattice and* f_A, f_B *and* $f_C \in \mathcal{L}$*. If*

$$(f_A \curlyvee f_B) \curlywedge (f_A \curlyvee f_C) \preceq f_A \curlyvee (f_A \curlywedge f_C)$$

or

$$f_A \curlywedge (f_A \curlyvee f_C) \preceq (f_A \curlywedge f_B) \curlyvee (f_A \curlywedge f_C),$$

then $\mathcal{L}$ *is called a one-side distributive soft lattice.*

Theorem 3.26. *Every soft lattice is a one-side distributive soft lattice.*

Proof. Let $f_A, f_B, f_C \in \mathcal{L}$. From Theorem 3.2 and 3.5, we have $f_A \curlywedge f_B \preceq f_A$ and $f_A \curlywedge f_B \preceq f_B \preceq f_B \curlyvee f_C$. Since $f_A \curlywedge f_B \preceq f_A$ and $f_A \curlywedge f_B \preceq f_B \curlyvee f_C$, then

$$f_A \curlywedge f_B = (f_A \curlywedge f_B) \curlywedge (f_A \curlywedge f_B) \preceq f_A \curlywedge (f_B \curlyvee f_C) \tag{3}$$

and also we have $f_A \curlywedge f_C \preceq f_A$ and $f_A \curlywedge f_C \preceq f_C \preceq f_B \curlyvee f_C$. Since $f_A \curlywedge f_C \preceq f_A$ and $f_A \curlywedge f_C \preceq f_B \curlyvee f_C$, then

$$f_A \curlywedge f_C = (f_A \curlywedge f_C) \curlywedge (f_A \curlywedge f_C) \preceq f_A \curlywedge (f_B \curlyvee f_C) \tag{4}$$

From (3) and (4), we get the result,

$$(f_A \curlywedge f_B) \curlyvee (f_A \curlywedge f_C) \preceq f_A \curlywedge (f_B \curlyvee f_C)$$

□

Definition 3.27. *Let* $(\mathcal{L}, \curlyvee, \curlywedge, \preceq)$ *be a soft lattice. If* $\mathcal{L}$ *satisfies the following axioms, it is called distributive soft lattice:*

$$f_A \curlywedge (f_B \curlyvee f_C) = (f_A \curlywedge f_B) \curlyvee (f_A \curlywedge f_C)$$

$$f_A \curlyvee (f_B \curlywedge f_C) = (f_A \curlyvee f_B) \curlywedge (f_A \curlyvee f_C)$$

for all f_A, f_B *and* $f_C \in \mathcal{L}$.

Theorem 3.28. $(\mathcal{L}, \tilde{\cup}, \tilde{\cap}, \tilde{\subseteq})$ *is a soft distributive lattice.*

Proof. Since soft intersection is distributive over soft union operation, the proof is trivial □

Example 3.29. *Let* $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ *and* $\mathcal{L} = \{f_\emptyset, f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}, f_{A_5}\}$ *Then,* $\mathcal{L} \subseteq S(U)$ *is a soft lattice with the operations* $\tilde{\cup}$ *and* $\tilde{\cap}$*. Assume that,*

$$\begin{array}{rcl}
f_{A_1} & = & \{(e_1, \{u_1, u_2, u_3, u_4\}), (e_2, \{u_3, u_5\}), (e_3, \{u_1, u_3, u_4\})\} \\
f_{A_2} & = & \{(e_1, \{u_1, u_2, u_4\}), (e_2, \{u_3, u_5\}), (e_3, \{u_1, u_3\})\} \\
f_{A_3} & = & \{(e_1, \{u_1, u_2, u_3\}), (e_2, \{u_3, u_5\}), (e_3, \{u_1, u_4\})\} \\
f_{A_4} & = & \{(e_1, \{u_4\}), (e_3, \{u_1, u_3\})\} \\
f_{A_5} & = & \{(e_1, \{u_1, u_2\}), (e_2, \{u_3, u_5\})\} \\
f_\emptyset & = & \emptyset
\end{array}$$

$(\mathcal{L}, \tilde{\cup}, \tilde{\cap}, \tilde{\subseteq})$ *is a soft distributive lattice. The Hasse Diagram of it appears in Figure 2.*

