ISSN: **1304-7981** Number: **9**, Year:**2015**, Pages: **57-65** **http://jnrs.gop.edu.tr**

Gaziosmanpasa University
Graduate School of Natural and Applied Sciences
Journal of New Results in Science

Received: 15.09.2015
Accepted: 19.11.2015

Editors-in-Chief: *Bilge Hilal Cadırcı*
Area Editor: *Serkan Demiriz*

Some operators on generalised fuzzy soft topological spaces

Prakash Mukherjee[a,1] (prakashmukherjee25@gmail.com)

[a] *Hijli College, Midnapore – 721301, West Bengal, India*

Abstract - In this paper, we defined difference of two generalised fuzzy soft sets, generalised fuzzy soft exterior, generalised fuzzy soft boundary and studied some of their basic properties. Finally, we introduced the notion of generalised soft quasi-coincidence for generalised fuzzy soft sets and studied some basic properties of this concept.

Keywords - *Generalised fuzzy soft set, Generalised fuzzy soft topology, Generalised fuzzy soft exterior, Generalised fuzzy soft boundary, Generalised soft quasi-coincidence.*

1 Introduction

In many complicated problems of the fields of engineering, social sciences, economics, computer science, medical science, environmental science etc, the associated data are not necessarily crisp, precise and deterministic because of their vague nature. To handle such vagueness, L.A. Zadeh [10] in 1965, was the first to come up with his remarkable theory of fuzzy set. Zadeh's theory brought a grand paradigmatic change in mathematics but this theory has its inherent difficulties possibly due to the inadequacy of parameterization tool of the theories as pointed out by Molodtsov in [6]. To deal with uncertainties and imprecisions, in 1999, Molodtsov [6] introduced a new mathematical tool called "soft set theory". This new concept is free from the above mentioned difficulties.

In recent times, the process of fuzzification of soft set theory is rapidly progressed. In 2001, Maji et al.[3] introduced the fuzzy soft set. Topological structure of fuzzy soft sets was started by Tanay and Burc kandemir [8]. The study was pursued by some others [2, 7]. In 2010, Majumdar and Samanta [4] introduced generalised fuzzy soft sets and successfully applied their notion in a decision making problem. Yang [9] pointed out that some results of Majumdar and Samanta [4] which are not valid in general. Chakraborty and Mukherjee [1] introduced the generalised fuzzy soft union,

[1]Corresponding Author

generalised fuzzy soft intersection and several other properties of generalised fuzzy soft sets. In the same paper they introduced "generalised fuzzy soft topological spaces" over some soft universe with a fixed set of parameters. In our present article, we introduced difference of two generalised fuzzy soft sets, some operators like generalized fuzzy soft exterior, generalised fuzzy soft boundary and studied some of their basic properties. Finally, we introduced the notion of generalised soft quasi-coincidence for generalised fuzzy soft sets and studied some of its basic properties.

2 Preliminaries

Throughout this paper X denotes the initial universe, E denotes the set of all possible parameters for X. $P(X)$ denotes the power set of X, I^X denotes the set of all fuzzy sets on X, I^E denotes the collection of all fuzzy sets on E, (X, E) denotes the soft universe and I stands for $[0, 1]$.

Definition 2.1 [10] A fuzzy set A in X is defined by a membership function $\mu_A : X \to [0, 1]$ whose value $\mu_A(x)$ represents the "grade of membership" of x in A for $x \in X$.

If A, $B \in I^X$ then from [10] we have the following:

(i) $A \leq B \Leftrightarrow \mu_A(x) \leq \mu_B(x),\ \forall\ x \in X$;
(ii) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x),\ \forall\ x \in X$;
(iii) $C = A \vee B \Leftrightarrow \mu_C(x) = \max(\mu_A(x), \mu_B(x)),\ \forall\ x \in X$;
(iv) $D = A \wedge B \Leftrightarrow \mu_D(x) = \min(\mu_A(x), \mu_B(x)),\ \forall\ x \in X$;
(v) $E = A^c \Leftrightarrow \mu_E(x) = 1 - \mu_A(x),\ \forall\ x \in X$.

Definition 2.2 [5] For two fuzzy sets A and B in X, we write AqB to mean that A is quasi-coincident with B, i.e., there exists at least one point $x \in X$ such that $\mu_A(x) + \mu_B(x) > 1$. If A is not quasi-coincident with B, then we write $A\overline{q}B$.

Definition 2.3 [6] Let $A \subseteq E$. A pair (f, A) is called a soft set over X if f is a mapping from A into $P(X)$, i.e., $f : A \to P(X)$.

In other words, a soft set is a parameterized family of subsets of the set X. For $e \in A$, $f(e)$ may be considered as the set of $e-$approximate elements of the soft set (f, A).

Definition 2.4 [3] Let $A \subseteq E$. A pair (F, A) is called a fuzzy soft set over X if $F : A \to I^X$ is a function, i.e., for each $a \in A$, $F(a) = F_a : X \to [0, 1]$ is a fuzzy set on X.

Definition 2.5 [4] Let X be the universal set of elements and E be the universal set of parameters for X. Let $F : E \to I^X$ and μ be a fuzzy subset of E, i.e., $\mu : E \to I = [0, 1]$. Let F_μ be the mapping $\tilde{F}_\mu : E \to I^X \times I$ be a function defined as follows: $\tilde{F}_\mu(e) = (F(e), \mu(e))$, where $F(e) \in I^X$ and $\mu(e) \in I^E$. Then $\tilde{F}_\mu$ is called a generalised fuzzy soft set ($GFSS$ in short) over (X, E).

Here for each parameter $e \in E$, $\tilde{F}_\mu(e) = (F(e), \mu(e))$ indicates not only the degree of belongingness of the elements of X in $F(e)$ but also the degree of possibility of such belongingness which is represented by $\mu(e)$.

