Gaziosmanpasa University
Graduate School of Natural and Applied Sciences
Journal of New Results in Science

ISSN: 1304-7981 http://jnrs.gop.edu.tr

Received: 23.03.2012
Accepted: 11.09.2012

Editors-in-Chief: Naim Çağman
Area Editor: Oktay Muhtaroğlu

Measure of Noncompactness of Matrix Operators on Some New Difference Sequence Spaces of Order m^{th}

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Abstract

Kızmaz [13] studied the difference sequence spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$. Several papers dealt with the sets of sequences the m^{th} order difference of which are bounded, convergent or convergent to zero. Aydın and Başar [6] introduced the sequence spaces a_0^r and a_c^r. The main purpose of the present paper is to introduce the spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$ consisting of all sequences whose m^{th} order differences are in the spaces a_0^r and a_c^r, respectively. Furthermore, the basis for the difference spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$, and the $\alpha-, \beta-$ and $\gamma-$duals of the difference spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$ have been determined. Moreover, the matrix classes $(a_c^r(\Delta^{(m)}) : \ell_p)$ and $(a_c^r(\Delta^{(m)}) : c)$ have been characterized. Finally, we have characterized the subclasses $K(a_c^r(\Delta^{(m)}), Y)$ of compact operators by applying the Hausdorff measure of noncompactness; where Y is one of the spaces $c_0, c, \ell_\infty, \ell_1, bv$ and $c(\Delta)$

Keywords: *Difference sequence spaces of order m, $\alpha-, \beta-$ and $\gamma-$ dual, matrix transformations, Hausdorff measure of noncompactness.*

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1 Preliminaries, background and notations

By ω, we shall denote the space of all real valued sequences. Any vector subspace of ω is called as a *sequence space*. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p ; we denote the spaces of all bounded, convergent, absolutely and $p-$ absolutely convergent series, respectively; $1 < p < \infty$.

A sequence space λ with a linear topology is called a K-space provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; where $\mathbb{C}$ and $\mathbb{N}$ denote the complex field and the set of naturel numbers, respectively. A K-space λ is called an FK-space provided λ is a complete linear metric space. An FK-space whose topology is normable is called a BK-space.

Let λ, μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the $A-$ transform of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk} x_k, \qquad (n \in \mathbb{N}). \tag{1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be $A-$ summable to α if Ax converges to α which is called as the $A-$ limit of x.

Let us give the definition of some triangle limitation matrices which are needed in the text. Let $q = (q_k)$ be a sequence of positive reals and write

$$Q_n = \sum_{k=0}^{n} q_k, \quad (n \in \mathbb{N}).$$

Then the Cesàro mean of order one, Riesz mean with respect to the sequence $q = (q_k)$ and A_r- mean with $0 < r < 1$ are respectively defined by the matrices $C = (c_{nk})$, $R^q = (r^q_{nk})$ and $A_r = (a^r_{nk})$; where

$$c_{nk} = \begin{cases} \dfrac{1}{n+1}, & (0 \le k \le n), \\ 0, & (k > n), \end{cases} \qquad r^q_{nk} = \begin{cases} \dfrac{q_k}{Q_n}, & (0 \le k \le n), \\ 0, & (k > n), \end{cases}$$

and

$$a^r_{nk} = \begin{cases} \dfrac{1+r^k}{1+n}, & (0 \le k \le n), \\ 0, & (k > n), \end{cases}$$

for all $k, n \in \mathbb{N}$. Additionally, define the summation $S = (s_{nk})$ and the difference matrices $\Delta^{(1)} = (\delta_{nk})$ and $\Delta^1 = (d_{nk})$ by

$$s_{nk} = \begin{cases} 1, & (0 \le k \le n), \\ 0, & (k > n), \end{cases} \qquad \text{and} \qquad \delta_{nk} = \begin{cases} (-1)^{n-k}, & (n-1 \le k \le n), \\ 0, & (0 \le k < n-1 \text{ or } k > n), \end{cases}$$

and
$$d_{nk} = \begin{cases} (-1)^{n-k}, & (n \leq k \leq n+1), \\ 0, & (0 \leq k < n \text{ or } k > n+1), \end{cases}$$
for all $k, n \in \mathbb{N}$.

For a sequence space λ, the matrix domain λ_A of an infinite matrix A is defined by
$$\lambda_A = \{x = (x_k) \in \omega : \quad Ax \in \lambda\}, \tag{2}$$
which is a sequence space. Although in the most cases the new sequence space λ_A generated in the limitation matrix A from a sequence space λ is the expansion or the contraction of the original space λ, it may be observed in some cases that those spaces overlap. Indeed, one can deduce that the inclusions $\lambda_S \subset \lambda$ strictly holds for $\lambda \in \{\ell_\infty, c, c_0\}$. As this, one can deduce that the inclusions $\ell_p \subset bv_p$ and $\lambda \subset \lambda_{\Delta^1}$ also strictly hold for $\lambda \in \{c, c_0\}$, where $1 \leq p \leq \infty$ and the space $(\ell_p)_{\Delta^{(1)}} = bv_p$ has been studied by Başar and Altay [8], (see also Çolak and Et and Malkowsky [9]). However, if we define $\lambda = c_0 \oplus z$ with $z = ((-1)^k)$, that is, $x \in \lambda$ if and only if $x = s + \alpha z$ for some $s \in c_0$ and some $\alpha \in \mathbb{C}$, and consider the matrix A with the rows A_n defined by $A_n = (-1)^n e^{(n)}$ for all $n \in \mathbb{N}$, we have $Ae = z \in \lambda$ but $Az = e \notin \lambda$ which lead us to the consequences that $z \in \lambda \setminus \lambda_A$ and $e \in \lambda_A \setminus \lambda$, where $e^{(n)}$ denotes the sequence whose only non-zero term is a 1 in n^{th} place for each $n \in \mathbb{N}$ and $e = (1, 1, 1, ...)$. That is to say that the sequence spaces λ_A and λ overlap but neither contains the other.

We denote throughout that the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$ and use the convention that any term with negative subscript is equal to naught and $0 < r < 1$.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has been recently employed by Wang [27], Ng and Lee [23], Aydın and Başar [6], Demiriz and Çakan [11] and Altay et all. [3]. They introduced the sequence spaces $(\ell_\infty)_{N_q}$ and c_{N_q} in [27], $(\ell_p)_{C_1} = X_p$ in [23], $(c_0)_{A_r} = a_0^r$ and $c_{A_r} = a_c^r$ in [6], and $(\ell_p)_{E^r} = e_p^r$ in [3]; where $1 \leq p < \infty$ and N_q, C_1 and E^r denote the Nörlund means, Cesàro means of order 1, Euler means of order r, respectively where $0 < r < 1$. The main purpose of the present paper is to introduce the spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$ consisting of all sequences whose m^{th} order differences are in the spaces a_0^r and a_c^r, respectively and is to derive some related results that fill up the gap in the existing literature. Moreover we give some topological properties, determine the $\alpha-$, $\beta-$ and $\gamma-$ duals of the spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$ and the Schauder basis of this spaces. Besides this, we essentially characterize the matrix classes $(a_c^r(\Delta^{(m)}) : \ell_p)$ and $(a_c^r(\Delta^{(m)}) : c)$, and also derive the characterizations of some other classes by means of a given basic lemma, where $1 \leq p \leq \infty$. Finally, the characterization of the subclasses $K(X, Y)$ of compact operators have been examined; where X is $a_c^r(\Delta^{(m)})$ and Y is one of the spaces $c_0, c, \ell_\infty, \ell_1, bv$ and $c(\Delta)$ by applying the Hausdorff measure of noncompactness.