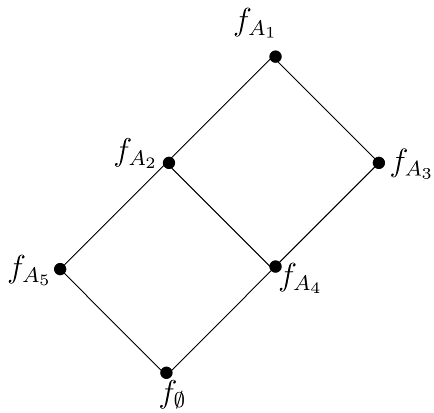

Figure 2: A soft distributive lattice structure

Definition 3.30. *Let $(\mathcal{L}, \curlyvee, \curlywedge, \preceq)$ be a soft lattice. Then $\mathcal{L}$ is called soft modular lattice, If it satisfies the following axiom,*

$$f_C \preceq f_A \Rightarrow f_A \curlywedge (f_B \curlyvee f_C) = (f_A \curlywedge f_B) \curlyvee f_C$$

for all f_A, f_B and $f_C \in \mathcal{L}$.

Theorem 3.31. *A distributive soft lattice, is a soft moduler lattice.*

Proof. It is clear from Definition 3.27. □

Note that, modular soft lattice may not be a distributive soft lattice

Proof. Let $(\mathcal{L}, \curlyvee, \curlywedge, \preceq)$ be a distributive soft lattice. Then $f_A \curlywedge (f_B \curlyvee f_C) = (f_A \curlywedge f_B) \curlyvee (f_A \curlywedge f_C)$. Hence, from Theorem 3.4, $f_C \preceq f_A \Rightarrow f_A \curlywedge (f_B \curlyvee f_C) = (f_A \curlywedge f_B) \curlyvee f_C$. □

Corollary 3.32. $(\mathcal{L}, \tilde{\cup}, \tilde{\cap}, \tilde{\subseteq})$ *is a soft moduler lattice.*

Example 3.33. *Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $\mathcal{L} = \{f_\emptyset, f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}\}$ .Then $\mathcal{L}$ is a soft lattice with the operations $\tilde{\cup}$ and $\tilde{\cap}$. Assume that,*

$$\begin{aligned} f_{A_1} &= \{(e_1, \{u_1, u_2\}), (e_2, \{u_3\}), (e_3, \{u_2, u_4\}), (e_4, \{u_5\})\} \\ f_{A_2} &= \{(e_1, \{u_1, u_2\}), (e_2, \{u_3\})\} \\ f_{A_3} &= \{(e_3, \{u_2, u_4\})\} \\ f_{A_4} &= \{(e_4, \{u_1, u_5\})\} \\ f_\emptyset &= \emptyset \end{aligned}$$

$(\mathcal{L}, \tilde{\cup}, \tilde{\cap}, \tilde{\subseteq})$ *is a soft modular lattice. The Hasse Diagram of it appears in Figure 3.*

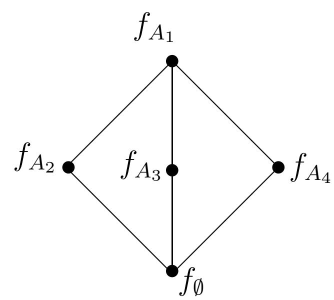

Figure 3: A soft moduler lattice structure

Theorem 3.34. *Let* $(\mathcal{L}, \curlyvee, \curlywedge, \preceq)$ *be a modular soft lattice. Then*

$$f_A \preceq f_B \Rightarrow f_A \preceq f_B \curlywedge (f_A \curlyvee f_C)$$

for all f_A, f_B *and* $f_C \in \mathcal{L}$.

Proof. The theorem is clearly from Definition 3.30. □

Example 3.35. *Assume that,* $(\mathcal{L}, \tilde{\cup}, \tilde{\cap}, \tilde{\subseteq})$ *is given as a modular soft lattice. Then*

$$f_A \tilde{\subseteq} f_B \Rightarrow f_A \tilde{\subseteq} f_B \tilde{\cap} (f_A \tilde{\cup} f_C).$$

Note that, modular soft lattice may not be a distributive soft lattice.

Example 3.36. *In Example 3.33, since* $f_{A_2} \cap (f_{A_3} \cup f_{A_4}) \neq (f_{A_2} \cap f_{A_3}) \cup (f_{A_3} \cap f_{A_4})$, *although* $(\mathcal{L}, \tilde{\cup}, \tilde{\cap}, \tilde{\subseteq})$ *is a modular soft lattice, it is not a distributive soft lattice.*

4 Conclusion

The soft set theory has been applied to many fields from theoretical to practical. In this study, we defined the concept of soft lattice as an algebraic structure and as a set-theoretic and shown that these definitions are equivalent. We then investigated several related properties and some characterization theorems.

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