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Multiple Sets

Shijina.V[a] (shijichan@gmail.com)
Sunil Jacob John[a,1] (sunil@nitc.ac.in)
Anitha Sara Thomas[a] (ani_dayana@yahoo.co)

[a]*Department of Mathematics, National Institute of Technology, 673 601 Calicut, India*

Abstract - In this paper, the concept of multiple sets is introduced as a generalization of fuzzy sets, multi fuzzy sets, fuzzy multisets and multisets. The standard operations of union, intersection, complement of multiple sets are discussed. Also it is proved that multiple sets are L-fuzzy sets in the sense of Goguen.

Keywords - *Multiple sets, Fuzzy set, Multi set, Fuzzy multiset, Multi fuzzy set, L-fuzzy set.*

1 Introduction

The volume of data that is being handled in various fields of social life and industries has increased tremendously with the advent of internet and database technologies. The traditional statistical techniques and data management tools are no longer effective in analyzing these huge data sets, which are often regarded as rich in data but poor in knowledge. A more efficient data processing approach is needed to deal with these large volumes of data in order to extract useful knowledge. For this, a new generation of technologies and tools has emerged, which aims at finding hidden, previously unknown, and potentially useful knowledge from the data. Recently various soft computing methodologies have been applied to handle the different challenges posed by data mining. To help people further in intelligently analyze data and automatedly study a number of cases, there emerged a new generation of tools based on generalized set theoretic structures such as Multisets, Rough Sets, Soft sets etc. and their hybridizations with existing structures like Fuzzy set theory, Grey system theory, Genetic algorithms, Neural networks etc. The relevance of these studies have two dimensions. Firstly, formulation of a more convenient and consistent model for representing these structures is mathematically important. Secondly, soft computing techniques based on

[1]Corresponding Author

these structures can be applied successfully in various applications so that a low-cost solution for Decision-making under uncertain conditions can be achieved.

The structures developed for the purpose of efficient modeling of uncertain situation includes Fuzzy sets by Zadeh[10], Multisets by Cerf et al. [2] and many more. Further, combinations and/or generalizations of these are also developed. They include Fuzzy Multisets by Yager [9] and Multi fuzzy sets by Sabu and Ramakrishnan [7]. In this paper we will introduce a new structure called Multiple sets, which unifies many of the above approaches in the sense that above mentioned cases becomes particular types of multiple sets. In multiple sets, multiple occurrences of elements are permitted in which each occurrence has a finite number of same or different membership values. That is, in multiple set theory, a multiple set of order (n, k) gives nk membership grades to each element x in the universal set X.

In this paper we will formally define a multiple set and show that ordinary sets, Zadeh type Fuzzy sets, Multi sets, Fuzzy Multi sets and Multi fuzzy sets can be considered as particular types of multiple sets. Further basic operations on multiple sets are suggested and finally it is proved that multiple sets can be considered as a particular type of L-fuzzy sets in the sense of Goguen [3].

2 Multiple Sets

As a new approach towards the representation of vague data, we defined a new mathematical model called multiple set. A multiple set gives degree of membership of each element in a multiple way;

Definition 2.1. *Let X be a non empty set called universe. A multiple set A drawn from X is an object of the form $\{(x, A(x))\,; x \in X\}$, where for each $x \in X$, its membership value is an $n \times k_x$ matrix*

$$A(x) = \begin{bmatrix} A_1^1(x) & A_1^2(x) & \cdots & A_1^{k_x}(x) \\ A_2^1(x) & A_2^2(x) & \cdots & A_2^{k_x}(x) \\ & \cdots & & \\ A_n^1(x) & A_n^2(x) & \cdots & A_n^{k_x}(x) \end{bmatrix}$$

where A_i, $i = 1, 2, ..., n$ are membership functions. For each $i = 1, 2, ..., n$, $A_i^j(x)$, $j = 1, 2, ..., k$ are membership values of the membership function A_i for the element $x \in X$, written in decreasing order. Then, by taking $k = sup\{k_x; x \in X\}$, we can make $A(x)$ as $n \times k$ matrix by adding sufficient number of columns of zeros. That is, for each $x \in X$

$$A(x) = \begin{bmatrix} A_1^1(x) & A_1^2(x) & \cdots & A_1^{k}(x) \\ A_2^1(x) & A_2^2(x) & \cdots & A_2^{k}(x) \\ & \cdots & & \\ A_n^1(x) & A_n^2(x) & \cdots & A_n^{k}(x) \end{bmatrix}$$

is the membership matrix of order $n \times k$. Then A is called multiple set of order (n, k).

