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Distributions of eigenvalues for Sturm-Liouville problem under jump conditions

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Abstract

This paper deals with the asymptotic estimates of eigenvalues for Sturm-Liouville problems with two supplementary transmission conditions at one interior point. By modifying some known techniques the existence and uniqueness results of solutions are obtained for the considered problem.

Keywords: *Sturm-Liouville problems, transmission conditions, eigenvalue, eigenfunction.*

1 Introduction

A Sturm-Liouville differential equation on a finite interval together with boundary conditions arises from the infinitesimal, vertical vibrations of a string with the ends subject to various constraints. The coefficient (also called potential) function in the differential equation is in a close relationship with the density of the string, and the eigenvalues of the Sturm-Liouville problem are the square of the frequencies of oscillation of the string. The

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related methods continue to give rise to Sturm-Liouville problems which model many phenomena such as the earths seismic behavior, the propagation of sonar in the water stratified by varying density, and the stability and velocity of large-scale waves in atmosphere [7]. The computation of eigenvalues plays a rather important role in both mathematical and physical fields. In this paper we deal with one discontinuous eigenvalue problem which consists of Sturm-Liouville equation,

$$\tau u := -u''(x) + q(x)u(x) = \lambda u(x) \tag{1}$$

on $x \in [-1,0) \cup (0,1]$ subject to the transmission conditions at the inner point $x = 0$

$$u(+0) = \alpha u(-0), \tag{2}$$

$$u'(+0) = \beta u'(-0) \tag{3}$$

and the boundary conditions at the end points $x = -1$ and $x = 1$

$$u(-1) = u'(1) = 0, \tag{4}$$

where the potential $q(x)$ is real-valued, continuous in each interval $[-1,0)$
and $(0,1]$ and has a finite limits $q(\mp c)$; $\alpha,\ \beta$ are real numbers; λ is a complex eigenparameter. Problems of this type arise from the method of separation of variables applied to mathematical models for certain physical problems including heat conduction and wave propagation, etc. [3]. Sturm-Liouville problems with impulse effects (also known as interface conditions, transmission conditions, discontinuity conditions) arise in many applications (e.g., thermal conduction in a thin laminated plate made up of layers of different materials). They have been the object of several investigations recently [1, 2, 4, 5, 6, 9] in addition to an earlier attempt [10]. In this paper we obtain asymptotic formulas for the eigenvalues and eigenfunctions of the second order boundary-value problem (2)-(3). For second order differential equations, similar asymptotic formulas were obtained in [2, 4, 6].

2 Definition of Fundemental Solutions

In this section we shall define two basic solutions $\phi(x,\lambda)$ and $\chi(x,\lambda)$ by own technique as follows. At first, let us consider the initial-value problem on the left part $[-1,0)$ of the considered interval $[-1,0) \cup (0,1]$

$$-y'' + q(x)y = \lambda y,\ x \in [-1,0) \tag{5}$$

$$\phi_1(-1,\lambda) = 0,\ \ \frac{\partial \phi_1(-1,\lambda)}{\partial x} = 1 \tag{6}$$

By virtue of well-known existence and uniqueness theorem of ordinary differential equation theory this initial-value problem for each λ has a unique solution $\phi_1(x,\lambda)$. Moreover this solution is an entire function of λ for each fixed $x \in [-1,0)$ (see, [8]). By using this solution we shall construct the initial-value problem on the right part $(0,1]$ of the considered interval $[-1,0) \cup (0,1]$ as

$$-y'' + q(x)y = \lambda y,\ x \in (0,-1] \tag{7}$$

$$\phi_2(+0,\lambda) = \alpha\phi_1(-0,\lambda), \qquad \frac{\partial\phi_2(+0,\lambda)}{\partial x} = \beta\frac{\partial\phi_1(-0,\lambda)}{\partial x}. \tag{8}$$

Define a sequence of functions $y_n(x,\lambda), n = 0,1,2,...$ on interval $(0,1]$ by the following equations:

$$\begin{aligned} y_0(x,\lambda) &= \beta\frac{\partial\phi_1(-0,\lambda)}{\partial x}x + \alpha\phi_1(-0,\lambda) \end{aligned}$$

$$y_n(x,\lambda) = y_0(x,\lambda) + \int_0^x (x-z)(q(z)-\lambda)y_{n-1}(z,\lambda)dz,\ n = 1,2,... \tag{9}$$