2 The Spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$ difference sequences

In the present section, subsequent to giving brief information on the spaces of difference sequences, we shall introduce the difference sequence spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$ of order m and investigate some topological properties of them.

The difference spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$, consisting of all sequences $x = (x_k)$ such that $\Delta^1 x = (x_k - x_{k+1})$ in the sequence spaces ℓ_∞, c and c_0, were defined by Kızmaz [13]. Let $p = (p_k)$ be an arbitrary bounded sequence of positive real numbers. Then, the linear spaces $\ell_\infty(p), c(p)$ and $c_0(p)$ were defined by Maddox [17] as follows:

$$\ell_\infty(p) = \{x = (x_k) \in \omega : \sup_{k\in\mathbb{N}} |x_k|^{p_k} < \infty\},$$

$$c(p) = \{x = (x_k) \in \omega : \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{R}\}$$

and

$$c_0(p) = \{x = (x_k) \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0\}.$$

Let ν denotes one of the sequence spaces ℓ_∞, c or c_0. In [1], Ahmad and Mursaleen defined the paranormed spaces of the difference sequences

$$\Delta\nu(p) = \left\{x = (x_k) \in \omega : \Delta^1 x = (x_k - x_{k+1}) \in \nu(p)\right\}.$$

The idea of difference sequences was generalized by Çolak and Et [10]. They defined the sequence spaces

$$\Delta^m \nu = \{x = (x_k) \in \omega : \Delta^m x \in \nu\},$$

where $\Delta^m x = \Delta^1(\Delta^{m-1}x)$ for $m = 1, 2, ...$ In [18], Malkowsky and Parashar defined the sequence spaces

$$\Delta^{(m)}\nu = \left\{x = (x_k) \in \omega : \Delta^{(m)} x \in \nu\right\},$$

where $m \in \mathbb{N}$ and $\Delta^{(m)}x = \Delta^{(1)}(\Delta^{(m-1)}x)$.

Başar and Altay [8] recently defined the space of sequences of $p-$ bounded variation, which is the difference spaces of the sequence spaces ℓ_p and ℓ_∞, as follows:

$$bv_p = \left\{x = (x_k) \in \omega : \sum_k |x_k - x_{k-1}|^p < \infty\right\} \quad (1 \le p < \infty),$$

and

$$bv_\infty = \left\{x = (x_k) \in \omega : \sup_{k\in\mathbb{N}} |x_k - x_{k-1}| < \infty\right\}.$$

Recently, Altay [2] has extended the space bv_p to the difference space $\ell_p(\Delta^{(m)})$ of order m and given some topological properties and inclusion relations, a Schauder basis and determine the $\alpha-, \beta-, \gamma-$ and $f-$ duals of the space $\ell_p(\Delta^{(m)})$, and characterized the matrix mappings on the sequence space $\ell_p(\Delta^{(m)})$.

The sequence spaces a_0^r and a_c^r were recently defined by Aydın and Başar [6], as follows;

$$\begin{aligned} a_0^r &= \left\{x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} (1 + r^k)x_k = 0\right\} \\ a_c^r &= \left\{x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} (1 + r^k)x_k \quad \text{exists}\right\}. \end{aligned}$$

Our main focus in this study is on the triangle matrix $\Delta^{(m)} = (\delta_{nk}^{(m)})$ is defined by

$$\delta_{nk}^{(m)} = \begin{cases} (-1)^{n-k}\begin{pmatrix} m \\ n-k \end{pmatrix} & (\max\{0, n-m\} \leq k \leq n), \\ 0 & (0 \leq k < \max\{0, n-m\} \text{ or } k > n), \end{cases}$$

for all $k, n \in \mathbb{N}$ and for any fixed $m \in \mathbb{N}$. We treat slightly more different than Kızmaz [13] and other authors following him, and employ the technique obtaining a new sequence space by the matrix domain of a triangle limitation method. Following this way, we introduce the sequence spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$ as the set of all sequences such that $\Delta^{(m)}-$ transforms of them are in the spaces a_0^r and a_c^r respectively, that is

$$\begin{aligned} a_0^r(\Delta^{(m)}) &= \left\{x = (x_k) \in \omega : \Delta^{(m)}x \in a_0^r\right\}, \\ a_c^r(\Delta^{(m)}) &= \left\{x = (x_k) \in \omega : \Delta^{(m)}x \in a_c^r\right\}. \end{aligned}$$

With the notation of (2), we can redefine the spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$ respectively as

$$a_0^r(\Delta^{(m)}) = (a_0^r)_{\Delta^{(m)}} \quad \textit{and} \quad a_c^r(\Delta^{(m)}) = (a_c^r)_{\Delta^{(m)}}. \tag{3}$$

It is obvious that the space $\lambda(\Delta^{(m)})$ is reduced in the case $m = 1$ to the space $\lambda(\Delta^{(1)})$ of Aydın and Başar [7]; where λ denotes any one of the sequence spaces a_0^r and a_c^r of non-absolute type.

Let us define the sequence $y = (y_k)$, which will be frequently used, as the $B(m, r)-$ transform of a sequence $x = (x_k)$, that is,

$$y_k = (A_r\Delta^{(m)}x)_k = \sum_{j=0}^{k} \frac{1}{k+1} \sum_{i=j}^{k} \begin{pmatrix} m \\ i-j \end{pmatrix} (-1)^{i-j}(1+r^i)x_k; \quad (k, m \in \mathbb{N}). \tag{4}$$

Here and after by $B(m, r)$, we denote the matrix $B(m, r) = (b_{nk}(m, r))$ defined by

$$b_{nk}(m, r) = \begin{cases} \dfrac{1}{k+1} \displaystyle\sum_{i=k}^{n} \begin{pmatrix} m \\ i-k \end{pmatrix} (-1)^{i-k}(1+r^i), & (0 \leq k \leq n), \\ 0, & (k > n), \end{cases} \tag{5}$$

for all $k, m, n \in \mathbb{N}$. Now, we may begin with the following theorem which essential in the text.

Theorem 1. *Let $\lambda \in \{a_0^r, a_c^r\}$. Then the set $\lambda(\Delta^{(m)})$ becomes a linear space with the coordinatewise addition and scalar multiplication which is the BK-space with the norm $\|x\|_{\lambda(\Delta^{(m)})} = \|\Delta^{(m)}x\|_\lambda$.*

Proof. The firs part of the theorem is a routine verification and so we omit it. As (3) holds and λ is a BK-space with respect to its natural norm ([6],Theorem 2.1) and $\Delta^{(m)}$ is a triangle, Theorem 4.3.2 of Wilansky [28] gives the fact that the space $\lambda(\Delta^{(m)})$ is a BK-space.

