Journal of Naval Science and Engineering
2015, Vol.11, No.1, pp.53-67

FINITE DIFFERENCE MODEL OF A CIRCULAR FIN WITH RECTANGULAR PROFILE

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Abstract

Numerical methods are commonly used in engineering where the analytical results are not reached or as a support of experimental studies. Various techniques are being used as a numeritical method as finite difference, finite volume or finite elements, etc. In this study, numerical solutions are obtained for a circular fin of rectangular profile using finite difference method, and the results are compared to the analytical solutions. It is seen that the analytical solution and numerical results are found to be compatible.

DİKDÖRTGEN KESİTLİ BİR DAİRESEL KANATÇIĞIN SONLU FARK MODELİ

Özetçe

Analitik çözümün mümkün olmadığı durumlarda veya deneysel çalışmalara destek olmak amacıyla sayısal yöntemler mühendislikte yaygın olarak kullanılmaktadır. Sayısal yöntemler olarak sonlu farklar, sonlu hacim, sonlu eleman metodları gibi çeşitli yöntemler kullanılmaktadır. Bu çalışmada sonlu fark yöntemi kullanılarak dikdörtgen kesitli dairesel bir kanatçık için sayısal çözüm elde edilmiş ve hesaplanan sonuçlar analitik çözümle karşılaştırılmıştır. Analitik ve sayısal sonuçların birbirleriyle oldukça uyumlu oldukları görülmüştür.

Keywords*: Circular Fin, Numerical Methods, Finite Difference Method.*
Anahtar Kelimeler: *Dairesel Kanatçık, Sayısal Yöntemler, Sonlu Fark Yöntemi.*

1. INTRODUCTION

Numerical analysis is the combination of mathematics and computer programming that creates and implements algorithms for solving the problems of continuous mathematics. These problems occur throughout the natural sciences, social sciences, engineering, and the other fields. The growth in power and availability of digital computers has led to an increasing use of numerical solution of the models in science and engineering.

Numerical methods are commonly used in engineering where the analytical results are not reached, or as a support of experimental studies. Various techniques are used to solve the tough partial differential equations which cannot be solved analytically. Most common used numerical method for solving a partial differential equation is the finite difference approach. In this study, finite difference method is used to get the numerical solution of heat transfer inside a circular fin. The temperature distribution inside a circular fin is governed by the general heat conduction equation. This equation is a three dimensional equation that has both a source term and a transient component. But a fin can be assumed steady if the base temperature, ambient fluid temperature and combined convection-radiation heat transfer coefficient are constant. Therefore, a one dimensional steady simplified conduction equation is used with no heat source.

2. FINITE DIFFERENCE METHOD

Finite-difference methods are numerical methods for solving differential equations by approximating them with difference equations. The derivatives are approximated by finite differences, so finite difference methods are discretization methods. Today, these methods are the most used approach in numerical solutions of partial differential equations [1]. The finite difference approach is based upon converting the differential equations to finite difference equations using the numerical expressions of the derivatives.

The error in an approximation is defined as the difference between the approximation and the exact analytical solution. The two sources of error in finite difference methods are round-off error and the discretization error. The round-off error is the loss of precision due to computer rounding of decimal quantities, where the discretization error is the difference between the exact solution of the finite difference equation and the exact quantity assuming perfect arithmetic.

The finite difference formulas for the first and second derivatives can be obtained from Taylor series expansion.