Definition 2.6 [4] Let $\tilde{F}_\mu$ and $\tilde{G}_\delta$ be two $GFSS$ over (X, E). Now $\tilde{F}_\mu$ is said to be a GFS subset of $\tilde{G}_\delta$ or $\tilde{G}_\delta$ is said to be a GFS super set of $\tilde{F}_\mu$ if

(i) μ is a fuzzy subset of δ;
(ii) $F(e)$ is also a fuzzy subset of $G(e)$, $\forall e \in E$.

In this case we write $\tilde{F}_\mu \sqsubseteq \tilde{G}_\delta$.

Definition 2.7 [4] Let $\tilde{F}_\mu$ be a $GFSS$ over (X, E). Then the complement of $\tilde{F}_\mu$, is denoted by $\tilde{F}_\mu{}^c$ and is defined by $\tilde{F}_\mu{}^c = \tilde{G}_\delta$, where $\delta(e) = \mu^c(e)$ and $G(e) = F^c(e), \forall e \in E$.
Obviously $(\tilde{F}_\mu{}^c)^c = \tilde{F}_\mu$.

Definition 2.8 [1] Union of two $GFSS$ $\tilde{F}_\mu$ and $\tilde{G}_\delta$, denoted by $\tilde{F}_\mu \sqcup \tilde{G}_\delta$, is a $GFSS\, \tilde{H}_\nu$, defined as $\tilde{H}_\nu : E \to I^X \times I$ such that $\tilde{H}_\nu(e) = (H(e), \nu(e))$, where $H(e) = F(e) \vee G(e)$ and $\nu(e) = \mu(e) \vee \delta(e)$, $\forall e \in E$.

Let $\{(\tilde{F}_\mu)_\lambda, \lambda \in \Lambda\}$, where Λ is an index set, be a family of $GFSSs$. The union of these family is denoted by $\tilde{\bigsqcup}_{\lambda\in\Lambda} (\tilde{F}_\mu)_\lambda$, is a $GFSS\, \tilde{H}_\nu$, defined as $\tilde{H}_\nu : E \to I^X \times I$ such that $\tilde{H}_\nu(e) = (H(e), \nu(e))$, where $H(e) = \bigvee_{\lambda\in\Lambda} (F(e))_\lambda$ and $\nu(e) = \bigvee_{\lambda\in\Lambda} (\mu(e))_\lambda, \quad \forall e \in E$.

Definition 2.9 [1] Intersection of two $GFSS$ $\tilde{F}_\mu$ and $\tilde{G}_\delta$, denoted by $\tilde{F}_\mu \sqcap \tilde{G}_\delta$, is a $GFSS\, \tilde{M}_\sigma$, defined as $\tilde{M}_\sigma : E \to I^X \times I$ such that $\tilde{M}_\sigma(e) = (M(e), \sigma(e))$, where $M(e) = F(e) \wedge G(e)$ and $\sigma(e) = \mu(e) \wedge \delta(e)$, $\forall e \in E$.

Definition 2.10 [4] A $GFSS$ is said to be a generalised null fuzzy soft set, denoted by $\tilde{\Phi}_\theta$, if $\tilde{\Phi}_\theta : E \to I^X \times I$ such that $\tilde{\Phi}_\theta(e) = (F(e), \theta(e))$, where $F(e) = \overline{0}\ \forall e \in E$ and $\theta(e) = 0\ \forall e \in E$ (where $\overline{0}$ denotes the null fuzzy set).

Definition 2.11 [4] A $GFSS$ is said to be a generalised absolute fuzzy soft set, denoted by $\tilde{1}_\Delta$, if $\tilde{1}_\Delta : E \to I^X \times I$ such that $\tilde{1}_\Delta(e) = (1(e), \Delta(e))$, where $1(e) = \overline{1}\ \forall e \in E$ and $\Delta(e) = 1\ \forall e \in E$ (where $\overline{1}(x) = 1\ \forall x \in X$).

Proposition 2.12 [1] Let $\tilde{F}_\mu$, $\tilde{G}_\delta$ and $\tilde{H}_\nu$ be any three $GFSS$ over (X, E), then the following holds:
(1) $\tilde{F}_\mu \sqcap (\tilde{G}_\delta \sqcup \tilde{H}_\nu) = (\tilde{F}_\mu \sqcap \tilde{G}_\delta) \sqcup (\tilde{F}_\mu \sqcap \tilde{H}_\nu)$;
(2) $\tilde{F}_\mu \sqcup (\tilde{G}_\delta \sqcap \tilde{H}_\nu) = (\tilde{F}_\mu \sqcup \tilde{G}_\delta) \sqcap (\tilde{F}_\mu \sqcup \tilde{H}_\nu)$.

Proposition 2.13 [1] Let $\tilde{F}_\mu$ and $\tilde{G}_\delta$ are two $GFSS$ over (X, E). Then the following holds:
(1) $(\tilde{F}_\mu \sqcap \tilde{G}_\delta)^c = \tilde{F}^c_\mu \sqcup \tilde{G}^c_\delta$;
(2) $(\tilde{F}_\mu \sqcup \tilde{G}_\delta)^c = \tilde{F}^c_\mu \sqcap \tilde{G}^c_\delta$.

Definition 2.14 [1] Let T be a collection of generalised fuzzy soft sets over (X, E). Then T is said to be a generalised fuzzy soft topology(GFS topology, in short) over (X, E) if the following conditions are satisfied:
(i) $\tilde{\Phi}_\theta$ and $\tilde{1}_\Delta$ are in T;
(ii) Arbitrary unions of members of T belong to T;
(iii) Finite intersections of members of T belong to T.

The triplet (X, T, E) is called a generalised fuzzy soft topological space($GFST$-space, in short) over (X, E).

Definition 2.15 [1] Let (X, T, E) be a $GFST$-space over (X, E), then the members of T are said to be a GFS open sets in (X, T, E).