Therefore, one can easily check that the absolute property does not hold on the space $\lambda(\Delta^{(m)})$, that is, $\|x\|_{\lambda(\Delta^{(m)})} \neq \||x|\|_{\lambda(\Delta^{(m)})}$ for at least one sequence in the space $\lambda(\Delta^{(m)})$, and this means that $\lambda(\Delta^{(m)})$ is a sequence space of nonabsolute type, where $|x| = (|x_k|)$.

Theorem 2. *The spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$ are linearly isomorphic to the spaces c_0 and c, respectively; that is, $a_0^r(\Delta^{(m)}) \cong c_0$ and $a_c^r(\Delta^{(m)}) \cong c$.*

Proof. To prove this, we should show the existence of a linear bijection between the spaces $a_0^r(\Delta^{(m)})$ and c_0. Consider the transformation T defined, with the notation of (4), from $a_0^r(\Delta^{(m)})$ to c_0 by $x \mapsto y = Tx$. The linearity of T is clear. Further, it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective.

Let $y \in c_0$ and define the sequence $x = \{x_n(r)\}$ by

$$x_n(r) = \sum_{k=0}^{n} \sum_{j=k-1}^{k} \binom{m+n-k-1}{n-k} (-1)^{k-j} \frac{1+j}{1+r^k} y_j; \quad (m, n \in \mathbb{N}). \tag{6}$$

Then, we have

$$\begin{aligned} \lim_{n\to\infty} A_r \Delta^{(m)} x_n &= \sum_{j=0}^{n} \frac{1}{n+1} \sum_{i=j}^{n} \binom{m}{i-j} (-1)^{i-j} (1+r^i) x_n \\ &= \lim_{n\to\infty} y_n = 0 \end{aligned}$$

which says us that $x \in a_0^r(\Delta^{(m)})$. Additionally, we observe that

$$\begin{aligned} \|x\|_{a_0^r(\Delta^{(m)})} &= \sup_{n\in\mathbb{N}} \left| \sum_{j=0}^{n} \frac{1}{n+1} \sum_{i=j}^{n} \binom{m}{i-j} (-1)^{i-j} (1+r^i) x_n \right| \\ &= \sup_{n\in\mathbb{N}} |y_n| = \|y\|_{c_0} < \infty. \end{aligned}$$

Consequently, we see from here that T is surjective and is norm preserving. Hence, T is a linear bijection which therefore shows us that the spaces $a_0^r(\Delta^{(m)})$ and c_0 are linearly isomorphic, as was desired.

It is clear here that if the spaces $a_0^r(\Delta^{(m)})$ and c_0 are respectively replaced by the spaces $a_c^r(\Delta^{(m)})$ and c, then we obtain the fact that $a_c^r(\Delta^{(m)}) \cong c$. This completes the proof.

Before giving the basis of the sequence spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$, we require to define the concept of the Schauder basis. If a normed sequence space λ contains a sequence (b_n) with the property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + ... + \alpha_n b_n)\| = 0,$$

then (b_n) is called a Schauder basis (or briefly basis) for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum \alpha_k b_k$. Now, we may give the sequences of the points of the spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$, which form the Schauder basis for those spaces. Because of the isomorphism T defined, with the notation of (4), by $x \mapsto y = Tx$ from the sequence spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$ to the spaces c_0 and c is onto, the inverse image of the basis of the sequence spaces c_0 and c are the basis of our new spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$, respectively. Therefore, we have:

Theorem 3. *Define the sequence* $b^{(k)}(m,r) = \{b_n^{(k)}(m,r)\}_{n\in\mathbb{N}}$ *of the elements of the space* $a_0^r(\Delta^{(m)})$ *for every fixed* $k \in \mathbb{N}$ *by*

$$b_n^{(k)}(m,r) = \begin{cases} 0, & (n < k), \\ \sum_{j=k-1}^{k} \binom{m+n-k-1}{n-k} (-1)^{k-j} \frac{1+j}{1+r^k}, & (n \geq k). \end{cases} \tag{7}$$

Then,

(a) the sequence $\{b^{(k)}(m,r)\}_{k\in\mathbb{N}}$ *is a basis for the space* $a_0^r(\Delta^{(m)})$ *and any* $x \in a_0^r(\Delta^{(m)})$ *has a unique representation of the form*

$$x = \sum_k \lambda_k(m,r) b^{(k)}(m,r),$$

where $\lambda_k(m,r) = (A_r\Delta^{(m)}x)_k$.

(b) the set $\{z, b^{(1)}(m,r), b^{(2)}(m,r), ...\}$ *is a basis for the space* $a_c^r(\Delta^{(m)})$ *and any* $x \in a_c^r(\Delta^{(m)})$ *has a unique representation of the form*

$$x = lz + \sum_k [\lambda_k(m,r) - l] b^{(k)}(m,r),$$

where $z = \{z_n(m,r)\}$ *with*

$$z_n(m,r) = \sum_{k=0}^{n} \sum_{j=k-1}^{k} \binom{m+n-k-1}{n-k} (-1)^{k-j} \frac{1+j}{1+r^k},$$

and

$$l = \lim_{k\to\infty} (A_r\Delta^{(m)}x)_k.$$

3 The $\alpha-, \beta-$ and $\gamma-$ duals of the spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$

In this section, we state and prove the theorems determining the $\alpha-, \beta-$ and $\gamma-$ duals of the spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$.

For the sequence spaces λ and μ, define the set $S(\lambda,\mu)$ by

$$S(\lambda,\mu) = \{z = (z_k) \in \omega : xz = (x_k z_k) \in \mu \text{ for all } x \in \lambda\}. \tag{8}$$

With the notation of (8), $\alpha-, \beta-$ and $\gamma-$ duals of a sequence space λ, which are respectively denoted by $\lambda^\alpha, \lambda^\beta$ and λ^γ, are defined by

$$\lambda^\alpha = S(\lambda, \ell_1), \quad \lambda^\beta = S(\lambda, cs) \quad \text{and} \quad \lambda^\gamma = S(\lambda, bs).$$

It is well-known for the sequence spaces λ and μ that $\lambda^\alpha \subseteq \lambda^\beta \subseteq \lambda^\gamma$ and $\lambda^\eta \supset \mu^\eta$ whenever $\lambda \subset \mu$, where $\eta \in \{\alpha, \beta, \gamma\}$.

We shall begin with to quote the lemmas, due to Sieglitz and Tietz [26], which are needed in proving Theorems 4 and 5, below.