The univeral set X and empty set Φ can be consider as multiple sets with the membership matrix in which all entries are one and zero respectively.

Remark 2.2. *Let A and B be two multiple sets of order (n_1, k_1) and (n_2, k_2) respectively. Take $n = \max\{n_1, n_2\}$ and $k = \max\{k_1, k_2\}$. Then, for each $x \in X$, both $A(x)$ and $B(x)$ can be viewed as $n \times k$ matrices by adding sufficient number of rows and columns of zeroes, if necessary. In this way, we can consider both A and B as multiple sets of order (n, k).*

Denote the set of all multiple sets drawn from X by $MS(X)$ and set of all multiple sets of order (n, k) drawn from X by $MS_{(n,k)}(X)$.

Remark 2.3. *A multiple set A can be viewed as a function $A : X \to \mathbb{M}$, where $\mathbb{M} = \mathbb{M}_{n\times k}([0,1])$ is the set of all matrices of order $n \times k$ with entries from $[0,1]$, which maps each $x \in X$ to its membership matrix $A(x)$.*

2.1 Operations on Multiple Sets

In this section we introduce the standard multiple set operations, that are the generalization of the standard operations complement, intersection and union on crisp sets.

Definition 2.4. *Let X be a universal set. Let $A, B \in MS_{(n,k)}(X)$ and*

$$A(x) = \begin{bmatrix} A_1^1(x) & A_1^2(x) & \cdots & A_1^k(x) \\ A_2^1(x) & A_2^2(x) & \cdots & A_2^k(x) \\ & \cdots & & \\ A_n^1(x) & A_n^2(x) & \cdots & A_n^k(x) \end{bmatrix}$$

$$B(x) = \begin{bmatrix} B_1^1(x) & B_1^2(x) & \cdots & B_1^k(x) \\ B_2^1(x) & B_2^2(x) & \cdots & B_2^k(x) \\ & \cdots & & \\ B_n^1(x) & B_n^2(x) & \cdots & B_n^k(x) \end{bmatrix}$$

be the membership matrices for x in A and B respectively.

1. ***Subset**: $A \subseteq B$ if and only if $A_i^j(x) \leq B_i^j(x)$ for every $x \in X$, $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$.*
2. ***Equality**: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. That is, if and only if $A_i^j(x) = B_i^j(x)$ for every $x \in X$, $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$.*
3. ***Standard Union**: The union of A and B, denoted as $A \cup B$, is a multiple set whose membership matrix for each $x \in X$, is given by $(A \cup B)_i^j(x) = \max\left\{A_i^j(x), B_i^j(x)\right\}$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$.*
4. ***Standard Intersection**: The intersection of A and B, denoted as $A \cap B$, is a multiple set whose membership matrix for each $x \in X$, is given by $(A \cap B)_i^j(x) = \min\left\{A_i^j(x), B_i^j(x)\right\}$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$.*
5. ***Standard Complement**: The standard complement of A, denoted as $\bar{A}$, is a multiple set whose membership matrix for each $x \in X$, is given by $\bar{A}_i^j(x) = 1 - A_i^j(x)$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$.*

2.2 Properties of Multiple Sets

Multiple sets satisfies all the fundemental properties of the set operations, except the law of contradiction and law of excluded middle.

1. **Involution**
   $\overline{\left(\overline{A}\right)} = A$
   Proof: Let $x \in X$. For $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$, we have

$$\begin{aligned} \overline{\left(\overline{A}\right)}_i^j(x) &= 1 - \left(\overline{A}\right)_i^j(x) \\ &= 1 - \left(1 - A_i^j(x)\right) \\ &= A_i^j(x) \end{aligned}$$

   That is, $\overline{\left(\overline{A}\right)}_i^j(x) = A_i^j(x)$ for every $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$. Since x is arbitrary, we get $\overline{\left(\overline{A}\right)} = A$.