Definition 2.16 [1] Let (X, T, E) be a $GFST$-space over (X, E). A $GFSS$ $\tilde{F}_\mu$ over (X, E) is said to be a GFS closed in (X, T, E), if its complement $\tilde{F}^c_\mu$ belongs to T.

Definition 2.17 [1] Let (X, T, E) be a $GFST$-space and $\tilde{F}_\mu$ be a $GFSS$ over (X, E). Then the generalised fuzzy soft closure of $\tilde{F}_\mu$, denoted by $\overline{\tilde{F}_\mu}$, is the intersection of all GFS closed supper sets of $\tilde{F}_\mu$.

Clearly, $\overline{\tilde{F}_\mu}$ is the smallest GFS closed set over (X, E) which contains $\tilde{F}_\mu$.

Definition 2.18 [1] Let $\tilde{F}_\mu$ be a $GFSS$ over (X, E). We say that $(e^\alpha_x, e_\lambda)\tilde{\in}\tilde{F}_\mu$ read as (e_x^α, e_λ) belongs to the $GFSS$ $\tilde{F}_\mu$ if $F(e)(x) = \alpha\,(0 < \alpha \leq 1)$ and $F(e)(y) = 0$, $\forall y \in X \setminus \{x\}$, $\mu(e) > \lambda$.

Definition 2.19 [1] A $GFSS$ $\tilde{F}_\mu$ in a $GFST$-space (X, T, E) is called a generalised fuzzy soft neighbourhood of the generalised fuzzy soft point $(e_x^\alpha, e_\lambda)\tilde{\in}\tilde{1}_\Delta$ if there exists a GFS open set $\tilde{G}_\delta$ such that $(e_x^\alpha, e_\lambda)\tilde{\in}\tilde{G}_\delta \sqsubseteq \tilde{F}_\mu$.

Definition 2.20 [1] Let (X, T, E) be a $GFST$-space. Let $\tilde{F}_\mu$ be a $GFSS$ over (X, E). The generalised fuzzy soft interior of $\tilde{F}_\mu$, denoted by $\tilde{F}_\mu^0$, is the union of all GFS open subsets of $\tilde{F}_\mu$.

Clearly, $\tilde{F}_\mu^0$ is the largest GFS open set over (X, E) which contained in $\tilde{F}_\mu$.

Theorem 2.21 [1] Let (X, T, E) be a $GFST$-space. Let $\tilde{F}_\mu$ and $\tilde{G}_\delta$ are $GFSS$ over (X, E). Then

(1) $(\tilde{\Phi}_\theta)^0 = \tilde{\Phi}_\theta$, $(\tilde{1}_\Delta)^0 = \tilde{1}_\Delta$;
(2) $(\tilde{F}_\mu)^0 \sqsubseteq \tilde{F}_\mu$;
(3) $\tilde{F}_\mu$ is GFS open if and only if $(\tilde{F}_\mu)^0 = \tilde{F}_\mu$;
(4) $(\tilde{F}_\mu^0)^0 = (\tilde{F}_\mu)^0$;
(5) $\tilde{F}_\mu \sqsubseteq \tilde{G}_\delta \Rightarrow (\tilde{F}_\mu)^0 \sqsubseteq (\tilde{G}_\delta)^0$;
(6) $(\tilde{F}_\mu \sqcap \tilde{G}_\delta)^0 = \tilde{F}_\mu^0 \sqcap \tilde{G}_\delta^0$;
(7) $\tilde{F}_\mu^0 \sqcup \tilde{G}_\delta^0 \sqsubseteq (\tilde{F}_\mu \sqcup \tilde{G}_\delta)^0$.

Theorem 2.22 [1] Let (X, T, E) be a $GFST$-space and $\tilde{F}_\mu$ be a $GFSS$ over (X, E). Then

(1) $(\overline{\tilde{F}_\mu})^c = (\tilde{F}_\mu^c)^0$;
(2) $(\tilde{F}_\mu^0)^c = \overline{(\tilde{F}_\mu^c)}$;
(3) $(\tilde{F}_\mu)^0 = (\overline{\tilde{F}_\mu^c})^c$.

3 Generalised Fuzzy Soft Exterior, Generalised Fuzzy Soft Boundary

In this section the concept of difference of two generalised fuzzy soft sets, generalised fuzzy soft exterior and generalised fuzzy soft boundary are introduced and some of its basic properties are studied.

Definition 3.1 Difference of two $GFSS$ $\tilde{F}_\mu$ and $\tilde{G}_\delta$, denoted by $\tilde{F}_\mu \setminus \tilde{G}_\delta$, is a $GFSS$ $\tilde{H}_\nu = \tilde{F}_\mu \sqcap \tilde{G}_\delta^c$, defined as $H(e) = F(e) \wedge G^c(e)$ and $\nu(e) = \mu(e) \wedge \delta^c(e)$, $\forall e \in E$.

Example 3.2 Let $X = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$

Let us consider the following $GFSS$ over (X, E).

$\tilde{F}_\mu = \{F_\mu(e_1) = (\{x_1/0.5, x_2/0.4, x_3/0.3\}, 0.6), F_\mu(e_2) = (\{x_1/0.7, x_2/0.2, x_3/0.3\}, 0.7)\}$
$\tilde{G}_\delta = \{G_\delta(e_1) = (\{x_1/0.3, x_2/0.5, x_3/0.4\}, 0.7), G_\delta(e_2) = (\{x_1/0.6, x_2/0.4, x_3/0.1\}, 0.4)\}$
$\tilde{G}_\delta^c = \{{G_\delta}^c(e_1) = (\{x_1/0.7, x_2/0.5, x_3/0.6\}, 0.3), {G_\delta}^c(e_2) = (\{x_1/0.4, x_2/0.6, x_3/0.9\}, 0.6)\}$
$\tilde{F}_\mu \backslash \tilde{G}_\delta = \tilde{F}_\mu \sqcap \tilde{G}_\delta^c = \tilde{H}_\nu = \{H_\nu(e_1) = (\{x_1/0.5, x_2/0.4, x_3/0.3\}, 0.3), H_\nu(e_2) = (\{x_1/0.4, x_2/0.2, x_3/0.3\}$

Definition 3.3 Let (X, T, E) be a $GFST$-space. Let $\tilde{F}_\mu$ be a $GFSS$ over (X, E). The generalised fuzzy soft exterior of $\tilde{F}_\mu$, denoted by $ext(\tilde{F}_\mu)$, is defined as $ext(\tilde{F}_\mu) = (\tilde{F}_\mu^c)^0$.