Lemma 1. *$A \in (c_0 : \ell_1) = (c : \ell_1)$ if and only if*

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} a_{nk} \right| < \infty.$$

Lemma 2. *$A \in (c_0 : c)$ if and only if*

$$\lim_{n\to\infty} a_{nk} = \alpha_k; \quad k \in \mathbb{N}, \tag{9}$$

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty. \tag{10}$$

Lemma 3. *$A \in (c : c)$ if and only if (9) and (10) hold, and*

$$\lim_{n\to\infty} \sum_k a_{nk} = \alpha. \tag{11}$$

Lemma 4. *$A \in (c_0 : \ell_\infty) = (c : \ell_\infty)$ if and only if (10) holds.*

Theorem 4. *Define the set D by*

$$D = \left\{ a = (a_k) \in \omega : \sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} d_{nk}(m,r) \right| < \infty \right\},$$

where $D(m,r) = (d_{nk}(m,r))$ is defined via the sequence $a = (a_n)$ by

$$d_{nk}(m,r) = \begin{cases} \displaystyle\sum_{j=k-1}^{k} \binom{m+n-k-1}{n-k} (-1)^{k-j} \frac{1+j}{1+r^k}, & (0 \le k \le n), \\ 0, & (k > n), \end{cases}$$

where $m, n, k \in \mathbb{N}$. Then, $\{a_0^r(\Delta^{(m)})\}^\alpha = \{a_c^r(\Delta^{(m)})\}^\alpha = D$.

Proof. Bearing in mind the relation (6) we immediately derive that

$$\begin{aligned} a_n x_n &= \sum_{k=0}^{n} \left[\sum_{j=k-1}^{k} \binom{m+n-k-1}{n-k} (-1)^{k-j} \frac{1+j}{1+r^k} a_n \right] y_k \\ &= \sum_{k=0}^{n} d_{nk}(m,r) = \{D(m,r)y\}_n. \end{aligned} \tag{12}$$

We therefore observe by (12) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in a_0^r(\Delta^{(m)})$ or $a_c^r(\Delta^{(m)})$ if and only if $D(m,r)y \in \ell_1$ whenever $y \in c_0$ or c. Then, we derive by Lemma 1 that

$$\{a_0^r(\Delta^{(m)})\}^\alpha = \{a_c^r(\Delta^{(m)})\}^\alpha = D.$$

Theorem 5. *Define the sets C_1, C_2 and C_3 by*

$$\begin{aligned} C_1 &= \left\{ a=(a_k)\in\omega : \sup_{n\in\mathbb{N}} \sum_k |c_{nk}(m,r)| < \infty \right\}, \\ C_2 &= \left\{ a=(a_k)\in\omega : \lim_{n\to\infty} c_{nk}(m,r) \text{ exists for all } k\in\mathbb{N} \right\}, \\ C_3 &= \left\{ a=(a_k)\in\omega : \lim_{n\to\infty} \sum_k c_{nk}(m,r) - \text{exists} \right\} \end{aligned}$$

where $C(m,r)=(c_{nk}(m,r))$ is defined by

$$c_{nk}(m,r) = \begin{cases} (k+1)\left[\dfrac{a_k}{1+r^k} + \left(\dfrac{1}{1+r^k} - \dfrac{1}{1+r^{k+1}}\right) \displaystyle\sum_{j=k+1}^{n} \binom{m+n-j-1}{n-j} a_j\right], & 0\le k\le n, \\ 0, & k>n, \end{cases}$$

for all $n,k\in\mathbb{N}$. Then, $\{a_0^r(\Delta^{(m)})\}^\beta = C_1\cap C_2$ and $\{a_c^r(\Delta^{(m)})\}^\beta = C_1\cap C_2\cap C_3$.

Proof. Consider the equation obtained by using the relation (4)

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{i=j-1}^{j} \binom{m+k-j-1}{k-j} (-1)^{j-i} \frac{1+i}{1+r^j} a_k y_i \\ &= \sum_{k=0}^{n} (k+1)\left[\frac{a_k}{1+r^k} + \left(\frac{1}{1+r^k} - \frac{1}{1+r^{k+1}}\right) \sum_{j=k+1}^{n} \binom{m+n-j-1}{n-j} a_j\right] y_k \\ &= \sum_{k=0}^{n} c_{nk}(m,r) = \{C(m,r)y\}_n; \quad (n\in\mathbb{N}). \end{aligned} \tag{13}$$

Thus we deduce from Lemma 2 together with (13) that $ax=(a_kx_k)\in cs$ whenever $x=(x_k)\in a_0^r(\Delta^{(m)})$ if and only if $C(m,r)y\in c$ whenever $y=(y_k)\in c_0$. That is to say that $a=(a_k)\in\{a_0^r(\Delta^{(m)})\}^\beta$ if and only if $C(m,r)\in(c_0:c)$ which yields us $\{a_0^r(\Delta^{(m)})\}^\beta = C_1\cap C_2$.

Since the reader may prove the fact about the $\beta-$ dual of the sequence space $a_c^r(\Delta^{(m)})$ in the similar fashion, we omit the proof.

Theorem 6. *The $\gamma-$ dual of the spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$ is the set C_1.*

Proof. This is obtained in a similar way used in the proof of Theorem 5 with Lemma 4 instead of Lemma 2 and so we leave the detail to the reader.

4 Certain Matrix Mappings on the Sequence Space $a_c^r(\Delta^{(m)})$

In this section, we directly prove the theorems which characterize the classes $(a_c^r(\Delta^{(m)}) : \ell_p)$ and $(a_c^r(\Delta^{(m)}) : c)$ and derive the characterizations of the matrix mappings from $a_c^r(\Delta^{(m)})$ into some of the known sequence spaces and into the Euler, difference, Riesz, Cesàro sequence spaces by means of a given basic lemma, where $1 \leq p \leq \infty$.

We shall write throughout for brevity that $T(m,r) = (t_{nk}(m,r))$ by

$$t_{nk}(m,r) = (k+1)\left[\frac{a_{nk}}{1+r^k} + \left(\frac{1}{1+r^k} - \frac{1}{1+r^{k+1}}\right)\sum_{j=k+1}^{\infty}\binom{m+n-j-1}{n-j}a_{nj}\right]$$

for all $k, m, n \in \mathbb{N}$.

We shall begin with two lemmas due to Wilansky [28] which are needed in the proof of our theorems.

Lemma 5. *The matrix mappings between the BK-spaces are continuous.*

Lemma 6. $A \in (c : \ell_p)$ *if and only if*

$$\sup_{F\in\mathcal{F}}\sum_n\left|\sum_{k\in F}a_{nk}\right|^p < \infty, \quad 1 \leq p < \infty. \tag{14}$$

Theorem 7. *(i) Let* $1 \leq p < \infty$. *Then,* $A \in (a_c^r(\Delta^{(m)}) : \ell_p)$ *if and only if*

$$\sup_{s\in\mathbb{N}}\sum_k |t_{nk}^s(m,r)| < \infty \quad \textit{for any} \quad n \in \mathbb{N}, \tag{15}$$

$$\lim_{s\to\infty}\sum_k t_{nk}^s(m,r) \quad \textit{exists for any} \quad n \in \mathbb{N}, \tag{16}$$

$$\lim_{s\to\infty} t_{nk}^s(m,r) = t_{nk}(m,r) \quad \textit{exists for any} \quad k,n \in \mathbb{N}, \tag{17}$$

$$\sup_{F\in\mathcal{F}}\sum_n\left|\sum_{k\in F}t_{nk}(m,r)\right|^p < \infty, \quad 1 \leq p < \infty; \tag{18}$$

where

$$t_{nk}^s(m,r) = (k+1)\left[\frac{a_{nk}}{1+r^k} + \left(\frac{1}{1+r^k} - \frac{1}{1+r^{k+1}}\right)\sum_{j=k+1}^{s}\binom{m+n-j-1}{n-j}a_{nj}\right].$$