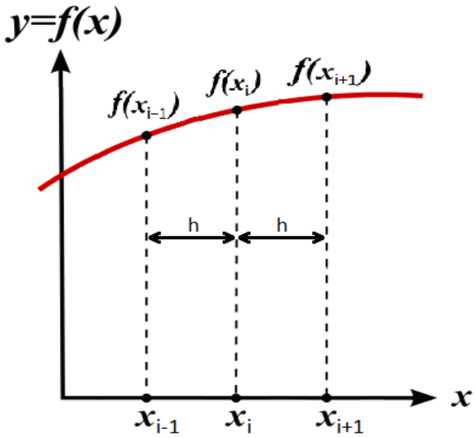

Figure 1 The function y=f(x)

The Taylor Series Expansion for the point x_{i+1} and x_{i-1} from the Figure 1:

$$f(x_{i+1}) = f(x_i) + h.f'(x_i) + \frac{1}{2!}h^2.f''(x_i) + \frac{1}{3!}h^3.f'''(x_i) + \ldots \quad (1)$$

$$f(x_{i-1}) = f(x_i) - h.f'(x_i) + \frac{1}{2!}h^2.f''(x_i) - \frac{1}{3!}h^3.f'''(x_i) + \ldots \quad (2)$$

if the first derivatives at x_i are expressed from the equations above:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{1}{2!}h.f''(x_i) - \frac{1}{3!}h^2.f'''(x_i) + \ldots \tag{3}$$

or

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + o(h) \tag{4}$$

and the second expression can be defined as:

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + \frac{1}{2!}h.f''(x_i) - \frac{1}{3!}h^2.f'''(x_i) + \ldots \tag{5}$$

or

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + o(h) \tag{6}$$

The *o(h)* term on the right hand side is the truncation error. The finite difference equations for the first derivative are called forward difference expression with error of order *h*:

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{h} \tag{7}$$

and backward difference expression with error of order h:

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{h} \tag{8}$$

If equation 2 is subtracted from equation 1, and the first derivative is derived from the result, the central difference equation for the first derivative with error order of h^2:

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_{i-1})}{2.h} \tag{9}$$

is obtained.

If the equations 1 and 2 are added together, the second derivative finite difference equation can be written as:

$$f''(x_i) \approx \frac{f(x_{i+1}) + f(x_{i-1}) - 2.f(x_i)}{h^2} \quad (10)$$

with error order of h^2.

The error order of h means if you decrease h to half, the error also be expected to decrease to half. But if the error is order of h^2, it means that if you decrease the h to half, the error is expected to decrease to $1/h^2$ times. Therefore, to use the expressions with error of high order should be preffered. But such expressions may be more complicated and they can increase the calculation time. The finite difference expressions with error order oh, h, h^2, h^4 can be found in the literature. In this study, it has been avoided using the finite difference expressions with error order of h, because it is needed much larger grid points to decrease the truncation error into the acceptable limits. Thus, the first order forward and backward difference expressions with error order of h^2:

$$f'(x_i) \approx \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} \quad (11)$$

$$f'(x_i) \approx \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h} \quad (12)$$

has been preferred to the equations 7 and 8 [2].

To use a finite difference method to find a solution to a problem, at first the problem's domain must be discretized. This is usually done by dividing the domain into a uniform grid (Fig.2).

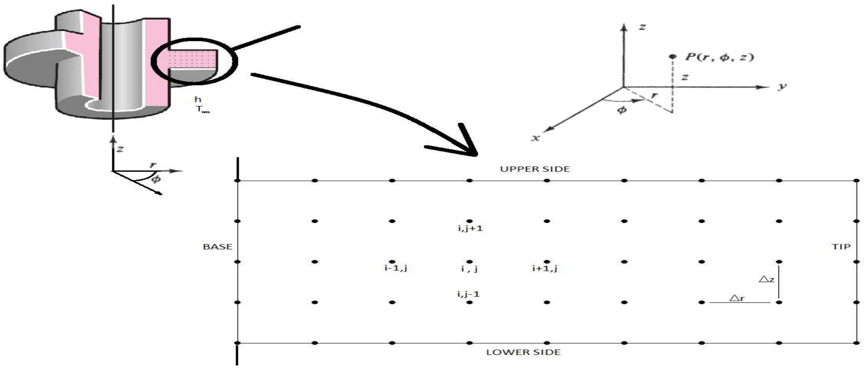

Figure 2 The discretized problem domain

The grid may be 1, 2 or 3 dimensional with respect to the nature of the problem. A 2-dimensional grid is used in this study because temperature change with respect to ϕ axis, $\frac{\partial T}{\partial \phi}$ is zero.