Thus (e_x^α, e_λ) is called a generalised fuzzy soft exterior point of $\tilde{F}_\mu$ if there exists a GFS open set $\tilde{G}_\delta$ such that $(e_x^\alpha, e_\lambda)\tilde{\in}\tilde{G}_\delta \sqsubseteq (\tilde{F}_\mu)^c$.

Clearly, $ext(\tilde{F}_\mu)$ is the largest GFS open set contained in $(\tilde{F}_\mu)^c$.

Example 3.4 Let $X = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$
Let us consider the following $GFSS$ over (X, E).
$\tilde{F}_\mu = \{F_\mu(e_1) = (\{x_1/0.5, x_2/0.4, x_3/0.3\}, 0.6), F_\mu(e_2) = (\{x_1/0.7, x_2/0.2, x_3/0.3\}, 0.7)\}$
$\tilde{G}_\delta = \{G_\delta(e_1) = (\{x_1/0.3, x_2/0.5, x_3/0.4\}, 0.7), G_\delta(e_2) = (\{x_1/0.6, x_2/0.4, x_3/0.1\}, 0.4)\}$
$\tilde{H}_\nu = \{H_\nu(e_1) = (\{x_1/0.5, x_2/0.5, x_3/0.4\}, 0.7), H_\nu(e_2) = (\{x_1/0.7, x_2/0.4, x_3/0.3\}, 0.7)\}$
$\tilde{J}_\sigma = \{J_\sigma(e_1) = (\{x_1/0.3, x_2/0.4, x_3/0.3\}, 0.6), J_\sigma(e_2) = (\{x_1/0.6, x_2/0.2, x_3/0.1\}, 0.4)\}$
Let us consider the $GFST$ $T = \{\tilde{\Phi}_\theta, \tilde{1}_\Delta, \tilde{F}_\mu, \tilde{G}_\delta, \tilde{H}_\nu, \tilde{J}_\sigma\}$ over (X, E).
Let us consider the following $GFSS$ over (X, E).
$\tilde{M}_\eta = \{M_\eta(e_1) = (\{x_1/0.6, x_2/0.5, x_3/0.4\}, 0.2), M_\eta(e_2) = (\{x_1/0.2, x_2/0.4, x_3/0.6\}, 0.5)\}$
Then $\tilde{M}^c_\eta = \{{M_\eta}^c(e_1) = (\{x_1/0.4, x_2/0.5, x_3/0.6\}, 0.8), {M_\eta}^c(e_2) = (\{x_1/0.8, x_2/0.6, x_3/0.4\}, 0.5)\}$
Then $(\tilde{M}^c_\eta)^0$, is the union of all fuzzy soft open sets contained in $\tilde{M}^c_\eta$.
That is $ext(\tilde{M}_\eta) = (\tilde{M}^c_\eta)^0 = \tilde{\Phi}_\theta \sqcup \tilde{G}_\delta \sqcup \tilde{J}_\sigma = \tilde{G}_\delta$.

Theorem 3.5 Let (X, T, E) be a $GFST$-space. Let $\tilde{F}_\mu$ and $\tilde{G}_\delta$ are $GFSS$ over (X, E). Then

(1) $ext(\tilde{\Phi}_\theta) = \tilde{1}_\Delta$, $ext(\tilde{1}_\Delta) = \tilde{\Phi}_\theta$;
(2) $\tilde{F}_\mu \sqsubseteq \tilde{G}_\delta \Rightarrow ext(\tilde{G}_\delta) \sqsubseteq ext(\tilde{F}_\mu)$;
(3) $(\tilde{F}_\mu)^0 \sqsubseteq ext(ext(\tilde{F}_\mu))$;
(4) $ext(\tilde{F}_\mu) = (\tilde{F}^c_\mu)^0$;
(5) $ext(\tilde{F}_\mu \sqcup \tilde{G}_\delta) = ext(\tilde{F}_\mu) \sqcap ext(\tilde{G}_\delta)$;
(6) $ext(\tilde{F}_\mu) \sqcup ext(\tilde{G}_\delta) \sqsubseteq ext(\tilde{F}_\mu \sqcap \tilde{G}_\delta)$.

proof: (1) Obvious.
(2) Let $\tilde{F}_\mu \sqsubseteq \tilde{G}_\delta \Rightarrow (\tilde{G}_\delta)^c \sqsubseteq (\tilde{F}_\mu)^c \Rightarrow (\tilde{G}^c_\delta)^0 \sqsubseteq (\tilde{F}^c_\mu)^0$ (by theorem 2.21(5)). This implies $ext(\tilde{G}_\delta) \sqsubseteq ext(\tilde{F}_\mu)$.
(3) Since $ext(\tilde{F}_\mu) = (\tilde{F}^c_\mu)^0 \sqsubseteq \tilde{F}^c_\mu$ (by theorem 2.21(2))
By (2), $ext(\tilde{F}^c_\mu) \sqsubseteq ext(ext(\tilde{F}_\mu))$. But $(\tilde{F}_\mu)^0 = ext(\tilde{F}^c_\mu)$.
Hence $(\tilde{F}_\mu)^0 \sqsubseteq ext(ext(\tilde{F}_\mu))$.
(4) It follows from the respective definitions.
(5) $ext(\tilde{F}_\mu \sqcup \tilde{G}_\delta) = ((\tilde{F}_\mu \sqcup \tilde{G}_\delta)^c)^0 = (\tilde{F}^c_\mu \sqcap \tilde{G}^c_\delta)^0$, by proposition 2.13
$= (\tilde{F}^c_\mu)^0 \sqcap (\tilde{G}^c_\delta)^0$, by theorem 2.21(6)
$= ext(\tilde{F}_\mu) \sqcap ext(\tilde{G}_\delta)$.
(6) $ext(\tilde{F}_\mu) \sqcup ext(\tilde{G}_\delta) = (\tilde{F}^c_\mu)^0 \sqcup (\tilde{G}^c_\delta)^0 \sqsubseteq (\tilde{F}^c_\mu \sqcup \tilde{G}^c_\delta)^0$, by theorem 2.21(7)
$= ((\tilde{F}_\mu \sqcap \tilde{G}_\delta)^c)^0$, by proposition 2.13
$= ext(\tilde{F}_\mu \sqcap \tilde{G}_\delta)$.