(ii)$A \in (a_c^r(\Delta^{(m)}) : \ell_\infty)$ *if and only if (15)-(17) hold and*

$$\sup_{n\in\mathbb{N}}\sum_k |t_{nk}(m,r)| < \infty. \tag{19}$$

Proof. (i) Suppose that the conditions (15)-(18) hold and take any $x \in a_c^r(\Delta^{(m)})$. Then, $\{a_{nk}\}_{k\in\mathbb{N}} \in \{a_c^r(\Delta^{(m)})\}^\beta$ for any $n \in \mathbb{N}$ and this implies that Ax exists. In this situation, since (14) is satisfied, $T(m,r) = (t_{nk}(m,r)) \in (c : \ell_p)$. Let us consider the following equality obtained from the s^{th} partial sum of the series $\sum_k a_{nk}x_k$ by using the relation (4):

$$\sum_{k=0}^{s} a_{nk}x_k = \sum_{k=0}^{s} t_{nk}^s(m,r)y_k \qquad (m,n,s \in \mathbb{N}). \tag{20}$$

Therefore, passing to limit in (20) as $s \to \infty$ we derive that

$$\sum_k a_{nk}x_k = \sum_k t_{nk}^s(m,r)y_k \qquad (m,n,\in \mathbb{N}); \tag{21}$$

which yields the fact that $A \in (a_c^r(\Delta^{(m)}) : \ell_p)$, since $T(m,r)y \in \ell_p$ whenever $y \in c$.

Conversely, suppose that $A \in (a_c^r(\Delta^{(m)}) : \ell_p)$. Then, since $a_c^r(\Delta^{(m)})$ and ℓ_p are the BK-spaces, we have from Lemma 5 that there exists some real constant $K > 0$ such that

$$\|Ax\|_{\ell_p} \leq K \cdot \|x\|_{a_c^r(\Delta^{(m)})} \tag{22}$$

for all $x \in a_c^r(\Delta^{(m)})$. Since the inequality (22) is also satisfied for the sequence $x = (x_k) = \sum_{k\in F} b^{(k)}(r)$ belonging to the space $a_c^r(\Delta^{(m)})$, where $b^{(k)}(r) = \{b_n^{(k)}(r)\}$ is defined by (7), we have for any $F \in \mathcal{F}$ that

$$\|Ax\|_{\ell_p} = \left(\sum_n \left|\sum_{k\in F} t_{nk}(m,r)\right|^p\right)^{1/p} \leq K \cdot \|x\|_{a_c^r(\Delta^{(m)})}$$

which shows the necessity of (18).

Because A is applicable to the space $a_c^r(\Delta^{(m)})$ by the hypothesis, the necessities of (15)-(17) are trivial. This completes the proof of the part (i) of Theorem.

Since the part (ii) may also be proved in the similar way that of the part (i), we leave the detailed proof to the reader.

Theorem 8. *$A \in (a_c^r(\Delta^{(m)}) : c)$ if and only if (15)-(17) and (19) hold, and*

$$\lim_{n\to\infty} t_{nk}(m,r) = \alpha_k \quad for\ each\ k \in \mathbb{N}, \tag{23}$$

$$\lim_{n\to\infty} \sum_k t_{nk}(m,r) = \alpha. \tag{24}$$

Proof. Suppose that A satisfies the conditions (15)-(17),(19),(23) and (24). Let us take any $x = (x_k)$ in $a_c^r(\Delta^{(m)})$. Then, Ax exists and it is trivial that the sequence $y = (y_k)$ connected with the sequence $x = (x_k)$ by the relation (4) is in c such that $y_k \to l$ as $k \to \infty$. At this stage, we observe from (23) and (19) that

$$\sum_{j=0}^{k} |\alpha_j| \leq \sup_{n\in\mathbb{N}} \sum_j |t_{nj}(m,r)| < \infty$$

holds for every $k \in \mathbb{N}$. This leads us to the consequence that $(\alpha_k) \in \ell_1$. Considering (21), let us write

$$\sum_k a_{nk}x_k = \sum_k t_{nk}(m,r)(y_k - l) + l\sum_k t_{nk}(m,r). \tag{25}$$

In this situation, by letting $n \to \infty$ in (25) we see that the first term on the right tends to $\sum_k \alpha_k(y_k - l)$ by (19) and (23), and the second term tends to $l\alpha$ by (24) and we thus have that

$$(Ax)_n \to \sum_k \alpha_k(y_k - l) + l\alpha$$

which shows that $A \in (a_c^r(\Delta^{(m)}) : c)$.

Conversely, suppose that $A \in (a_c^r(\Delta^{(m)}) : c)$. Then, since the inclusion $c \subset \ell_\infty$ holds, the necessities of (15)-(17) and (19) are immediately obtained from Theorem 7. To prove the necessity of (23), consider the sequence $x = x^{(k)} = \{x_n^{(k)}(m,r)\}_{n\in\mathbb{N}} \in a_c^r(\Delta^{(m)})$ defined by

$$x_n^{(k)}(m,r) = \begin{cases} 0, & (n<k), \\ \sum_{j=k-1}^{k} \binom{m+n-k-1}{n-k}(-1)^{k-j}\frac{1+j}{1+r^k}, & (n \geq k), \end{cases}$$

for each $k \in \mathbb{N}$. Since Ax exists and is in c for every $x \in a_c^r(\Delta^{(m)})$, one can easily see that $Ax^{(k)} = \{t_{nk}(m,r)\}_{n\in\mathbb{N}} \in c$ for each $k \in \mathbb{N}$ which shows the necessity of (23).

Similarly by putting $x = e$ in (21), we also obtain that $Ax = \{\sum_k t_{nk}(m,r)\}_{n\in\mathbb{N}}$ belongs to the space c and this shows the necessity of (24).

Now, we may present our basic lemma given by Başar and Altay [3, Lemma 4.5.6] which is useful for obtaining the characterization of some new matrix classes from Theorem 7 and Theorem 8.

Lemma 7. *Let λ, μ be any two sequence spaces, A be an infinite matrix and B a triangle matrix. Then, $A \in (\lambda : \mu_B)$ if and only if $BA \in (\lambda : \mu)$.*

It is trivial that Lemma 7 has several consequences. Indeed, combining the Lemma 7 with Theorems 7 and 8, one can easily derive the following results:

Corollary 1. *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $C = (c_{nk})$ by*

$$c_{nk} = \sum_{j=0}^{n}\binom{n}{j}(1-r)^{n-j}r^j a_{jk}; \quad n,k \in \mathbb{N}.$$

Then, the necessary and sufficient conditions in order for A belongs to anyone of the classes $(a_c^r(\Delta^{(m)}) : e_\infty^r), (a_c^r(\Delta^{(m)}) : e_p^r)$ and $(a_c^r(\Delta^{(m)}) : e_c^r)$ are obtained from the respective ones in Theorems 7 and 8 by replacing the entries of the matrix A by those of the matrix C; where e_c^r denotes the Euler space of all sequences whose E^r-transforms are in the space c and is studied by Altay and Başar in a separate paper [4].