It is easy to see that each of $y_n(x,\lambda)$ is an entire function of λ for each $x \in (0,1]$. Consider the series

$$y_0(x,\lambda) + \sum_{n=1}^{\infty}(y_n(x,\lambda) - y_{n-1}(x,\lambda)) \tag{10}$$

Denoting

$$q_1 = \max_{x\in(c,b]}|q(x)| \text{ and } Y(\lambda) = \max_{x\in(c,b]}|y_0(x,\lambda)|,$$

we can show that

$$|y_n(x,\lambda) - y_{n-1}(x,\lambda)| \le \frac{1}{(2n)!}Y(\lambda)(q_1 + |\lambda|^n)(x)^{2n} \tag{11}$$

for each $n = 1,2,....$ Because of this inequality the series (10) is uniformly convergent with respect to the variable x on $(0,1]$, and with respect to the variable λ on every closed bar $|\lambda| \le R$. Let $\phi_2(x,\lambda)$ be the sum of the series (10). Consequently $\phi_2(x,\lambda)$ is an entire function of λ for each fixed $x \in (0,1]$. Since for $n \ge 2$

$$y_n'(x,\lambda) - y_{n-1}'(x,\lambda) = \int_0^x (q(z)-\lambda)(y_{n-1}(z,\lambda) - y_{n-2}(z,\lambda))dz$$

and

$$y_n''(x,\lambda) - y_{n-1}''(x,\lambda) = (q(x)-\lambda)(y_{n-1}(x,\lambda) - y_{n-2}(x,\lambda))$$

the first and second differentiated series also converge uniformly with respect to x. Taking into account the last equality we have

$$\begin{aligned} \varphi_2''(x,\lambda) &= y_1''(x,\lambda) + \sum_{n=2}^{\infty}(y_n''(x,\lambda) - y_{n-1}''(x,\lambda)) \\ &= (q(x)-\lambda)y_1(x,\lambda) \\ &\quad + \sum_{n=2}^{\infty}(q(x)-\lambda)(y_n(x,\lambda) - y_{n-1}(x,\lambda)) \\ &= (q(x)-\lambda)\phi_2(x,\lambda), \end{aligned}$$

so $\phi_2(x,\lambda)$ satisfies the equation (7). Moreover, since each $y_n(x,\lambda)$ satisfies the initial conditions (8), then the function $\phi(x,\lambda)$ defined by

$$\phi(x,\lambda) = \{ \begin{array}{c} \phi_1(x,\lambda) \text{ for} x \in [-1,0) \\ \phi_2(x,\lambda) \text{ for } x \in (0,1]. \end{array} \tag{12}$$

satisfies equation (1), first boundary condition and the both transmission conditions (2) and (3). Similarly let $\chi_2(x,\lambda)$ be solutions of equation (1) on $(0,1]$ subject to initial conditions

$$\chi_2(1,\lambda) = -1, \quad \frac{\partial \chi_2(1,\lambda)}{\partial x} = 0. \tag{13}$$

Again, by virtue of [8] this solution is entire function of λ for fixed x. By applying the same technique we can prove that there is an unique solution $\chi_1(x,\lambda)$ of equation (1) on $[-1,0)$ satisfying the initial condition

$$\chi_1(-0,\lambda) = \frac{1}{\alpha}\chi_2(+0,\lambda), \quad \frac{\partial \chi_1(-0,\lambda)}{\partial x} = \frac{1}{\beta}\frac{\partial \chi_2(+0,\lambda)}{\partial x}. \tag{14}$$

By applying the similar technique as in [4] we can prove that the solution $\chi_1(x,\lambda)$ is also an entire function of parameter λ for each fixed x. Consequently, the function $\chi(x,\lambda)$ defined as

$$\chi(x,\lambda) = \begin{cases} \chi_1(x,\lambda), & x \in [-1,0) \\ \chi_2(x,\lambda), & x \in (0,1] \end{cases}$$

satisfies the equation (1) on whole $[-1,0) \cup (0,1]$, the other boundary condition $u'(1) = 0$ and the both transmission conditions (2) and (3).