Definition 3.6 Let (X, T, E) be a $GFST$-space. Let $\tilde{F}_\mu$ be a $GFSS$ over (X, E). The generalised fuzzy soft boundary of $\tilde{F}_\mu$, denoted by $(\tilde{F}_\mu)^b$, is defined as
$(\tilde{F}_\mu)^b = \overline{\tilde{F}_\mu} \sqcap \overline{\tilde{F}^c_\mu}$.

Clearly, $(\tilde{F}_\mu)^b$ is the smallest GFS closed set over (X, E) which contains $\tilde{F}_\mu$.

Remark 3.7 It follows from the above definition that the $GFSS$ $\tilde{F}_\mu$ and $\tilde{F}^c_\mu$ will have same generalised fuzzy soft boundary.

Example 3.8 Let $X = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$
Let us consider the following $GFSS$ over (X, E).
$\tilde{F}_\mu = \{F_\mu(e_1) = (\{x_1/0.7, x_2/0.3, x_3/0.2\}, 0.4), F_\mu(e_2) = (\{x_1/0.6, x_2/0.1, x_3/0.4\}, 0.5)\}$
$\tilde{G}_\delta = \{G_\delta(e_1) = (\{x_1/0.6, x_2/0.4, x_3/0.5\}, 0.5), G_\delta(e_2) = (\{x_1/0.5, x_2/0.3, x_3/0.1\}, 0.3)\}$
$\tilde{H}_\nu = \{H_\nu(e_1) = (\{x_1/0.7, x_2/0.4, x_3/0.5\}, 0.5), H_\nu(e_2) = (\{x_1/0.6, x_2/0.3, x_3/0.4\}, 0.5)\}$
$\tilde{J}_\sigma = \{J_\sigma(e_1) = (\{x_1/0.6, x_2/0.3, x_3/0.2\}, 0.4), J_\sigma(e_2) = (\{x_1/0.5, x_2/0.1, x_3/0.1\}, 0.3)\}$

Let us consider the $GFST$ $T = \{\tilde{\Phi}_\theta, \tilde{1}_\Delta, \tilde{F}_\mu, \tilde{G}_\delta, \tilde{H}_\nu, \tilde{J}_\sigma\}$ over (X, E).
Now, $\tilde{F}_\mu^c = \{{F_\mu}^c(e_1) = (\{x_1/0.3, x_2/0.7, x_3/0.8\}, 0.6), {F_\mu}^c(e_2) = (\{x_1/0.4, x_2/0.9, x_3/0.6\}, 0.5)\}$
$\tilde{G}_\delta^c = \{{G_\delta}^c(e_1) = (\{x_1/0.4, x_2/0.6, x_3/0.5\}, 0.5), {G_\delta}^c(e_2) = (\{x_1/0.5, x_2/0.7, x_3/0.9\}, 0.7)\}$
$\tilde{H}_\nu^c = \{{H_\nu}^c(e_1) = (\{x_1/0.3, x_2/0.6, x_3/0.5\}, 0.5), {H_\nu}^c(e_2) = (\{x_1/0.4, x_2/0.7, x_3/0.6\}, 0.5)\}$
$\tilde{J}_\sigma^c = \{{J_\sigma}^c(e_1) = (\{x_1/0.4, x_2/0.7, x_3/0.8\}, 0.6), {J_\sigma}^c(e_2) = (\{x_1/0.5, x_2/0.9, x_3/0.9\}, 0.7)\}$
Clearly, $\tilde{F}_\mu^c, \tilde{G}_\delta^c, \tilde{H}_\nu^c, \tilde{J}_\sigma^c$ are GFS closed sets.
Let us consider the following $GFSS$ over (X, E).
$\tilde{M}_\eta = \{M_\eta(e_1) = (\{x_1/0.6, x_2/0.5, x_3/0.6\}, 0.4), M_\eta(e_2) = (\{x_1/0.7, x_2/0.4, x_3/0.3\}, 0.7)\}$
$\tilde{M}_\eta^c = \{{M_\eta}^c(e_1) = (\{x_1/0.4, x_2/0.5, x_3/0.4\}, 0.6), {M_\eta}^c(e_2) = (\{x_1/0.3, x_2/0.6, x_3/0.7\}, 0.3)\}$
Then the GFS closure of $\tilde{M}_\eta$, denoted by $\overline{\tilde{M}_\eta}$, is the intersection of all GFS closed sets containing $\tilde{M}_\eta$.
That is, $\overline{\tilde{M}_\eta} = \tilde{1}_\Delta$.
Again, the GFS closure of $\tilde{M}_\eta^c$, denoted by $\overline{\tilde{M}_\eta^c}$, is the intersection of all GFS closed sets containing $\tilde{M}_\eta^c$.
That is, $\overline{\tilde{M}_\eta^c} = \tilde{J}_\sigma^c \sqcap \tilde{1}_\Delta = \tilde{J}_\sigma^c$.
Then the generalised fuzzy soft boundary of $\tilde{M}_\eta$, denoted by $(\tilde{M}_\eta)^b$ is given by
$(\tilde{M}_\eta)^b = \overline{\tilde{M}_\eta} \sqcap \overline{\tilde{M}_\eta^c} = \tilde{1}_\Delta \sqcap \tilde{J}_\sigma^c = \tilde{J}_\sigma^c$.