Corollary 2. *Let $A = (a_{nk})$ be an infinite matrix and define the matrices $C = (c_{nk})$ and $D = (d_{nk})$ by $c_{nk} = a_{nk} - a_{n+1,k}$ and $d_{nk} = a_{nk} - a_{n-1,k}$ for all $n,k \in \mathbb{N}$. Then, the necessary and sufficient conditions in order for A belongs to anyone of the classes $(a_c^r(\Delta^{(m)}) : \ell_\infty(\Delta)), (a_c^r(\Delta^{(m)}) : c(\Delta))$ and $(a_c^r(\Delta^{(m)}) : bv_p)$ are obtained from the respective ones in Theorems 7 and 8 by replacing the entries of the matrix A by those of the matrices C and D.*

Corollary 3. *Let $A = (a_{nk})$ be an infinite matrix and $t = (t_k)$ be a sequence of positive numbers and define the matrix $C = (c_{nk})$ by*

$$c_{nk} = \frac{1}{T_n} \sum_{j=0}^{n} t_j a_{jk}; \quad (n, k \in \mathbb{N}),$$

where $T_n = \sum_{k=0}^{n} t_k$ for all $n \in \mathbb{N}$. Then, the necessary and sufficient conditions in order for A belongs to anyone of the classes $(a_c^r(\Delta^{(m)}) : r_\infty^t), (a_c^r(\Delta^{(m)}) : r_p^t)$ and $(a_c^r(\Delta^{(m)}) : r_c^t)$ are obtained from the respective ones in Theorems 7 and 8 by replacing the entries of the matrix A by those of the matrix C; where r_p^t is defined in [5] as the space of all sequences whose R^t- transforms are in the space ℓ_p and is derived from the paranormed space $r^t(p)$ in the case $p_k = p$ for all $k \in \mathbb{N}$ and r_∞^t, r_c^t are defined in [19] as the spaces of all sequences whose R^t- transforms are in the spaces ℓ_∞ and c, respectively.

Since the spaces r_∞^t and r_p^t reduce in the case $t = e$ to the Cesàro sequence spaces X_∞ and X_p of non-absolute type, respectively. Corollary 3 also includes the characterizations of the classes $(a_c^r(\Delta^{(m)}) : X_\infty)$ and $(a_c^r(\Delta^{(m)}) : X_p)$.

5 The Hausdorff measure of noncompactness

The theory of FK spaces is the most important tool in the characterization of matrix transformations between certain sequence spaces. The most important result is that matrix transformations between FK spaces are continuous. It is quite natural to find conditions for a matrix map between FK spaces to define a compact operator. This can be achieved by applying the Hausdorff measure of noncompactness. For details on noncompactness and the recent studies we refer [20, 14, 15, 16].

Let X and Y be Banach spaces. A linear operator $L : X \to Y$ is called compact if its domain is all of X and for every bounded sequence (x_k) in X, the sequence $(L(x_k))$ has a convergent subsequence in Y. We denote the class of such operators by $K(X, Y)$.

If $X \supset \phi$ is a BK space and $a \in \omega$ we write

$$\|a\|_X^* = \sup\left\{\left|\sum_{k=0}^{\infty} a_k x_k\right| : \|x\| = 1\right\}.$$

Throughout this section, $B = B(m, r) = (b_{nk}(m, r))_{n,k=0}^{\infty}$ is a triangle (as defined by (5)), that is $b_{nk}(m, r) = 0$ for $k > n$ and $b_{nn}(m, r) \neq 0$ $(n = 0, 1, 2, ...)$, S is its inverse and N is a finite subset of $\mathbb{N}$.

For our investigation we also need the following results:

Theorem 9. *[20, Theorem 1.23] Let X and Y be FK spaces. Then, we have $(X, Y) \subset B(X, Y)$, that is, every $A \in (X, Y)$ defines a linear operator $L_A \in B(X, Y)$; where $L_A(x) = A(x)$; $(x \in X)$.*

Theorem 10. *[21, Proposition 3.2] Let $X \supset \phi$ and Y be BK spaces. Then, we have $A \in (X, \ell_\infty)$ if and only if*

$$\|A\|_X^* = \sup_n \|A_n\|_X^* < \infty.$$

Furthermore, if $A \in (X, \ell_\infty)$ then it follows that $\|L_A\| = \|A\|_X^*$.

Theorem 11. *[22, Satz 1] Let X be a BK space. Then, $A \in (X, \ell_1)$ if and only if*

$$\|A\|_{X,1}^* = \sup_{N \subset \mathbb{N}_0} \left\| \left(\sum_{n \in N} a_{nk} \right)_{k=0}^{\infty} \right\|_X^* < \infty.$$

Moreover, if $A \in (X, \ell_1)$ then

$$\|A\|_{X,1}^* \leq \|L_A\| \leq 4 \cdot \|A\|_{X,1}^*.$$

Theorem 12. *[12, Theorem 2.6] Let X be a BK space with AK and $R = S^T$, the transpose of S. If $a \in (X_B)^\beta$ then, we have*

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} R_k(a) B_k(x); \quad for\ all\ x \in X_B.$$

Remark 1. *[12, Remark 2.7] The conclusion of Theorem 12 also holds for $X = c$ and $X = \ell_\infty$.*

Now, we are going to characterize some classes of compact operators on $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$ by using the Hausdorff measure of noncompactness. It is clear that the spaces $a_0^r(\Delta^{(m)})$ and $a_c^r(\Delta^{(m)})$ are the domains of the matrix $B = B(m, r)$ in c_0 and c, respectively. Also; by a straightforward computation, one can see that the inverse $S = S(m, r) = (s_{nk}(m, r))$ of B is as follows:

$$s_{nk}(m, r) = \begin{cases} (k+1) \left[\dfrac{a_k}{1+r^k} + \left(\dfrac{1}{1+r^k} - \dfrac{1}{1+r^{k+1}} \right) \displaystyle\sum_{j=k+1}^{n} \binom{m+n-j-1}{n-j} a_j \right], & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for $n = 0, 1, 2, ...$

Let us recall some definitions and well-known results.

Definition 1. *Let (X, d) be a metric space, Q a bounded subset of X and $K(x, r) = \{y \in X : d(x, y) < r\}$. Then, the Hausdorff measure of noncompactness of Q, denoted by $\chi(Q)$, is defined by*

$$\chi(Q) = \inf \left\{ \varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} K(x_i, r_i),\ x_i \in X,\ r_i < \varepsilon\ (i = 1, 2, ...),\ n \in \mathbb{N}_0 \right\}.$$

The following results and more properties of the measure of noncompactness can be found in [20, 24].

If Q, Q_1 and Q_2 are bounded subsets of the metric space (X, d), then we have

$\chi(Q) = 0$ if and only if Q is a totaly bounded set,

$\chi(Q) = \chi(\overline{Q})$,

$Q_1 \subset Q_2$ implies $\chi(Q_1) \leq \chi(Q_2)$,

$\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\}$
and

$\chi(Q_1 \cap Q_2) \leq \min\{\chi(Q_1), \chi(Q_2)\}$.