3 Asymptotic Behaviour of Fundemental Solutions

Let $\lambda = s^2$. By applying the method of variation of parameters we can prove that the next integral and integro-differential equations are hold for $k = 0$ and $k = 1$.

$$\frac{d^k}{dx^k}\phi_1(x,\lambda) = -\frac{1}{s}\frac{d^k}{dx^k}\sin\left[s\left(x+1\right)\right] + \frac{1}{s}\int_{-1}^{x}\frac{d^k}{dx^k}\sin\left[s\left(x-z\right)\right]q(z)\phi_1(z,\lambda)dz \tag{15}$$

$$\begin{aligned} \frac{d^k}{dx^k}\chi_1(x,\lambda) &= \frac{1}{\alpha}\frac{d^k}{dx^k}\cos sx\chi_2(+0,\lambda) + \frac{\chi_2'(+0,\lambda)}{\beta s}\frac{d^k}{dx^k}\sin sx \\ &+ \frac{1}{s}\int_{x}^{0}\frac{d^k}{dx^k}\sin\left[s\left(x-z\right)\right]q(z)\chi_1(z,\lambda)dz \end{aligned} \tag{16}$$

for $x \in [-1, 0)$ and

$$\begin{aligned} \frac{d^k}{dx^k}\phi_2(x,\lambda) &= \alpha\frac{d^k}{dx^k}\cos sx\phi_1(-0,\lambda) + \frac{\beta\phi_1'(-0,\lambda)}{s}\frac{d^k}{dx^k}\sin sx \\ &+ \frac{1}{s}\int\limits_0^x \frac{d^k}{dx^k}\sin\left[s\left(x-z\right)\right]q(z)\phi_2(z,\lambda)dz \end{aligned} \tag{17}$$

$$\frac{d^k}{dx^k}\chi_2(x,\lambda) = -\frac{d^k}{dx^k}\cos\left[s\left(x-1\right)\right] + \frac{1}{s}\int\limits_x^1 \frac{d^k}{dx^k}\sin\left[s\left(x-z\right)\right]q(z)\chi_2(z,\lambda)dz \tag{18}$$

for $x \in (0, 1]$. Now we are ready to prove the following theorems.

Theorem 3.1. *Let $\lambda = s^2$, $Ims = t$. Then*

$$\frac{d^k}{dx^k}\phi_1(x,\lambda) = -\frac{1}{s}\frac{d^k}{dx^k}\sin\left[s(x+1)\right] + O\left(|s|^{k-2}e^{|t|(x+1)}\right) \tag{19}$$

$$\begin{aligned} \frac{d^k}{dx^k}\phi_2(x,\lambda) &= -\frac{1}{s}(\alpha\sin s\frac{d^k}{dx^k}\cos sx + \beta\cos s\frac{d^k}{dx^k}\sin sx) \\ &+O\left(|s|^{k-2}e^{|t|(x+1)}\right) \end{aligned} \tag{20}$$

as $|\lambda| \to \infty$ $(k = 0, 1)$. Each of this asymptotic equalities hold uniformly for x.

Proof. The asymptotic formula (19) follows immediately from the Titchmarsh's Lemma on the asymptotic behavior of $\phi_\lambda(x)$ ([8], Lemma 1.7). But the corresponding formulas for $\phi_2(x, \lambda)$ need individual consideration.

Substituting (19) in (17) we have the next "asymptotic integral equation"

$$\begin{aligned} \phi_2(x,\lambda) &= \frac{1}{s}\int\limits_0^x \frac{d^k}{dx^k}\sin\left[s\left(x-z\right)\right]q(z)\phi_2(z,\lambda)dz + O\left(\frac{1}{|s|^2}e^{|t|(x+1)}\right) \\ &+ -\frac{\alpha}{s}\sin s\cos sx - \frac{1}{\beta s}\cos s\sin sx \end{aligned} \tag{21}$$

It is easy to derive that

$$\frac{1}{s}\int\limits_0^x \frac{d^k}{dx^k}\sin\left[s\left(x-z\right)\right]q(z)\phi_2(z,\lambda)dz = O\left(\frac{1}{|s|^2}e^{|t|(x+1)}\right). \tag{22}$$

Substituting the equation (22) in the equality (21) we obtain (20) for the case $k = 0$. The case $k = 1$ of the eguality (20) follows at once on differentiating (17) and making the same procedure as in the case $k = 0$. □

Similarly we can easily obtain the following Theorem for $\chi_i(x, \lambda)(i = 1, 2)$.