Theorem 3.9 Let (X, T, E) be a $GFST$-space. Let $\tilde{F}_\mu$ be a $GFSS$ over (X, E). Then

(1) $((\tilde{F}_\mu)^b)^c = \tilde{F}_\mu^0 \sqcup (\tilde{F}_\mu^c)^0 = \tilde{F}_\mu^0 \sqcup ext(\tilde{F}_\mu)$;

(2) $(\tilde{F}_\mu)^b = \overline{\tilde{F}_\mu} \sqcap \overline{\tilde{F}_\mu^c} = \overline{\tilde{F}_\mu} \setminus \tilde{F}_\mu^0$.

proof: (1) $\tilde{F}_\mu^0 \sqcup (\tilde{F}_\mu^c)^0 = ((\tilde{F}_\mu^0)^c)^c \sqcup (((\tilde{F}_\mu^c)^0)^c)^c = [(\tilde{F}_\mu^0)^c) \sqcap ((\tilde{F}_\mu^c)^0)^c]^c$, {by proposition 2.13.} $= [\overline{\tilde{F}_\mu^c} \sqcap \overline{\tilde{F}_\mu}]^c = ((\tilde{F}_\mu)^b)^c$.
(2) $(\tilde{F}_\mu)^b = \overline{\tilde{F}_\mu} \setminus \tilde{F}_\mu^0 = \overline{\tilde{F}_\mu} \sqcap (\tilde{F}_\mu^0)^c = \overline{\tilde{F}_\mu} \sqcap \overline{\tilde{F}_\mu^c}$.

Theorem 3.10 Let (X, T, E) be a $GFST$-space. Let $\tilde{F}_\mu$ be a $GFSS$ over (X, E). Then

(1) $(\tilde{F}_\mu)^b \sqsubseteq \overline{\tilde{F}_\mu}$;

(2) $(\tilde{F}_\mu)^b = \overline{\tilde{F}_\mu} \setminus \tilde{F}_\mu^0$.

proof: (1) It follows from the definition of generalised fuzzy soft boundary.
(2) Obvious.

Theorem 3.11 Let (X, T, E) be a $GFST$-space. Let $\tilde{F}_\mu$ and $\tilde{G}_\delta$ are $GFSS$ over (X, E). Then

(1) $(\tilde{F}_\mu \sqcup \tilde{G}_\delta)^b \sqsubseteq ((\tilde{F}_\mu)^b \sqcap \overline{\tilde{G}_\delta^c}) \sqcup [(\tilde{G}_\delta)^b \sqcap \overline{\tilde{F}_\mu^c}]$;

(2) $(\tilde{F}_\mu \sqcap \tilde{G}_\delta)^b \sqsubseteq [(\tilde{F}_\mu)^b \sqcap \overline{\tilde{G}_\delta}] \sqcup [(\tilde{G}_\delta)^b \sqcap \overline{\tilde{F}_\mu}]$.

proof: (1) $(\tilde{F}_\mu \sqcup \tilde{G}_\delta)^b = \overline{(\tilde{F}_\mu \sqcup \tilde{G}_\delta)} \sqcap \overline{(\tilde{F}_\mu \sqcup \tilde{G}_\delta)^c} = (\overline{\tilde{F}_\mu} \sqcup \overline{\tilde{G}_\delta}) \sqcap \overline{[\tilde{F}_\mu^c \sqcap \tilde{G}_\delta^c]}$
$\sqsubseteq (\overline{\tilde{F}_\mu} \sqcup \overline{\tilde{G}_\delta}) \sqcap [\overline{\tilde{F}_\mu^c} \sqcap \overline{\tilde{G}_\delta^c}]$
$= [\overline{\tilde{F}_\mu} \sqcap \overline{\tilde{F}_\mu^c} \sqcap \overline{\tilde{G}_\delta^c}] \sqcup [\overline{\tilde{G}_\delta} \sqcap \overline{\tilde{F}_\mu^c} \sqcap \overline{\tilde{G}_\delta^c}]$
$= [(\overline{\tilde{F}_\mu} \sqcap \overline{\tilde{F}_\mu^c}) \sqcap \overline{\tilde{G}_\delta^c}] \sqcup [(\overline{\tilde{G}_\delta} \sqcap \overline{\tilde{G}_\delta^c}) \sqcap \overline{\tilde{F}_\mu^c}]$
$= ((\tilde{F}_\mu)^b \sqcap \overline{\tilde{G}_\delta^c}) \sqcup [(\tilde{G}_\delta)^b \sqcap \overline{\tilde{F}_\mu^c}]$.
(2) $(\tilde{F}_\mu \sqcap \tilde{G}_\delta)^b = \overline{(\tilde{F}_\mu \sqcap \tilde{G}_\delta)} \sqcap \overline{(\tilde{F}_\mu \sqcap \tilde{G}_\delta)^c}$
$\sqsubseteq (\overline{\tilde{F}_\mu} \sqcap \overline{\tilde{G}_\delta}) \sqcap [\overline{\tilde{F}_\mu^c} \sqcup \overline{\tilde{G}_\delta^c}]$

$= [\overline{\tilde{F}_\mu} \sqcap \overline{\tilde{G}_\delta} \sqcap \overline{\tilde{F}^c_\mu}] \sqcup [\overline{\tilde{F}_\mu} \sqcap \overline{\tilde{G}_\delta} \sqcap \overline{\tilde{G}^c_\delta}]$
$= [(\tilde{F}_\mu)^b \sqcap \overline{\tilde{G}_\delta}] \sqcup [(\tilde{G}_\delta)^b \sqcap \overline{\tilde{F}_\mu}]$.