3. GEOMETRY

In thermal engineering, circular fins are widely used to enhance the heat transfer from the surfaces. Adding a circular fin to an object increases the amount of surface area in contact with the surrounding fluid, which increases the convective and radiative heat transfer between the object and surrounding fluid and the surfaces. The radiative heat transfer usually can be neglected if the convection is forced convection. Because the surface area increases as length from the object increases, a circular fin transfers more heat than a similar pin fin at any given length. Circular fins are often used to increase the heat transfer in liquid–gas heat exchanger systems. A schematic diagram for a circular fin of rectangular profile is given in Figure 3 [3]:

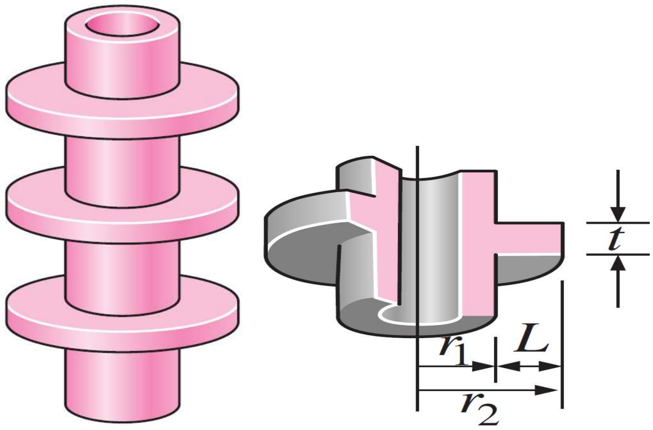

Figure 3 Schemetic Diagram of a Circular Fin with rectangular profile

4. GOVERNING EQUATION

The general heat conduction equation in a medium can be expressed in rectangular, cylindrical and spherical coordinate systems. Cylindrical coordinates conduction equation is used in this study since the problem is 2-dimensional if this coordinate system is chosen. If rectangular or Cartesian coordinate system is would be chosen, the problem would be 3-dimensional ant it would be much more complicated to solve the problem.

The general heat conduction equation in cylindrical coordinates is given as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t} \qquad (13)$$

where k is thermal conductivity, ρ is density, c is specific heat, and $\dot{e}_{gen}$ is the heat generated in a unit volume.

The base temperature, ambient fluid temperature, the combined convection-radiation coefficient, and thermal conductivity of the fin material are assumed as constant. The problem is a steady, and there is no heat generation inside the fin. Under these assumptions, the governing equation becomes:

$$\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial r^2}+\frac{\partial^2 T}{\partial z^2}=0 \qquad (14)$$

For the governing equation, the central difference expressions can be written as:

$$\frac{\partial T}{\partial r}\approx\frac{T_{i+1,j}-T_{i-1,j}}{2..\Delta r} \qquad (15)$$

$$\frac{\partial^2 T}{\partial r^2}\approx\frac{T_{i+1,j}+T_{i-1,j}-2T_{i,j}}{(\Delta r)^2} \qquad (16)$$

$$\frac{\partial^2 T}{\partial z^2}\approx\frac{T_{i,j+1}+T_{i,j-1}-2T_{i,j}}{(\Delta z)^2} \qquad (17)$$

Using these finite difference equations, for the internal grid points, the governing equation (14) is discretized as:

$$T_{i,j}=\frac{T_{i+1,j}\left(\frac{1}{2r_i}+\frac{1}{\Delta r}\right)+T_{i-1,j}\left(\frac{1}{\Delta r}-\frac{1}{2r_i}\right)+\frac{T_{i,j+1}}{\Delta z}+\frac{T_{i,j-1}}{\Delta z}}{\left(\frac{2}{\Delta r}+\frac{2}{\Delta z}\right)} \qquad (18)$$

Boundary Conditions

At the base of the fine, the temperature is constant, and it the base temperature. So, the boundary condition for the base is:

$$T_{i,j} = T_b \tag{19}$$

as seen in Figure 4.