If Q, Q_1 and Q_2 are bounded subsets of the normed space X, then we have

$\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2)$,

$\chi(Q + x) = \chi(Q) \quad (x \in X)$
and

$\chi(\lambda Q) = |\lambda|\chi(Q)$ for all $\lambda \in \mathbb{C}$.

Definition 2. *Let X and Y be Banach spaces and χ_1 and χ_2 be Hausdorff measures on X and Y. Then the operator $L : X \to Y$ is called (χ_1, χ_2)-bounded if $L(Q)$ is a bounded subset of Y for every bounded subset Q of X and there exists a positive constant K such that $\chi_2(L(Q)) \leq K \cdot \chi_1(Q)$ for every bounded subset Q of X. If an operator L is (χ_1, χ_2)-bounded then, the number*

$$\|L\|_{(\chi_1,\chi_2)} = \inf\{K > 0 : \chi_2(L(Q)) \leq K \cdot \chi_1(Q)\}$$

for all bounded $Q \subset X$, is called (χ_1, χ_2)-measure of noncompactness of L. In particular, if $\chi_1 = \chi_2 = \chi$, then we write $\|L\|_{(\chi,\chi)} = \|L\|_\chi$.

Theorem 13. *[20, Theorem 2.25] Let X and Y be Banach spaces, $S_X = \{x \in X : \|x\| = 1\}$, $K_X = \{x \in X : \|x\| \leq 1\}$ and $A \in B(X, Y)$. Then the Hausdorff measure of noncompactness of the operator A, denoted by A_χ, is given by*

$$\|A\|_\chi = \chi(AK) = \chi(AS).$$

Furthermore, A is compact if and only if $\|A\|_\chi = 0$ (see [20]). The Hausdorff measure of noncompactness satisfies the inequality $\|A\|_\chi \leq \|A\|$ (see [20]).

Theorem 14. *[20, Theorem 2.23] Let X be a Banach space with Schauder basis $\{e_1, e_2, ...\}$, Q be a bounded subset of X, and $P_n : X \to X$ be the projector onto the linear span of $\{e_1, e_2, ..., e_n\}$. Then we have*

$$\frac{1}{a} \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I - P_n)x\| \right) \leq \chi(Q) \leq \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I - P_n)x\| \right),$$

where $a = \limsup_{n\to\infty} \|I - P_n\|$.

Theorem 15. *[25, Theorem 2.8] Let Q be a bounded subset of the normed space X, where X is ℓ_p for $1 \leq p < \infty$ or c_0. If $P_n : X \to X$ is the operator defined by $P_n(x) = (x_0, x_1, ..., x_n, 0, 0, ...)$ for $(x_0, x_1, ..., x_n, 0, 0, ...) \in X$, then*

$$\chi(Q) = \lim_{n\to\infty} \left(\sup_{x\in Q} \|(I - P_n)x\| \right).$$

Theorem 16. *[12, Remark 2.13] Let X be a normed sequence space and χ_B and χ denote the Hausdorff measures of noncompactness on M_{X_B} and M_X, the collections of all bounded sets in X_B and X, respectively. Then $\chi\chi_B(Q) = \chi(B(Q))$ for all $Q \in M_{X_B}$.*

Theorem 17. *(a) Let $A \in (a_c^r(\Delta^{(m)}), \ell_\infty)$, the matrix $\widetilde{A} = \widetilde{A}(m, r) = (\tilde{a}_{nk}(m, r))_{n,k=0}^\infty$ be defined by*

$$\tilde{a}_{nk}(m, r) = (k + 1) \left[\frac{a_{nk}}{1 + r^k} + \left(\frac{1}{1 + r^k} - \frac{1}{1 + r^{k+1}} \right) \sum_{j=k+1}^{\infty} \binom{m + n - j - 1}{n - j} a_{nj} \right]$$

for all $n, k = 0, 1, ...$ and

$$\|A\|_{(a_c^r(\Delta^{(m)}),\infty)} = \sup_n \left(\sum_{k=0}^{\infty} |\tilde{a}_{nk}(m, r)| \right).$$

Then, we have

$$\|L_A\| = \|A\|_{(a_c^r(\Delta^{(m)}),\infty)}. \tag{26}$$

(b) Let $A \in (a_c^r(\Delta^{(m)}), \ell_1)$ and

$$\|A\|_{(a_c^r(\Delta^{(m)}),\ell_1)} = \sup_{N\subset\mathbb{N}_0} \left(\sum_k \left| \sum_{n\in N} \tilde{a}_{nk}(m, r) \right| \right).$$

Then, we have

$$\|A\|_{(a_c^r(\Delta^{(m)}),1)} \leq \|L_A\| \leq 4 \cdot \|A\|_{(a_c^r(\Delta^{(m)}),1)}. \tag{27}$$

Proof. (a) We assume that $A \in (a_c^r(\Delta^{(m)}), \ell_\infty)$, and omit the subscripts $a_c^r(\Delta^{(m)})$ for the norms in the proof. Then we have $A_n \in \{a_c^r(\Delta^{(m)})\}^\beta$ for all $n = 0, 1, ...$, and it follows from Theorem 12 and Remark 1 that for all $x \in a_c^r(\Delta^{(m)})$ and for all $n = 0, 1, ...$,

$$A_n(x) = \sum_{k=0}^{\infty} R_k(A_n)B_k(x), \tag{28}$$

where for all n and k

$$\begin{aligned} R_k(A_n) &= \sum_{j=0}^{\infty} r_{kj}(m,r)a_{nj} \\ &= \sum_{j=k}^{\infty} s_{jk}(m,r)a_{nj} \\ &= (k+1)\left[\frac{a_{nk}}{1+r^k} + \left(\frac{1}{1+r^k} - \frac{1}{1+r^{k+1}}\right)\sum_{j=k+1}^{\infty}\binom{m+n-j-1}{n-j}a_{nj}\right] \\ &= \tilde{a}_{nk}(m,r). \end{aligned}$$

Since $a_c^r(\Delta^{(m)})$ is a BK space, Theorem 10 yields

$$\|A\|^* = \sup_n \|A_n\|^* = \|L_A\|. \tag{29}$$

Furthermore, we have $x \in S_{a_c^r(\Delta^{(m)})}$ if and only if $y = B(x) \in S_c$ by Theorem 1, and conclude from (28),(29) and the definition of the norms $\|.\|^*$ and $\|.\|_c^*$, for all $n = 0, 1, \ldots$ that

$$\|A_n\|^* = \sup\{|A_n(x)| : x \in S_{a_c^r(\Delta^{(m)})}\} = \sup\{|\widetilde{A}_n(y)| : y \in S_c\} = \|\widetilde{A}_n\|_c^*. \tag{30}$$

Finally, since the continuous dual space of c and ℓ_1 are norm-isomorphic, (26) follows from (29) and (30).

Part (b) is proved in exactly the same way as part (a); we apply Theorem 11 instead of Theorem 10.

Now we give the main results.