2. **Commutativity**
   $A \cup B = B \cup A$
   $A \cap B = B \cap A$
   Proof: Let $x \in X$. For $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$, we have

$$\begin{aligned} (A \cup B)_i^j(x) &= \max\left\{A_i^j(x), B_i^j(x)\right\} \\ &= \max\left\{B_i^j(x), A_i^j(x)\right\} \\ &= (B \cup A)_i^j(x) \end{aligned}$$

   That is, $(A \cup B)_i^j(x) = (B \cup A)_i^j(x)$ for every $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$. Since x is arbitrary, we get $A \cup B = B \cup A$. Similarly, we can prove $A \cap B = B \cap A$.

3. **Associativity**:
   $(A \cup B) \cup C = A \cup (B \cup C)$
   $(A \cap B) \cap C = A \cap (B \cap C)$
   Proof: Let $x \in X$. For $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$, we have

$$\begin{aligned} [(A \cup B) \cup C]_i^j(x) &= \max\left\{(A \cup B)_i^j(x), C_i^j(x)\right\} \\ &= \max\left\{\max\left\{A_i^j(x), B_i^j(x)\right\}, C_i^j(x)\right\} \\ &= \max\left\{A_i^j(x), \max\left\{B_i^j(x), C_i^j(x)\right\}\right\} \\ &= \max\left\{A_i^j(x), (B \cup C)_i^j(x)\right\} \\ &= [A \cup (B \cup C)]_i^j(x) \end{aligned}$$

   That is, $[(A \cup B) \cup C]_i^j(x) = [A \cup (B \cup C)]_i^j(x)$ for every $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$. Since x is arbitrary, we get $(A \cup B) \cup C = A \cup (B \cup C)$. Similarly, we can prove $(A \cap B) \cap C = A \cap (B \cap C)$.

4. **Distributivity**:
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Proof: Let $x \in X$. For $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$, we have

$$\begin{aligned} [A \cap (B \cup C)]_i^j(x) &= \min\left\{A_i^j(x), (B \cup C)_i^j(x)\right\} \\ &= \min\left\{A_i^j(x), \max\left\{B_i^j(x), C_i^j(x)\right\}\right\} \\ &= \max\left\{\min\left\{A_i^j(x), B_i^j(x)\right\}, \min\left\{A_i^j(x), C_i^j(x)\right\}\right\} \\ &= \max\left\{(A \cap B)_i^j(x), (A \cap C)_i^j(x)\right\} \\ &= [(A \cap B) \cup (A \cap C)]_i^j(x) \end{aligned}$$

That is, $[A \cap (B \cup C)]_i^j(x) = [(A \cap B) \cup (A \cap C)]_i^j(x)$ for every $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$. Since x is arbitrary, we get $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Similarly, we can prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

5. **Idempotence**:
$A \cup A = A$
$A \cap A = A$
Proof: Let $x \in X$. For $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$, we have

$$\begin{aligned} (A \cup A)_i^j(x) &= \max\left\{A_i^j(x), A_i^j(x)\right\} \\ &= A_i^j(x) \end{aligned}$$

That is, $(A \cup A)_i^j(x) = A_i^j(x)$ for every $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$. Since x is arbitrary, we get $A \cup A = A$. Similarly, we can prove $A \cap A = A$.

6. **Absorption**:
$A \cup (A \cap B) = A$
$A \cap (A \cup B) = A$
Proof: Let $x \in X$. For $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$, we have

$$\begin{aligned} [A \cup (A \cap B)]_i^j(x) &= \max\left\{A_i^j(x), \min\left\{A_i^j(x), B_i^j(x)\right\}\right\} \\ &= A_i^j(x) \end{aligned}$$

That is, $[A \cup (A \cap B)]_i^j(x) = A_i^j(x)$ for every $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$. Since x is arbitrary, we get $A \cup (A \cap B) = A$. Similarly, we can prove $A \cap (A \cup B) = A$.

7. **Absorption by X and Φ**:
$A \cup X = X$
$A \cap \Phi = \Phi$
Proof: Let $x \in X$. For $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$, we have

$$\begin{aligned} (A \cup X)_i^j(x) &= \max\left\{A_i^j(x), X_i^j(x)\right\} \\ &= \max\left\{A_i^j(x), 1\right\} \\ &= 1 \\ &= X_i^j(x) \end{aligned}$$

That is, $(A \cup X)_i^j(x) = X_i^j(x)$ for every $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$. Since x is arbitrary, we get $A \cup X = X$. Similarly, we can prove $A \cap \Phi = \Phi$.