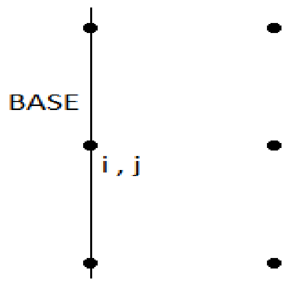

Figure 4 The boundary condition for the base of the fin

For the upper side, the heat conducted from the lower nodes should be equal to the convection to outside

$$-k\frac{\partial T}{\partial z} = h(T - T_\infty) \tag{20}$$

as seen in Figure 5, where T_∞ is the ambient fluid temperature.

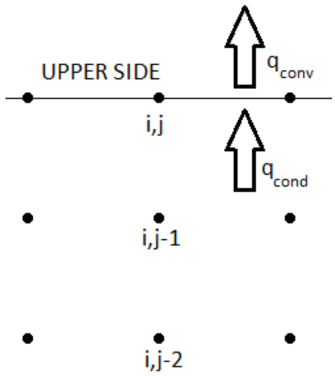

Figure 5 The boundary condition for the upper side of the fin

As this equation is dicretized by equation 11, the boundary condition becomes:

$$T_{i,j} = \frac{-\frac{k}{2.\Delta z}T_{i,j-2} + \frac{2k}{\Delta z}T_{i,j-1} + hT_{\infty}}{h + \frac{3}{2}\frac{k}{\Delta z}} \tag{21}$$

For the lower side, the heat conducted from the upper nodes should be equal to the convection to outside:

$$-k\frac{\partial T}{\partial z} = h(T - T_{\infty}) \tag{22}$$

as seen in Figure 6.

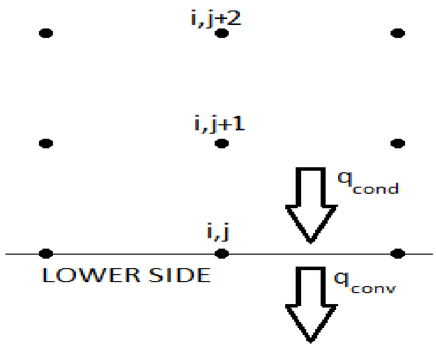

Figure 6 The boundary condition for the lower side of the fin

As this equation is dicretized by equation 12, the boundary condition becomes:

$$T_{i,j} = \frac{-\frac{k}{2.\Delta z}T_{i,j+2} + \frac{2k}{\Delta z}T_{i,j+1} + hT_{\infty}}{h + \frac{3}{2}\frac{k}{\Delta z}} \tag{23}$$

For the tip of the fin, the heat conducted from the inner nodes should be equal to the convection to outside:

$$-k\frac{\partial T}{\partial r} = h\left(T - T_{\infty}\right) \tag{24}$$

as seen in Figure 7.

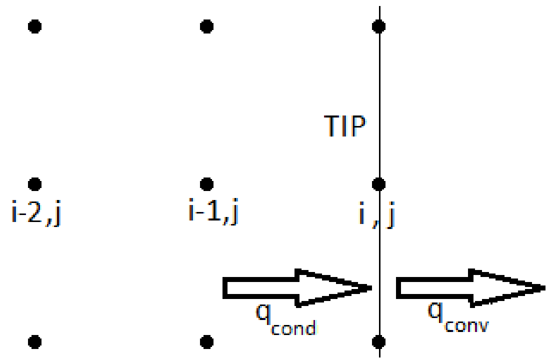

Figure 7 The boundary condition for the tip of the fin

As this equation is dicretized by equation 12, the boundary condition becomes:

$$T_{i,j} = \frac{-\frac{k}{2.\Delta r} T_{i-2,j} + \frac{2k}{\Delta r} T_{i-1,j} + hT_{\infty}}{h + \frac{3}{2}\frac{k}{\Delta r}} \tag{25}$$