Theorem 3.2. *Let $\lambda = s^2$, $Ims = t$. Then*

$$\frac{d^k}{dx^k}\chi_2(x,\lambda) = -\frac{d^k}{dx^k}\cos\left[s\left(x-1\right)\right] + O\left(|s|^{k-1}\, e^{|t|(x-1)}\right) \tag{23}$$

$$\begin{aligned}\frac{d^k}{dx^k}\chi_1(x,\lambda) &= -\left(\frac{\cos s}{\alpha}\frac{d^k}{dx^k}\cos sx + \frac{\sin s}{\beta}\frac{d^k}{dx^k}\sin sx\right) \\ &\quad + O\left(|s|^{k-1}e^{|t|(1-x)}\right)\end{aligned} \tag{24}$$

as $|\lambda| \to \infty$ (k=0,1). Each of this asymptotic equalities hold uniformly for x.

4 Distribution of Eigenvalues and Asymptotic Behavior of Eigenfunctions

It is well-known from ordinary differential equation theory that the Wronskians $W[\phi_1(\lambda), \chi_1(\lambda)]_x$ and $W[\phi_2(\lambda), \chi_2(\lambda)]_x$ are independent of variable x. By using (8) and (14) we have

$$\begin{aligned}w_1(\lambda) &= \phi_1(-0,\lambda)\frac{\partial\chi_1(-0,\lambda)}{\partial x} - \chi_1(-0,\lambda)\frac{\partial\phi_1(-0,\lambda)}{\partial x} \\ &= \frac{1}{\alpha\beta}\left(\phi_2(+0,\lambda)\frac{\partial\chi_2(+0,\lambda)}{\partial x} - \chi_2(+0,\lambda)\frac{\partial\phi_2(+0,\lambda)}{\partial x}\right) \\ &= \frac{1}{\alpha\beta}w_2(\lambda)\end{aligned}$$

for each $\lambda \in \mathbb{C}$. It is convenient to introduce the notation

$$w(\lambda) := \alpha\beta w_1(\lambda) = w_2(\lambda). \tag{25}$$

Now by modifying the standard method we prove that all eigenvalues of the problem $(1)-(4)$ are real.

Theorem 4.1. *The eigenvalues of the boundary-value-transmission problem* $(1)-(4)$ *are real.*

Proof. Let λ_0 be eigenvalue and y_0 be eigenfunction corresponding to this eigenvalue. By two partial integration we have

$$\begin{aligned}
& \int_{-1}^{0}(\lambda_0 y_0(x))\overline{y_0(x)}dx + \frac{1}{\alpha\beta}\int_{0}^{1}((\lambda_0 y_0(x))\overline{y_0(x)}dx \\
= & \int_{-1}^{0}(\tau y_0)(x)\overline{y_0(x)}dx + \frac{1}{\alpha\beta}\int_{0}^{1}(\tau y_0)(x)\overline{y_0(x)}dx \\
= & \int_{-1}^{0} y_0(x)\overline{(\tau y_0)(x)}dx + \frac{1}{\alpha\beta}\int_{0}^{1} y_0(x)\overline{(\tau y_0)(x)}dx + W[y_0,\overline{z};-0] \\
- & W[y_0,\overline{y_0};-1] + \frac{1}{\alpha\beta}W[y_0,\overline{y_0};1] - \frac{1}{\alpha\beta}\ W[y_0,\overline{y_0};+0] \\
= & \int_{-1}^{0} y_0(x)\overline{(\lambda_0 y_0)(x)}dx + \frac{1}{\alpha\beta}\int_{0}^{1} y_0(x)\overline{(\lambda_0 y_0)(x)}dx + \ W(y_0,\overline{z};-0) \\
- & W[y_0,\overline{y_0};-1] + \frac{1}{\alpha\beta}W[y_0,\overline{y_0};1] - \ \frac{1}{\alpha\beta}W[y_0,\overline{y_0};+0] \qquad (26)
\end{aligned}$$