8. **Identity:**
$A \cup \Phi = A$
$A \cap X = A$
Proof: Let $x \in X$. For $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$, we have

$$\begin{aligned} (A \cup \Phi)_i^j(x) &= \max\left\{A_i^j(x), \Phi_i^j(x)\right\} \\ &= \max\left\{A_i^j(x), 0\right\} \\ &= A_i^j(x) \end{aligned}$$

That is, $(A \cup \Phi)_i^j(x) = A_i^j(x)$ for every $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$. Since x is arbitrary, we get $A \cup \Phi = A$. Similarly, we can prove $A \cap X = A$.

9. **De Morgan's laws**:
$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$
$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$
Proof: Let $x \in X$. For $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$, we have

$$\begin{aligned} \overline{(A \cup B)}_i^j(x) &= 1 - (A \cup B)_i^j(x) \\ &= 1 - \max\left\{A_i^j(x), B_i^j(x)\right\} \\ &= \min\left\{1 - A_i^j(x), 1 - B_i^j(x)\right\} \\ &= \min\left\{\bar{A}_i^j(x), \bar{B}_i^j(x)\right\} \\ &= \left(\bar{A} \cap \bar{B}\right)_i^j(x) \end{aligned}$$

That is, $\overline{(A \cup B)}_i^j(x) = \left(\overline{A} \cap \overline{B}\right)_i^j(x)$ for every $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$. Since x is arbitrary, we get $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$. Similarly, we can prove $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$.

10. $A \subseteq A \cup B$
$B \subseteq A \cup B$
Proof: Let $x \in X$. For $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$, we have

$$A_i^j(x) \leq \max\left\{A_i^j(x), B_i^j(x)\right\} = (A \cup B)_i^j(x)$$

That is, $A_i^j(x) \leq (A \cup B)_i^j(x)$ for every $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$. Since x is arbitrary, we get $A \subseteq A \cup B$. Similarly, we can prove $B \subseteq A \cup B$.

11. $A \cap B \subseteq A$
$A \cap B \subseteq B$
Proof: Let $x \in X$. For $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$, we have

$$(A \cap B)_i^j(x) = \min\left\{A_i^j(x), B_i^j(x)\right\} \leq A_i^j(x)$$

That is, $(A \cap B)_i^j(x) \leq A_i^j(x)$ for every $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$. Since x is arbitrary, we get $A \cap B \subseteq A$. Similarly, we can prove $A \cap B \subseteq B$.

Remark 2.5. *Law of contradiction and law of excluded middle are violated for multiple sets. For, let $x \in X$. For $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$, we have*

$$\begin{aligned} \left(A \cap \overline{A}\right)_i^j(x) &= \min\left\{A_i^j(x), \overline{A}_i^j(x)\right\} \\ &= \min\left\{A_i^j(x), 1 - A_i^j(x)\right\} \end{aligned}$$

Thus $\left(A \cap \bar{A}\right)_i^j(x) = 0$ only when $A_i^j(x) = 0$ or 1. That is $\left(A \cap \bar{A}\right)_i^j(x)$ need not be equal to 0 always. Therefore law of contradiction is violated for multiple sets.
Similarly,

$$\begin{aligned} \left(A \cup \bar{A}\right)_i^j(x) &= \max\left\{A_i^j(x), \bar{A}_i^j(x)\right\} \\ &= \max\left\{A_i^j(x), 1 - A_i^j(x)\right\} \end{aligned}$$

Thus $\left(A \cup \overline{A}\right)_i^j(x) = 1$ only when $A_i^j(x) = 0$ or 1. That is $\left(A \cap \bar{A}\right)_i^j(x)$ need not be equal to 1 always. Therefore law of excluded middle are violated for multiple sets.

2.3 Multiple Set as a Unified Generalization Tool

Multiple set can be considered as a generalization of fuzzy sets, multi fuzzy sets, fuzzy multisets and multisets. Before considering it, a brief review of fuzzy sets, multi fuzzy sets, fuzzy multisets and multisets are useful. According to Zadeh, a fuzzy set is characterised by a membership function.