5. STUDY

A Matlab code has been written to calculate the temperature distribution inside the fin. Gauss-Seidel iterative method was used for iteration with an overrelaxation parameter, w, between 1 and 2 to speed up convergence:

$$T_{i,j} = wT_{(i,j)new} + (1-w)T_{(i,j)old} \tag{26}$$

After finding the temperature distribution, the heat transferred to ambient air from the fin has been calculated from the equation:

$$\dot{Q}_{fin} = -kA\frac{dT}{dr}\bigg|_{base} \tag{27}$$

since the heat transferred to ambient fluid is equal to the heat that is conducted from the base of the fin.

The fin efficiency is defined as:

$$\eta_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{fin,\max}} \tag{28}$$

where $\dot{Q}_{fin,\max}$ is the heat transfer from a perfect fin with an infinite thermal conductivity, which has a surface temperature equal to the base temperature. So $\dot{Q}_{fin,\max}$ is defined as:

$$\dot{Q}_{fin,\max} = hA_{fin}\left(T_{base} - T_{\infty}\right) \tag{29}$$

Heat transfer from the fin was calculated numerically and compared to the analytical solution exist in the literature. The analytical solution of efficiency is given as:

$$\eta_{fin} = C_2\,\frac{K_1(mr_1)I_1(mr_{2c}) - I_1(mr_1)K_1(mr_{2c})}{I_0(mr_1)K_1(mr_{2c}) + K_0(mr_1)I_1(mr_{2c})} \tag{30}$$

where $m = \sqrt{2h/kt}$, $C_2 = \dfrac{2r_1/m}{r_{2c}^2 - r_1^2}$, and I_0, I_1, K_0, K_1 are modified Bessel functions of the first and second kind.

6. RESULTS

The temperature distribution inside the fin was calculated numerically for 3 cases: $r_{2c}/r_1 = 2$, $r_{2c}/r_1 = 3$, and $r_{2c}/r_1 = 4$. The heat transfer from the fin was calculated numerically from the Eq.27, and fin efficiency was calculated from Eq.28, for different ξ values,

$$\xi = L_c^{3/2}\left(\frac{h}{kA_p}\right)^{1/2} \qquad (31)$$

where L_c is corrected length, $L_c = L + t/2$ and $A_p = L_c t$.

The results are given in Figure 8. The square, diamond, and triangle values are the numerical results while the continuous solid lines are analytical values from Equation 30. There is a good agreement between the numerical solution and the analytical solution as it is seen in the figure.

It is seen in the figure that fin efficiency approaches to 1 as the dimensionless variable ξ goes to zero.

As fin length L or convection coeffient h goes to zero, or thermal conductivity k of the fin is very large in Eq. 31, ξ approaches to 0, which means that the temperature of the fin is close to the temperature of the base, which means the efficiency is very close to 1, as expected.

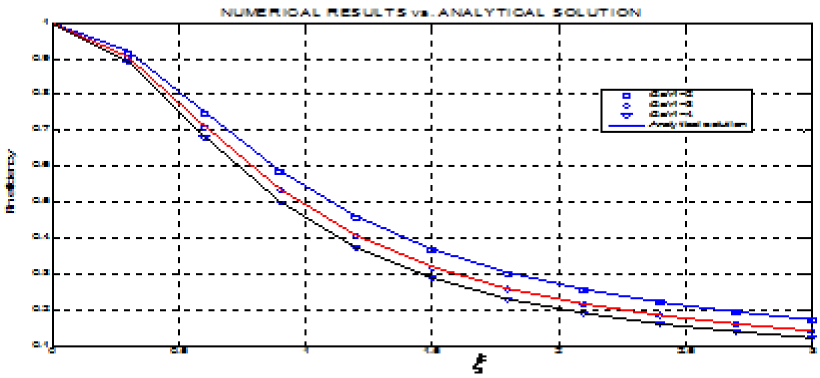

Figure 8 Numerical Results vs. Analytical Solution

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