Theorem 3.12 Let (X, T, E) be a $GFST$-space. Let $\tilde{F}_\mu$ be a $GFSS$ over (X, E). Then

$(\tilde{F}^0_\mu)^b \sqsubseteq (\tilde{F}_\mu)^b$.

proof: $(\tilde{F}^0_\mu)^b = \overline{(\tilde{F}_\mu)^0} \sqcap \overline{((\tilde{F}_\mu)^0)^c} \sqsubseteq \overline{(\tilde{F}_\mu)^0} \sqcap \overline{(\tilde{F}_\mu)^c} \sqsubseteq \overline{\tilde{F}_\mu} \sqcap \overline{(\tilde{F}_\mu)^c} = (\tilde{F}_\mu)^b$.

4 Generalised Soft Quasi-Coincidence

In this section, we introduce the notion of generalised soft quasi-coincidence for generalised fuzzy soft set and some of its basic properties are established.

Definition 4.1 For any two $GFSS$ $\tilde{F}_\mu$ and $\tilde{G}_\delta$ over (X, E). $\tilde{F}_\mu$ is said to be generalised soft quasi-coincident with $\tilde{G}_\delta$, denoted by $\tilde{F}_\mu \tilde{q} \tilde{G}_\delta$, if there exist $e \in E$ and $x \in X$ such that $F(e)(x) + G(e)(x) > 1$ and $\mu(e) + \delta(e) > 1$.

If $\tilde{F}_\mu$ is not generalised soft quasi-coincident with $\tilde{G}_\delta$, then we write $\tilde{F}_\mu \overline{\tilde{q}} \tilde{G}_\delta \Leftrightarrow$ For every $\mathrm{e} \in \mathrm{E}$ and every $\mathrm{x} \in \mathrm{X}$, $\mathrm{F(e)(x) + G(e)(x)} \leq 1$ or for every $\mathrm{e} \in \mathrm{E}$ and every $\mathrm{x} \in \mathrm{X}$, $\mu(\mathrm{e}) + \delta(\mathrm{e}) \leq 1$.

Definition 4.2 Let (e^α_x, e_λ) be a generalised fuzzy soft point and $\tilde{F}_\mu$ be a $GFSS$ over (X, E). (e^α_x, e_λ) is said to be generalised soft quasi-coincident with $\tilde{F}_\mu$, denoted by $(e^\alpha_x, e_\lambda)\tilde{q}\tilde{F}_\mu$, if and only if there exists an $e \in E$ such that $\alpha + F(e)(x) > 1$ and $\lambda + \mu(e) > 1$.

Proposition 4.3 Let $\tilde{F}_\mu$ and $\tilde{G}_\delta$ are $GFSS$ over (X, E). Then the followings are holds:

(1) $\tilde{F}_\mu \sqsubseteq \tilde{G}_\delta \Leftrightarrow \tilde{F}_\mu \overline{\tilde{q}} (\tilde{G}_\delta)^c$;
(2) $\tilde{F}_\mu \tilde{q} \tilde{G}_\delta \Rightarrow \tilde{F}_\mu \sqcap \tilde{G}_\delta \neq \tilde{\Phi}_\theta$;
(3) $(e^\alpha_x, e_\lambda)\overline{\tilde{q}}\tilde{F}_\mu \Leftrightarrow (e^\alpha_x, e_\lambda)\tilde{\in}(\tilde{F}_\mu)^c$;
(4) $\tilde{F}_\mu \overline{\tilde{q}} (\tilde{F}_\mu)^c$.

proof: (1) $\tilde{F}_\mu \sqsubseteq \tilde{G}_\delta \Leftrightarrow$ for all $\mathrm{e} \in \mathrm{E}$ and all $\mathrm{x} \in \mathrm{X}$, $\mathrm{F(e)(x) \leq G(e)(x)}, \mu(\mathrm{e}) \leq \delta(\mathrm{e})$
$\Leftrightarrow$ for all $\mathrm{e} \in \mathrm{E}$ and all $\mathrm{x} \in \mathrm{X}$, $\mathrm{F(e)(x) - G(e)(x)} \leq 0$, $\mu(\mathrm{e}) - \delta(\mathrm{e}) \leq 0$
$\Leftrightarrow$ for all $\mathrm{e} \in \mathrm{E}$ and all $\mathrm{x} \in \mathrm{X}$, $\mathrm{F(e)(x)}+1-\mathrm{G(e)(x)} \leq 1$, $\mu(\mathrm{e})+1-\delta(\mathrm{e}) \leq 1$
$\Leftrightarrow \tilde{F}_\mu \overline{\tilde{q}} (\tilde{G}_\delta)^c$.

(2) Let $\tilde{F}_\mu \tilde{q} \tilde{G}_\delta$. Then there exist an $e \in E$ and $x \in X$ such that $F(e)(x) + G(e)(x) > 1$ and $\mu(e) + \delta(e) > 1$. This implies that $F(e)(x) \neq \overline{0}, \mu(e) \neq 0$ and $G(e)(x) \neq \overline{0}, \delta(e) \neq 0$ for $e \in E$ and $x \in X$. Hence $\tilde{F}_\mu \sqcap \tilde{G}_\delta \neq \tilde{\Phi}_\theta$.

(3) $(e^\alpha_x, e_\lambda)\overline{\tilde{q}}\tilde{F}_\mu \Leftrightarrow$ for all $\mathrm{e} \in \mathrm{E}$ and $\mathrm{x} \in \mathrm{X}$, $\alpha + \mathrm{F(e)(x)} \leq 1, \lambda + \mu(\mathrm{e}) \leq 1$
$\Leftrightarrow$ for all $\mathrm{e} \in \mathrm{E}$ and $\mathrm{x} \in \mathrm{X}$, $\alpha \leq 1 - \mathrm{F(e)(x)}, \lambda \leq 1 - \mu(\mathrm{e})$
$\Leftrightarrow$ for all $\mathrm{e} \in \mathrm{E}$ and $\mathrm{x} \in \mathrm{X}$, $\alpha \leq \mathrm{F^c(e)(x)}, \lambda \leq \mu^c(\mathrm{e})$
$\Leftrightarrow (e^\alpha_x, e_\lambda)\tilde{\in}(\tilde{F}_\mu)^c$.