Definition 2.6. *[10] Let X be a given universal set, which is always a crisp set. A fuzzy set A on X is characterized by a function $A : X \to [0, 1]$ called fuzzy membership function, which assigns to each object a grade of membership ranging between zero and one. A fuzzy set A is defined as*

$$A = \{(x, A(x)); x \in X\}$$

where $A(x)$ is the fuzzy membership value of x in X.

If the membership function A takes only values 0 or 1, then a fuzzy set A coincides with the crisp set A.

Multisets are a generalization of sets in which elements can occur more than once. Jena et al. defined multi set as;

Definition 2.7. *[4] Let X be a non empty set, called universe. A multiset M drawn from X is represented by a count function $C_M : X \to \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of positive integers. For each $x \in X$, $C_M(x)$ indicates the number of occurrences of the element x in M. Then a multiset M can be expressed as $\{C_M(x)/x; x \in X\}$.*

Miyamoto discussed fuzzy multi set as a generalization of multisets;

Definition 2.8. *[5] For $x \in X$, the membership sequence of x is defined as a non increasing sequence of membership values of x and it is denoted by $(\mu_A^1(x), \mu_A^2(x), ..., \mu_A^k(x))$, such that $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^k(x)$, where μ_A is a membership function and μ_A^j, $j = 1, 2, ..., k$ are values(same or different) of membership function μ_A . A fuzzy multiset is a collection of all x together with its membership sequence.*

Multi-fuzzy set theory is an extension of fuzzy set theory, L-fuzzy set theory and Atanassov intuitionistic fuzzy set theory. The concept of multi fuzzy set was given by S. Sebastian and T.V. Ramakrishnan.

Definition 2.9. *[7] Let X be a non empty set and let $\{L_i; i \in \mathbb{N}\}$ be a family of complete lattices where $\mathbb{N}$ is the set of positive integers. A multi fuzzy set A in X is a set of ordered sequences*

$$A = \{(x, \mu_1(x), \mu_2(x), ...); x \in X\}$$

where $\mu_i \in L_i^X$ for $i \in \mathbb{N}$. The function $\mu_A = (\mu_1, \mu_2, ...)$ is called a multi membership function of multi fuzzy set A.

From the above discussed basic definitions, we can realise that fuzzy sets, multi fuzzy sets, fuzzy multisets and multisets can be considered as a special type of multiple sets. This fact is stated as the following theorem;

Theorem 2.10. Let A be a multiple set of order $n \times k$.

1. If $k = 1$ and $n = 1$, then A is a fuzzy set. This is a special case of crisp set.
2. If $n = 1$, then A is a fuzzy multi set.
3. If $k = 1$, then A is a multi fuzzy set (finite case).
4. If $A_i^j(x) = 0$ or 1 for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$, then A is multi set.

Proof. Let A be a multiple set of order $n \times k$. Then A is a function $A : X \to \mathbb{M}$, where $\mathbb{M}$ is the set of all matrices of order $n \times k$ with entries from $[0, 1]$.

1. If $k = 1$ and $n = 1$, then A is a function $A : X \to [0, 1]$. Thus A is a fuzzy set. If A takes only the values 0 or 1 , then A is a crisp set.
2. If $n = 1$, then A maps each $x \in X$ to a membership sequence $\left(A_1^1(x), A_1^2(x), \cdots, A_1^k(x)\right)$. Thus A is a fuzzy multi set.
3. If $k = 1$, then A maps each $x \in X$ to a membership sequence $\left(A_1(x), A_2(x), \cdots, A_k(x)\right)$. Thus A is a multi fuzzy set(finite case).
4. If $A_i^j(x) = 0$ or 1 for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$, then A maps each $x \in X$ to a matrix $A(x) = (a_{ij})$ whose entries are either 0 or 1. Now define $C(x) = \sum_{i,j} a_{ij}$ and map each $x \in X$ to $C(x)$. Therefore A is a multiset.

□

2.4 Multiple Sets as a Special Type of L-fuzzy Sets

Basic definitions of poset, lattice, complete lattice are given [6, 1, 8].