Theorem 18. *Let A be an infinite matrix and put*

$$\|A\|^{(\eta)}_{(a_c^r(\Delta^{(m)}),\infty)} = \sup_{n>\eta}\sum_{k=0}^{\infty}|\tilde{a}_{nk}(m,r)|.$$

(a) If $A \in (a_c^r(\Delta^{(m)}), c_0)$, then

$$\|L_A\|_\chi = \lim_{\eta\to\infty}\|A\|^{(\eta)}_{(a_c^r(\Delta^{(m)}),\infty)}. \tag{31}$$

(b) If $A \in (a_c^r(\Delta^{(m)}), c)$, then

$$\frac{1}{2}\cdot\lim_{\eta\to\infty}\|A\|^{(\eta)}_{(a_c^r(\Delta^{(m)}),\infty)} \le \|L_A\|_\chi \le \lim_{\eta\to\infty}\|A\|^{(\eta)}_{(a_c^r(\Delta^{(m)}),\infty)}. \tag{32}$$

(c) If $A \in (a_c^r(\Delta^{(m)}), \ell_\infty)$, then

$$0 \le \|L_A\|_\chi \le \lim_{\eta\to\infty}\|A\|^{(\eta)}_{(a_c^r(\Delta^{(m)}),\infty)}. \tag{33}$$

Proof. In the proof, we use the same technique as in the proof of [20]. Let us remark that limits in (31)-(33) exists. We write $K = \{x \in a_c^r(\Delta^{(m)}) : \|x\| \leq 1\}$.

(a) Applying Theorem 14, we have

$$\|L_A\|_\chi = \chi(AK) = \lim_{\eta\to\infty}\left[\sup_{x\in K}\|(I-P_\eta)Ax\|\right], \tag{34}$$

where $P_\eta : c_0 \to c_0$ ($\eta = 0, 1, 2, ...$) is the projector such that $P_\eta(x) = (x_0, x_1, ..., x_\eta, 0, 0, ...)$ for $x = (x_k)_{k=0}^\infty \in c_0$. It is known that $\|I - P_\eta\| = 1$ for all η. Let $A_{(\eta)} = (\hat{a}_{nk})_{n,k=0}^\infty$ be the infinite matrix with

$$\hat{a}_{nk} = \begin{cases} 0, & (0 \leq n \leq \eta), \\ \tilde{a}_{nk}, & (\eta < n). \end{cases}$$

Now, we have

$$\sup_{x\in K}\|(I-P_\eta)Ax\| = \|L_{A_{(\eta)}}\| = \|A_{(\eta)}\|_{(a_c^r(\Delta^{(m)}),\infty)} = \|A\|^{(\eta)}_{(a_c^r(\Delta^{(m)}),\infty)}, \tag{35}$$

and we obtain (31) from (34) and (35).

(b) This proof is similar as in (a). The only difference is the number a in Theorem 14; namely, if $P_\eta : c \to c$ ($\eta = 0, 1, 2, ...$) is the projector such that $P_\eta(x) = le + \sum_{k=0}^{\eta}(x_k - l)e^{(k)}$, then $\|I - P_\eta\| = 2$ for all η.

(c) Let us define the projector $P_\eta : \ell_\infty \to \ell_\infty$ ($\eta = 0, 1, 2, ...$) by $P_\eta(x) = (x_0, x_1, ..., x_\eta, 0, 0, ...)$ for $x = (x_k)_{k=0}^\infty \in \ell_\infty$. Since $AK \subset P_\eta(AK) + (I - P_\eta)(AK)$, applying the properties of χ, we obtain

$$\begin{aligned} \chi(AK) &\leq \chi(P_\eta(AK) + (I-P_\eta)(AK)) \\ &= \chi((I-P_\eta)(AK)) \\ &\leq \sup_{x\in K}\|(I-P_\eta)Ax\| \\ &= \|L_{A_{(\eta)}}\|. \end{aligned}$$

Corollary 4. *If either $A \in (a_c^r(\Delta^{(m)}), c)$ or $A \in (a_c^r(\Delta^{(m)}), c_0)$, then L_A is compact if and only if*

$$\lim_{\eta\to\infty}\|A\|^{(\eta)}_{(a_c^r(\Delta^{(m)}),\infty)} = 0. \tag{36}$$

If $A \in (a_c^r(\Delta^{(m)}), \ell_\infty)$, then L_A is compact if condition (36) holds.

Theorem 19. *Let A be an infinite matrix and put*

$$\|A\|^{(\eta)}_{(a_c^r(\Delta^{(m)}),1)} = \sup_{N\subset\mathbb{N}_0\setminus\{0,1,2,\ldots,\eta\}}\left(\sum_k\left|\sum_{n\in N}\tilde{a}_{nk}(m,r)\right|\right).$$

Then we have

$$\|L_A\|_\chi = \lim_{\eta\to\infty}\|A\|^{(\eta)}_{(a_c^r(\Delta^{(m)}),1)}.$$

Proof. Since the proof is similar to the that of the Theorem 18, we omit the details.

Corollary 5. *If $A \in (a_c^r(\Delta^{(m)}), \ell_1)$ then L_A is compact if and only if*

$$\lim_{\eta\to\infty} \|A\|^{(\eta)}_{(a_c^r(\Delta^{(m)}),1)} = 0.$$

Using Theorem 16 and our previous results, we obtain that next corollaries.

Corollary 6. *(a) If $A \in (a_c^r(\Delta^{(m)}), bv)$ then L_A is compact if and only if*

$$\lim_{\eta\to\infty} \sup_{N\subset\mathbb{N}_0\setminus\{0,1,2,\ldots,m\}} \left(\sum_k \left|\sum_{n\in N} (\tilde{a}_{nk}(m,r) - \tilde{a}_{n-1,k}(m,r))\right|\right) = 0.$$

(b) If $A \in (a_c^r(\Delta^{(m)}), c(\Delta))$ then L_A is compact if and only if

$$\lim_{\eta\to\infty} \sup_{n>\eta} \sum_{k=0}^{\infty} |\tilde{a}_{nk}(m,r) - \tilde{a}_{n-1,k}(m,r)| = 0.$$

Proof. The proof of the part (a) can be obtained by using the Theorems 19 and 16. Namely, if $A \in (a_c^r(\Delta^{(m)}), bv)$, then L_A is compact if and only if $\|L_A\|_\chi = 0$. Further, $\|L_A\|_\chi = \chi(AK)$, where $K = \{x \in a_c^r(\Delta^{(m)}) : \|x\| \leq 1\}$. Since the space bv is the matrix domain of the triangle Δ in ℓ_1 then by Theorem 16, we have

$$\chi_{bv}(AK) = \chi_{\ell_1}(\Delta(AK)) = \|L_{\Delta A}\|_\chi.$$

By Lemma 7, $\Delta A \in (a_c^r(\Delta^{(m)}), \ell_1)$. Now, by Theorem 19 we obtain

$$\chi_{bv}(AK) = \lim_{\eta\to\infty} \sup_{N\subset\mathbb{N}_0\setminus\{0,1,2,\ldots,\eta\}} \left(\sum_k \left|\sum_{n\in N} (\tilde{a}_{nk}(m,r) - \tilde{a}_{n-1,k}(m,r))\right|\right).$$

(b) The proof is similar to (a).

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