(4) Suppose that $\tilde{F}_\mu \tilde{q} (\tilde{F}_\mu)^c$. Then there exist $e \in E$ and $x \in X$ such that $F(e)(x) + F^c(e)(x) > 1, \mu(e) + \mu^c(e) > 1$. Then $F(e)(x) + 1 - F(e)(x) > 1, \mu(e) + 1 - \mu(e) > 1$. So, $F(e)(x) > F(e)(x), \mu(e) > \mu(e)$, which is contradiction.

Theorem 4.4 Let (X, T, E) be a $GFST$-space. Let $\tilde{F}_\mu$ be a $GFSS$ over (X, E). Then

(1) $(\tilde{F}_\mu)^b \overline{\tilde{q}} (\tilde{F}_\mu)^0$;
(2) $(\tilde{F}_\mu)^b \overline{\tilde{q}} ext(\tilde{F}_\mu)$.

proof: Straightforward.

Theorem 4.5 Let (X, T, E) be a $GFST$-space. Let $\tilde{F}_\mu$ be a $GFSS$ over (X, E). Then

(1) $\tilde{F}_\mu$ is a GFS open set over (X, E) if and only if $\tilde{F}_\mu \overline{\tilde{q}} (\tilde{F}_\mu)^b$;

(2) $\tilde{F}_\mu$ is a GFS closed set over (X, E) if and only if $(\tilde{F}_\mu)^b \overline{\tilde{q}} (\tilde{F}_\mu)^c$.

proof: (1) Let $\tilde{F}_\mu$ be a GFS open set over (X, E). Then $(\tilde{F}_\mu)^0 = \tilde{F}_\mu$
By theorem 4.4, $(\tilde{F}_\mu)^b \overline{\tilde{q}} (\tilde{F}_\mu)^0 = (\tilde{F}_\mu)^b \overline{\tilde{q}} \tilde{F}_\mu$

Conversely, let $\tilde{F}_\mu \overline{\tilde{q}} (\tilde{F}_\mu)^b$. Then $\tilde{F}_\mu \overline{\tilde{q}} (\overline{\tilde{F}_\mu} \sqcap \overline{(\tilde{F}_\mu)^c})$. That is $\tilde{F}_\mu \overline{\tilde{q}} \overline{(\tilde{F}_\mu)^c}$. So $\overline{(\tilde{F}_\mu)^c} \sqsubseteq (\tilde{F}_\mu)^c$ which implies that $(\tilde{F}_\mu)^c$ is a GFS closed set and hence $\tilde{F}_\mu$ is a GFS open set.

(2) Let $\tilde{F}_\mu$ be a GFS closed set over (X, E). Then $\overline{\tilde{F}_\mu} = \tilde{F}_\mu$. Now $(\tilde{F}_\mu)^b = \overline{\tilde{F}_\mu} \sqcap \overline{(\tilde{F}_\mu)^c} \sqsubseteq \overline{\tilde{F}_\mu} = \tilde{F}_\mu$. That is, $(\tilde{F}_\mu)^b \overline{\tilde{q}} (\tilde{F}_\mu)^c$.

Conversely, let $(\tilde{F}_\mu)^b \overline{\tilde{q}} (\tilde{F}_\mu)^c$. Since $(\tilde{F}_\mu)^b = (\tilde{F}_\mu^c)^b$. We have $(\tilde{F}_\mu^c)^b \overline{\tilde{q}} (\tilde{F}_\mu)^c$
Then by (1), $(\tilde{F}_\mu)^c$ is a GFS open set and hence $\tilde{F}_\mu$ is a GFS closed set (X, E).

5 Conclusion

In the present work, we have introduced difference of two generalised fuzzy soft sets, generalised fuzzy soft exterior, generalised fuzzy soft boundary and have established several interesting properties. Finally, we introduced the notion of generalised soft quasi-coincidence for generalised fuzzy soft sets and studied some basic properties of this concept. We hope that this study will be useful for research in theoretical as well as in a applicable directions.

References

[1] R. P. Chakraborty, P. Mukherjee, On generalised fuzzy soft topological spaces, Afr. J. Math. Comput. Sci. Res. 8 (2015) 1-11.

[2] R. P. Chakraborty, P. Mukherjee, P. K. Gain, A note on fuzzy soft semi open sets and fuzzy soft semi continuous functions, J. Fuzzy Math. 22 (2014) 973-989.

[3] P. K. Maji, R. Biswas, A. R. Roy, Fuzzy soft sets, J.Fuzzy Math 9(2001) 589-602.

[4] P.Majumdar, S.K.Samanta, Generalised fuzzy soft sets, Comput.Math.Appl 59(2010) 1425-1432.

[5] P.B.Ming, L.Y.Ming, Fuzzy topology I.Neighbourhood structure of a fuzzy point and Moore-Smith convergence, J.Math.Anal.Appl 76(1980) 571-599.

[6] D. Molodtsov, Soft set theory-First results, comput.Math.Appl 37(1999) 19-31.

[7] P.Mukherjee, R.P.Chakraborty, C.Park, Fuzzy soft $\theta-$ closure operator in fuzzy soft topological spaces, Electronic Journal of Mathematical Analysis and Applications 3(2015) 227-236.

[8] B. Tanay, M. Burc Kandemir, Topological structure of fuzzy soft sets, comput.Math.Appl 61(2011)2952-2957.

[9] H.L.Yang, notes on Generalised fuzzy soft sets, Journal of mathematical Research and exposition 31(2011) 567-570.

[10] L. A. Zadeh, Fuzzy sets, Inform and control 8(1965) 338-353.