A relation $\leq$ on a nonempty set P is called a partial order if $\leq$ is

1. Reflexive: $a \leq a$
2. Anti symmetric: $a \leq b$ and $b \leq a$ implies $a = b$
3. Transitive: $a \leq b$ and $b \leq c$ implies $a \leq c$

Partially ordered set(poset) is a structure $\langle P, \leq \rangle$, where P is a nonempty set and $\leq$ is a partial order relation on P. A poset P is a totally ordered poset if and only if for every $x, y \in P$ either $x \leq y$ or $y \leq x$, that is, if and only if any two elements $x, y \in P$ are comparable. An element $\mathbf{0} \in P$ is said to be least element or minimum element in P if and only if $\mathbf{0} \leq a$ for every $a \in P$. If the least element $\mathbf{0}$ exists, then it is unique. Similarly, if a poset contains an element $\mathbf{1} \in P$ such that $a \leq \mathbf{1}$ for every $a \in P$. Then that element is uniquely determined and will be called the greatest element or maximum element. A poset with the minimum element $\mathbf{0}$ and the maximum element $\mathbf{1}$ is denoted as $\langle P, \leq, \mathbf{0}, \mathbf{1} \rangle$. For all $b \in P$, $\mathbf{0} \leq b \leq \mathbf{1}$. So $\langle P, \leq, \mathbf{0}, \mathbf{1} \rangle$ is called a bounded poset.

Let S be a a nonempty subset of a poset P and $x \in P$. If $s \leq x$ for every $s \in S$, then x is said to be an upper bound of S and is called the least upper bound of S if it satisfies the condition: if $c \in P$ satisfies $s \leq c$ for every $s \in S$, then $x \leq c$. If $s \geq x$ for every $s \in S$, then x is said to be an lower bound of S and is called the greatest lower bound of S if it satisfies the condition: if $c \in P$ satisfies $s \geq c$ for every $s \in S$, then $x \geq c$.

A lattice is a poset $\langle L, \leq \rangle$ such that both $x \wedge y = \inf\{x, y\}$ (called lattice meet or conjunction) and $x \vee y = \sup\{x, y\}$ (called the lattice join or disjunction) exist for every pair of elements $x, y \in L$. A lattice L is complete if and only if both $\inf S$ and $\sup S$ exist for every nonempty subset S of L. A complete lattice with N is a triplet $\langle L, \leq, N \rangle$, where $\langle L, \leq \rangle$ is a complete lattice and N is a unary, involutive and order reversing operator on L.

Let $\mathbb{M} = \mathbb{M}_{n \times k}([0, 1])$ denote the set of all matrices of order $n \times k$, with entries from $[0, 1]$. For $A = (a_{ij}), B = (b_{ij})$ in $\mathbb{M}$, define the relation $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Then $\langle \mathbb{M}, \leq, N \rangle$ is complete lattice with the least element $\mathbf{0}$ equals to the zero matrix of order $n \times k$, greatest element $\mathbf{1}$ equals the $n \times k$ matrix with all elements 1 and the operator N is defined by

$$N(A) = B \text{ with } b_{ij} = 1 - a_{ij}.$$

for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$.

L-fuzzy sets are introduced by Goguen in 1967. The concept of L-fuzzy sets is a generalization of the concept of fuzzy sets and takes the latter as a special case when $L = [0, 1]$. An L-fuzzy set is defined as;

Definition 2.11. *[3] An L-fuzzy set on a universe X is a mapping $X \to L$, where $\langle L, \leq, N \rangle$ is a complete lattice provided with a unary, involutive and order reversing operator on L.*

Multiple sets can be viewed as special type of L-fuzzy sets;

Theorem 2.12. A multiple set of order $n \times k$ is a special type of L-fuzzy set.

Proof. We have $\langle \mathbb{M}, \leq, N \rangle$ is complete lattice and a multiple set of order $n \times k$ is a a function $A : X \to \mathbb{M}$. Thus a multiple set is a special type of L-fuzzy set. □

3 Conclusion

In this paper we have introduced multiple sets, a new mathematical approach to model vagueness and multiplicity. Multiple set is an extension of fuzzy set, multiset, fuzzy multiset and multi fuzzy set. Then we have reviewed L-fuzzy sets and concluded that multiple sets are special types of L-fuzzy sets.

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