From the boundary boundary conditions (2)-(3) it is follows obviously that

$$W(y_0,\overline{y_0};-1) = 0 \text{ and } W(y_0,\overline{y_0};1) = 0. \qquad (27)$$

The direct calculation gives

$$W(y_0,\overline{y_0};-0) = \frac{1}{\alpha\beta}W(y_0,\overline{y_0};+0). \qquad (28)$$

Substituting (27) and (28) in (26) we have the equality

$$(\lambda_0 - \overline{\lambda_0})[\int_{-1}^{0}(y_0(x))^2dx + \frac{1}{\alpha\beta}\int_{0}^{1}(y_0(x))^2dx] = 0$$

Thus, we get $\lambda_0 = \overline{\lambda_0}$ since $\alpha\beta > 0$. Consequently all eigenvalues of the problem $(1)-(4)$ are real. □

Corollary 4.2. *Let $u(x)$ and $v(x)$ be eigenfunctions corresponding to distinct eigenvalues. Then they are orthogonal in the sense of the following equality*

$$\int_{-1}^{0} u(x)v(x)dx + \frac{1}{\alpha\beta}\int_{0}^{1} u(x)v(x)dx = 0. \qquad (29)$$

Since the Wronskians of $\phi_2(x,\lambda)$ and $\chi_2(x,\lambda)$ are independent of x, in particular, by putting $x = 1$ we have

$$\begin{aligned}
w(\lambda) &= \phi_2(1,\lambda)\chi_2'(1,\lambda) - \phi_2'(1,\lambda)\chi_2(1,\lambda) \\
&= \phi_2'(1,\lambda). \qquad (30)
\end{aligned}$$

Let $\lambda = s^2$, $Ims = t$. By substituting (20) in (30) we obtain easily the following asymptotic representation

$$w(\lambda) = -\alpha\sin^2 s + \beta\cos^2 s + O\left(\frac{1}{s}e^{|2t|}\right) \qquad (31)$$

Now we are ready to derived the needed asymptotic formulas for eigenvalues and eigenfunctions.

Theorem 4.3. *The boundary-value-transmission problem* (1)-(4) *has an precisely numerable many real eigenvalues,* $\lambda_0, \lambda_1, \lambda_2 ...$ *for which the following asymptotic expression is hold*

$$s_n^{\pm} = \pi n \pm \arctan \sqrt{\frac{\beta}{\alpha}} + O\left(\frac{1}{n}\right) \tag{32}$$

where $s_n = \{s_n^-, s_n^+\}$.

Proof. By applying the well-known Rouche Theorem which asserts that if $f(z)$ and $g(z)$ are analytic inside and on a closed contour Γ, and $|g(z)| < |f(z)|$ on Γ then $f(z)$ and $f(z)+g(z)$ have the same number zeros inside Γ provided that the zeros are counted with multiplicity on a sufficiently large contour, it follows that $w(\lambda)$ has the same number of zeros inside the contour as the leading term $w_0(\lambda) = -\alpha \sin^2 s + \beta \cos^2 s$ in (31). Hence, if $\lambda_0 < \lambda_1 < \lambda_2 ...$ are the zeros of $w(\lambda)$ and $s_n = \lambda_n$, we have the needed asymptotic formulas (32). □

Using this asymptotic formulas for eigenvalues we can derive that the corresponding eigenfunctions may be expressed by the formula

$$\phi_n(x) = \begin{cases} \sin(\pi n \pm \arctan \sqrt{\frac{\beta}{\alpha}})(x+1) + O\left(\frac{1}{n}\right) \quad \text{for } x \in [-1, 0) \\ \alpha \sin(\pi n \pm \arctan \sqrt{\frac{\beta}{\alpha}}) \cos[(\pi n \pm \arctan \sqrt{\frac{\beta}{\alpha}})x] \\ -\beta \cos(\pi n \pm \arctan \sqrt{\frac{\beta}{\alpha}}) \sin[(\pi n \pm \arctan \sqrt{\frac{\beta}{\alpha}})x] \\ +O\left(\frac{1}{n}\right), \text{ for } x \in (0, 1] \end{cases}$$

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